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On Orders of Approximation of Function Classes in Lorentz Spaces with Anisotropic Norm

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Abstract—In this paper, we study the anisotropic Lorentz space of periodic functions. We establish a sharp estimate of the order of approximation for the Besov class by trigonometric polynomials in Lorentz spaces with anisotropic norm.

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Key words: *anisotropic Lorentz space, Besov class, approximation of function classes, trigonometric polynomial, periodic function, Lebesgue space, Hölder's inequality.*

Suppose that $\mathbb{R}^m$ is the m-dimensional Euclidean space of points $\overline{x} = (x_1, \dots, x_m)$ with real coordinates, $I^m = [0, 2\pi)^m$ is the m-dimensional cube, and we are given

$$\overline{q} = (q_1, \dots, q_m), \quad \overline{\theta} = (\theta_1, \dots, \theta_m), \qquad q_j, \theta_j \in [1, +\infty), \quad j = 1, \dots, m.$$

By $L_{\overline{q},\overline{\theta}}(I^m)$ we denote the anisotropic Lorentz space of Lebesgue measurable (2π)-periodic functions $f(\overline{x})$ for which the norm

$$\|f\|_{\overline{q},\overline{\theta}} = \left[\int_0^{2\pi} t_m^{\frac{\theta_m}{q_m}-1}\left[\cdots\left[\int_0^{2\pi}\left(f^{*_1,\dots,*_m}(t_1,\dots,t_m)\right)^{\theta_1} t_1^{\frac{\theta_1}{q_1}-1}\,dt_1\right]^{\frac{\theta_2}{\theta_1}}\cdots\right]^{\frac{\theta_m}{\theta_{m-1}}}dt_m\right]^{\frac{1}{\theta_m}}$$

is finite; here $f^{*_1,\dots,*_m}(t_1,\dots,t_m)$ is a nonincreasing rearrangement of the function $|f(\overline{x})|$ with respect to each variable x_j for fixed values of the other variables (first, with respect to x_1, then with respect to x_2 etc.; see [1], [2]). $L_{\overline{q},\infty}(I^m)$ is the Marcinkiewicz space of functions for which

$$\|f\|_{\overline{q},\infty} = \sup_{\overline{t}\in I^m} \prod_{j=1}^m t_j^{\frac{1}{q_j}} f^{*_1,\dots,*_m}(t_1,\dots,t_m) < +\infty.$$

The Lebesgue space $L_{\overline{p}}(I^m)$, $p_j \in [1, +\infty)$, $j = 1, \dots, m$, with mixed norm consists of all Lebesgue measurable functions $f(\overline{x})$ for which (see [3], [4])

$$\|f\|_{\overline{p}} = \left[\int_0^{2\pi}\left[\cdots\left[\int_0^{2\pi}|f(\overline{x})|^{p_1}\,dx_1\right]^{\frac{p_2}{p_1}}\cdots\right]^{\frac{p_m}{p_{m-1}}}dx_m\right]^{\frac{1}{p_m}} < +\infty.$$

$L^0_{\overline{p}}(I^m)$ is the set of all functions $f \in L_{\overline{p}}(I^m)$ such that

$$\int_0^{2\pi} f(\overline{x})\,dx_j = 0, \qquad j = 1, \dots, m.$$

Suppose we are given a vector $\overline{r} = (r_1, \dots, r_m)$, $r_j > 0$, $j = 1, \dots, m$. Consider the Nikol′skii class $H^{\overline{r}}_{\overline{p}}$ consisting of all functions $f \in L_{\overline{p}}(I^m)$ for which

$$\Omega_{\overline{l}}(f, \overline{t})_{\overline{p}} \le \prod_{j=1}^m t_j^{r_j}, \qquad \overline{t} \in [0,1]^m,$$

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where $l_j > r_j$, $j = 1, \dots, m$, and $\Omega_{\overline{l}}(f, \overline{t})_{\overline{p}}$ is the mixed modulus of smoothness of the function $f \in L_{\overline{p}}(I^m)$ (see [3]).

In what follows,

$$\delta_{\overline{s}}(f, \overline{x}) = \sum_{\overline{n} \in \rho(\overline{s})} a_{\overline{n}}(f) e^{i\langle \overline{n}, \overline{x} \rangle},$$

where

$$\langle \overline{y}, \overline{x} \rangle = \sum_{j=1}^{m} y_j x_j, \qquad \rho(\overline{s}) = \{\overline{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : 2^{s_j - 1} \le |k_j| < 2^{s_j},\ j = 1, \dots, m\},$$

and the $a_{\overline{n}}(f)$ are the Fourier coefficients of a function $f \in L_1(I^m)$ in the multiple system $\{e^{i\langle \overline{n}, \overline{x} \rangle}\}$.

Suppose we are given a vector $\overline{r} = (r_1, \dots, r_n)$. Set $\overline{\gamma} = \overline{r}/r_1$, $r_j > 0$, $j = 1, \dots, m$, and

$$Q_n^{\overline{\gamma}} = \bigcup_{\langle \overline{s}, \overline{\gamma} \rangle \le n} \rho(\overline{s}), \qquad T(Q_n^{\overline{\gamma}}) = \Bigg\{ t(\overline{x}) = \sum_{\overline{k} \in Q_n^{\overline{\gamma}}} b_{\overline{k}} e^{i\langle \overline{k}, \overline{x} \rangle} \Bigg\};$$

$E_n^{(\overline{\gamma})}(f)_{\overline{q}, \overline{\theta}}$ is the best approximation of a function $f \in L_{\overline{q}, \overline{\theta}}(I^m)$ by polynomials from the set $T(Q_n^{\overline{\gamma}})$.

By $C(\alpha, \beta, \dots)$ we denote positive values depending on the parameters (given in parentheses), which are, in general, different in different formulas.

For brevity, we write $A \asymp B$ to indicate that there exist positive constants c_1 and c_2 such that $c_1 \cdot A \le B \le c_2 \cdot A$.

The theory of embedding of function spaces $H_{\overline{p}}^{\overline{r}}$ was studied by Nikol′skii [3], Amanov [4], and other mathematicians (see the references in [3], [4]). In Lebesgue spaces with isotropic norm, the orders of approximation of H-classes by trigonometric polynomials with harmonics from hyperbolic crosses were studied in [5]–[12], while those in spaces with mixed norm were investigated in [13], [14]. In the present paper, we present approximation estimates for classes $H_{\overline{p}}^{\overline{r}}$ in Lorentz and Marcinkiewicz spaces with anisotropic norm.

Lemma A (see [15]). *Suppose we are given positive numbers* $0 < \theta < +\infty$ *and* $a_k, b_k,\ k = 1, 2, \dots$.

a) If $\sum_{k=1}^{n} b_k \le C \cdot b_n$, *then*

$$\sum_{n=1}^{\infty} a_n \bigg(\sum_{k=n}^{\infty} b_k \bigg)^{\theta} \le C \cdot \sum_{n=1}^{\infty} a_n b_n^{\theta}.$$

b) If $\sum_{k=n}^{\infty} b_k \le C \cdot b_n$, *then*

$$\sum_{n=1}^{\infty} a_n \bigg(\sum_{k=1}^{n} b_k \bigg)^{\theta} \le C \cdot \sum_{n=1}^{\infty} a_n \cdot b_n^{\theta}.$$

Lemma B (see [3, p. 133], [16]). *Suppose that* $1 \le p_j < \infty$, $j = 1, \dots, m$. *For any trigonometric polynomial*

$$T_{\overline{n}}(\overline{x}) = \sum_{\overline{k} \le \overline{n}} a_{\overline{k}} e^{i\langle \overline{k}, \overline{x} \rangle},$$

the following inequality holds:

$$\max_{\overline{x} \in I^m} |T_{\overline{n}}(\overline{x})| \le C \cdot \prod_{j=1}^{m} n_j^{\frac{1}{p_j}} \cdot \|T_{\overline{n}}\|_{\overline{p}}.$$

Let us now prove some auxiliary assertions.

Set

$$G_e(\overline{n}) = \{\overline{s} = (s_1, \dots, s_m) \in \mathbb{N}^m : s_j \le n_j,\ j \in e; \quad s_j > n_j,\ j \notin e\},$$

where $e \subset \{1, \dots, m\}$;

$$U_{\overline{n}}(f, \overline{x}) = \sum_{e \subset \{1,\dots,m\}} \sum_{\overline{s} \in G_e(\overline{n})} \delta_{\overline{s}}(f, \overline{x}).$$

Let there be given numbers $\theta_j \in [1, +\infty)$, $j = 1, \dots, m$. By $\ell_{\overline{\theta}}(\mathbb{Z}_+^m)$ we denote the space of all numerical sequences $\{a_{\overline{s}}\}_{\overline{s} \in \mathbb{Z}_+^m}$ for which

$$\|\{a_{\overline{s}}\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^m)} = \left\{ \sum_{s_m=0}^{\infty} \left[\cdots \left\{ \sum_{s_1=0}^{\infty} |a_{\overline{s}}|^{\theta_1} \right\}^{\frac{\theta_2}{\theta_1}} \cdots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}} < +\infty.$$

Set

$$Y^m(n, \overline{\gamma}) = \{\overline{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m : \langle \overline{s}, \overline{\gamma} \rangle > n\},$$

where $\overline{\gamma} = (\gamma_1, \dots, \gamma_m)$.

Lemma 1. *Suppose that $\overline{\theta} = (\theta_1, \dots, \theta_m)$, $1 \le \theta_j < +\infty$, $j = 1, \dots, m$, $\alpha \in [0, +\infty)$,*

$$b_{\overline{s}}(n, \overline{\gamma}) = \begin{cases} 2^{-\alpha\langle \overline{s}, \overline{\gamma}\rangle} & \text{for } \overline{s} \in Y^m(n, \overline{\gamma}), \\ 0 & \text{for } \overline{s} \notin Y^m(n, \overline{\gamma}). \end{cases}$$

Then the following inequality holds:

$$\|\{b_{\overline{s}}(n, \overline{\gamma})\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^m)} \le C \cdot 2^{-n\alpha} \cdot n^{\sum_{j=2}^{m} \frac{1}{\theta_j}}.$$

Proof. Let us prove the lemma by induction on the dimension m. For $m = 1$, the assertion of the lemma is obvious.

Suppose that $m = 2$. It follows from the definition of the set $Y^m(n, \overline{\gamma})$ and the numbers $b_{\overline{s}}(n, \overline{\gamma})$ that

$$\begin{aligned} &\|\{b_{\overline{s}}(n, \overline{\gamma})\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^2)} \\ &= \left\{ \sum_{s_2 < \frac{n}{\gamma_2}} \left[\sum_{s_1 > \frac{1}{\gamma_1}(n - \gamma_2 s_2)} |b_{\overline{s}}(n, \overline{\gamma})|^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} + \sum_{s_2 \ge \frac{n}{\gamma_2}} \left[\sum_{s_1=0}^{\infty} |b_{\overline{s}}(n, \overline{\gamma})|^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \right\}^{\frac{1}{\theta_2}} \\ &= \left\{ \sum_{s_2 < \frac{n}{\gamma_2}} \left[\sum_{s_1 > \frac{1}{\gamma_1}(n - \gamma_2 s_2)} 2^{-\alpha(s_1\gamma_1 + s_2\gamma_2)\theta_1} \right]^{\frac{\theta_2}{\theta_1}} + \sum_{s_2 \ge \frac{n}{\gamma_2}} \left[\sum_{s_1=0}^{\infty} 2^{-\alpha(s_1\gamma_1 + s_2\gamma_2)\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \right\}^{\frac{1}{\theta_2}} \\ &\le C(\gamma_1, \gamma_2, \theta_1, \theta_2) \cdot 2^{-n\alpha} \cdot \left\{ \sum_{s_2 < \frac{n}{\gamma_2}} 1 + 1 \right\}^{\frac{1}{\theta_2}} \le C(\gamma_1, \gamma_2, \theta_1, \theta_2) \cdot 2^{-n\alpha} \cdot n^{\frac{1}{\theta_2}}. \end{aligned}$$

Now suppose that the assertion of the lemma is valid for $m = \nu$, i.e.,

$$\|\{b_{\overline{s}}(n, \overline{\gamma})\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^{\nu})} \le C \cdot 2^{-n\alpha} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}. \tag{1}$$

Let us prove that the assertion of the lemma is also valid for $m = \nu + 1$. For brevity, set $\overline{s}(\nu) = (s_1, \dots, s_\nu)$. Then, by the definition of the norm, we have

$$\|\{b_{\overline{s}}(n, \overline{\gamma})\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^{\nu+1})} = \left\{ \sum_{s_{\nu+1}=0}^{\infty} \|\{b_{\overline{s}(\nu), s_{\nu+1}}(n, \overline{\gamma})\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^{\nu})}^{\theta_{\nu+1}} \right\}^{\frac{1}{\theta_{\nu+1}}}$$

$$= \left\{ \sum_{s_{\nu+1} \le \frac{n}{\gamma_{\nu+1}}} \|\{b_{\overline{s}(\nu),s_{\nu+1}}(n,\overline{\gamma})\}\|^{\theta_{\nu+1}}_{\ell_{\overline{\theta}}(\mathbb{Z}^\nu_+)} + \sum_{s_{\nu+1} > \frac{n}{\gamma_{\nu+1}}} \|\{b_{\overline{s}(\nu),s_{\nu+1}}(n,\overline{\gamma})\}\|^{\theta_{\nu+1}}_{\ell_{\overline{\theta}}(\mathbb{Z}^\nu_+)} \right\}^{\frac{1}{\theta_{\nu+1}}}. \qquad (2)$$

If $s_{\nu+1} \le n/\gamma_{\nu+1}$, then, by definition,

$$b_{\overline{s}}(n,\overline{\gamma}) = \begin{cases} 0 & \text{for } \sum_{j=1}^{\nu} \gamma_j s_j < n - \gamma_{\nu+1}\cdot s_{\nu+1}, \\ 2^{-\alpha\langle \overline{s},\overline{\gamma}\rangle} & \text{for } \sum_{j=1}^{\nu} \gamma_j s_j > n - \gamma_{\nu+1}\cdot s_{\nu+1}. \end{cases}$$

Therefore, by the assumption (1), we obtain

$$\sum_{s_{\nu+1} \le \frac{n}{\gamma_{\nu+1}}} \|\{b_{\overline{s}(\nu),s_{\nu+1}}(n,\overline{\gamma})\}\|^{\theta_{\nu+1}}_{\ell_{\overline{\theta}}(\mathbb{Z}^\nu_+)} \le C\cdot 2^{-n\alpha\theta_{\nu+1}} \sum_{s_{\nu+1} \le \frac{n}{\gamma_{\nu+1}}} (n - s_{\nu+1}\gamma_{\nu+1})^{\theta_{\nu+1}\sum_{j=2}^{\nu}\frac{1}{\theta_j}}$$
$$\le C\cdot 2^{-n\alpha\theta_{\nu+1}}\cdot n^{1+\sum_{j=2}^{\nu}\frac{\theta_{\nu+1}}{\theta_j}}. \qquad (3)$$

If $s_{\nu+1} > n/\gamma_{\nu+1}$, then

$$\sum_{j=1}^{\nu+1} s_j\gamma_j > n \qquad \forall\,(s_1,\dots,s_\nu)\in\mathbb{Z}^\nu_+.$$

Therefore,

$$\sum_{s_{\nu+1} > \frac{n}{\gamma_{\nu+1}}} \|\{b_{\overline{s}}(n,\overline{\gamma})\}\|^{\theta_{\nu+1}}_{\ell_{\overline{\theta}}(\mathbb{Z}^\nu_+)} = C(\gamma_1,\dots,\gamma_\nu,\theta_1,\dots,\theta_\nu) \sum_{s_{\nu+1} > \frac{n}{\gamma_{\nu+1}}} 2^{-s_{\nu+1}\alpha\gamma_{\nu+1}\theta_{\nu+1}}$$
$$\le C(\gamma_1,\dots,\gamma_\nu,\theta_1,\dots,\theta_\nu) 2^{-n\alpha\theta_{\nu+1}}. \qquad (4)$$

It follows from inequalities (2)–(4) that

$$\|\{b_{\overline{s}}(n,\overline{\gamma})\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}^{\nu+1}_+)} \le C(\gamma,\theta)\cdot 2^{-n\alpha}\cdot n^{\sum_{j=2}^{\nu+1}\frac{1}{\theta_j}}.$$

The lemma is proved. □

Lemma 2. *Suppose that* $\overline{\theta} = (\theta_1,\dots,\theta_m)$, $1 \le \theta_j < +\infty$, $j = 1,\dots,m$, $\overline{\gamma} = (\gamma_1,\dots,\gamma_m)$, $\overline{\gamma}' = (\gamma'_1,\dots,\gamma'_m)$ *satisfy* $\gamma_1 = \dots = \gamma_\nu = \gamma'_1 = \dots = \gamma'_\nu = 1 < \gamma'_j < \gamma_j$ *for any* $j = \nu+1,\dots,m$ *and*

$$b_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}') = \begin{cases} 2^{-\alpha\langle \overline{s},\overline{\gamma}\rangle} & \text{if } \overline{s}\in Y^m(n,\overline{\gamma}'), \\ 0 & \text{for } \overline{s}\notin Y^m(n,\overline{\gamma}'), \end{cases} \qquad \alpha > 0.$$

Then the following inequality holds:

$$\|\{b_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}^m_+)} \le C(\theta,\overline{\gamma},\overline{\gamma}')\cdot 2^{-n\alpha}\cdot n^{\sum_{j=2}^{\nu}\frac{1}{\theta_j}}.$$

Proof. The set $Y^m(n,\overline{\gamma}')$ can be expressed as

$$Y^m(n,\overline{\gamma}') = \left\{\overline{s}\in\mathbb{Z}^m_+ : \sum_{j=\nu+1}^{m} s_j\gamma'_j < n \text{ and } \sum_{j=1}^{\nu} s_j\gamma'_j > n - \sum_{j=\nu+1}^{m} s_j\gamma'_j\right\}$$
$$\cup \left\{\overline{s}\in\mathbb{Z}^m_+ : \sum_{j=\nu+1}^{m} s_j\gamma'_j \ge n,\ s_1,\dots,s_\nu \ge 0\right\} = Y^m_1(\overline{\gamma}') \cup Y^m_2(\overline{\gamma}').$$

We define the numbers $b^{(1)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')$ and $b^{(2)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')$ as follows:

$$b^{(1)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}') = \begin{cases} 2^{-\alpha\langle \overline{s},\overline{\gamma}\rangle}, & \overline{s}\in Y^m_1(\overline{\gamma}'), \\ 0, & \overline{s}\in Y^m_2(\overline{\gamma}'), \end{cases} \qquad b^{(2)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}') = b_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}') - b^{(1)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}').$$

Obviously, $b_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}') = b^{(2)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}') + b^{(1)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')$.

Set

$$Y_3^{m-\nu}(\overline{\gamma}',n) = \{(s_{\nu+1},\dots,s_m) \in \mathbb{Z}_+^{m-\nu} : \sum_{j=\nu+1}^{m} s_j\gamma_j' < n\}.$$

Since $\gamma_1 = \gamma_1' = \dots = \gamma_\nu = \gamma_\nu'$, $\gamma_j' < \gamma_j$ for each $j = \nu+1,\dots,m$ and $\alpha > 0$, by Lemma 1 (with m replaced by ν and n by $n - \sum_{j=\nu+1}^m s_j\gamma_j'$), we have

$$\begin{aligned}
&\|\{b^{(1)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^m)} \\
&= \left\| \left\{ 2^{-\alpha\sum_{j=\nu+1}^m s_j\gamma_j} \left\| \left\{ 2^{-\alpha\sum_{j=1}^{\nu} s_j\gamma_j} \right\} \right\|_{\ell_{\overline{\theta}}(Y^\nu(\overline{\gamma}',n-\sum_{j=\nu+1}^m s_j\gamma_j'))} \right\} \right\|_{\ell_{\overline{\theta}}(Y_3^{m-\nu}(\overline{\gamma}',n))} \\
&\le C(\theta,\gamma)2^{-n\alpha} \left\| \left\{ \left(n - \sum_{j=\nu+1}^m s_j\gamma_j' \right)^{\sum_{j=2}^{\nu}\frac{1}{\theta_j}} 2^{-\alpha\sum_{j=\nu+1}^m s_j(\gamma_j-\gamma_j')} \right\} \right\|_{\ell_{\overline{\theta}}(Y_3^{m-\nu}(\overline{\gamma}',n))} \\
&\le C(\theta,\gamma)2^{-n\alpha} n^{\sum_{j=2}^{\nu}\frac{1}{\theta_j}} \cdot \prod_{j=\nu+1}^m \left\{ \sum_{s_j=0}^{\infty} 2^{-\alpha s_j\theta_j(\gamma_j-\gamma_j')} \right\}^{\frac{1}{\theta_j}} \\
&= C(\theta,\gamma)2^{-n\alpha} n^{\sum_{j=2}^{\nu}\frac{1}{\theta_j}}. \qquad (5)
\end{aligned}$$

Further, by the definition of $b^{(2)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')$ for all $(s_{\nu+1},\dots,s_m)$ such that $\sum_{j=\nu+1}^m s_j\gamma_j' > n$, the following estimate holds:

$$b^{(2)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}') = 2^{-\alpha\langle\overline{s},\overline{\gamma}\rangle} \le 2^{-n\alpha} 2^{-\alpha\sum_{j=1}^{\nu} s_j\gamma_j} 2^{-\alpha\sum_{j=\nu+1}^m s_j(\gamma_j-\gamma_j')}.$$

Therefore, taking into account the fact that the norm is monotonic and also the inequality $\gamma_j' < \gamma_j$, $j = \nu+1,\dots,m$, we obtain

$$\begin{aligned}
\|\{b^{(2)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^m)} &= 2^{-n\alpha} \left\| \left\{ 2^{-\alpha\sum_{j=1}^{\nu} s_j\gamma_j} \cdot 2^{-\alpha\sum_{j=\nu+1}^m s_j(\gamma_j-\gamma_j')} \right\} \right\|_{\ell_{\overline{\theta}}(Y_2^m(\overline{\gamma}',n)} \\
&\le 2^{-n\alpha} \cdot \prod_{j=1}^{\nu} \left\{ \sum_{s_j=0}^{\infty} 2^{-\alpha\theta_j s_j} \right\}^{\frac{1}{\theta_j}} \cdot \prod_{j=\nu+1}^m \left\{ \sum_{s_j=0}^{\infty} 2^{-\alpha\theta_j s_j(\gamma_j-\gamma_j')} \right\}^{\frac{1}{\theta_j}} = C(\gamma,\theta)\cdot 2^{-n\alpha}. \qquad (6)
\end{aligned}$$

Now, by (5) and (6), we have

$$\begin{aligned}
\|\{b_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^m)} &\le \|\{b^{(1)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^m)} + \|\{b^{(2)}_{\overline{s}}(n,\overline{\gamma},\overline{\gamma}')\}\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^m)} \\
&\le C(\theta,\gamma)\cdot 2^{-n\alpha}\cdot n^{\sum_{j=2}^{\nu}\frac{1}{\theta_j}}.
\end{aligned}$$

The lemma is proved. □

Remark 1. For $\theta_1 = \dots = \theta_m$, Lemmas 1 and 2 imply Lemmas B and C from [7].

Set

$$\overline{f}(\overline{t}) = \sup_{|E_m|\ge t_m} \frac{1}{|E_m|} \int_{E_m} \cdots \sup_{|E_1|\ge t_1} \frac{1}{|E_1|} \int_{E_1} |f(x_1,\dots,x_m)|\,dx_1\cdots dx_m,$$

where $|E_j|$ is the Lebesgue measure of the set $E_j \in [0,2\pi)$.

Now, let us prove the main assertions.

Theorem 1. *Suppose that* $1 \le p_j < +\infty$, $j = 1, \dots, m$. *Then, for any function* $f \in L_{\overline{p}}(I^m)$, *the following inequality holds:*

$$\overline{f}(\overline{t}) \le C(p,m)\Big\{\prod_{j=1}^{m} t_j^{-\frac{1}{p_j}} \|f - U_{\overline{n}}(f)\|_{\overline{p}} + \sum_{e\subset\{1,\dots,m\}} \prod_{j\notin e} t_j^{-\frac{1}{p_j}} \sum_{\overline{s}\in G_e(\overline{n})} \prod_{j\in e} 2^{\frac{s_j}{p_j}} \|\delta_{\overline{s}}(f)\|_{\overline{p}}\Big\}$$

for all $t_j \in (2^{-n_j-1}, 2^{-n_j}]$, $n_j = 0, 1, \dots,$ $j = 1, \dots, m$.

Proof. Suppose that $E_j \subset [0, 2\pi]$ are Lebesgue measurable sets. Then, by the property of the integral, we have

$$\int_{E_m} \cdots \int_{E_1} |f(x_1,\dots,x_m)|\, dx_1 \cdots dx_m \le \int_{E_m} \cdots \int_{E_1} |f(\overline{x}) - U_{\overline{n}}(f,\overline{x})|\, dx_1 \cdots dx_m + \int_{E_m} \cdots \int_{E_1} |U_{\overline{n}}(f,\overline{x})|\, d\overline{x}. \tag{7}$$

Applying Hölder's integral inequality, we obtain

$$\int_{E_m} \cdots \int_{E_1} |f(\overline{x}) - U_{\overline{n}}(f,\overline{x})|\, d\overline{x} \le \prod_{j=1}^{m} |E_j|^{\frac{1}{p_j'}} \|f - U_{\overline{n}}(f)\|_{\overline{p}}, \tag{8}$$

where $p_j' = p_j/(p_j - 1)$, $j = 1, \dots, m$.

Suppose that $e = \{1, \dots, i\} \subset \{1, \dots, m\}$. Then, applying the inequality for different metrics for the polynomials in the trigonometric system (see Lemma B) for the variables $x_1, \dots, x_i$, and Hölder's inequality for the other variables, we obtain

$$\begin{aligned}
\int_{E_m} \cdots \int_{E_1} \Big| \sum_{\overline{s}\in G_e(\overline{n})} \delta_{\overline{s}}(f,\overline{x}) \Big|\, d\overline{x} &\le \int_{E_m} \cdots \int_{E_1} \sum_{\overline{s}\in G_e(\overline{n})} |\delta_{\overline{s}}(f,\overline{x})|\, d\overline{x} \\
&= \int_{E_m} \cdots \int_{E_1} \sum_{s_1=1}^{n_1} \cdots \sum_{s_i=1}^{n_i} \sum_{s_{i+1}=n_{i+1}+1}^{\infty} \cdots \sum_{s_m=n_m+1}^{\infty} |\delta_{\overline{s}}(f,\overline{x})|\, d\overline{x} \\
&\le C \cdot \sum_{s_1=1}^{n_1} \cdots \sum_{s_i=1}^{n_i} \sum_{s_{i+1}=n_{i+1}+1}^{\infty} \cdots \sum_{s_m=n_m+1}^{\infty} \prod_{j=1}^{i} 2^{\frac{s_j}{p_j}} \|\delta_{\overline{s}}(f)\|_{\overline{p}} \cdot \prod_{j=1}^{i} |E_j| \cdot \prod_{j=i+1}^{m} |E_j|^{\frac{1}{p_j'}} \\
&= C \cdot \prod_{j\in e} |E_j| \cdot \prod_{j\notin e} |E_j|^{\frac{1}{p_j'}} \sum_{\overline{s}\in G_e(\overline{n})} \prod_{j\in e} 2^{\frac{s_j}{p_j}} \|\delta_{\overline{s}}(f)\|_{\overline{p}}.
\end{aligned} \tag{9}$$

For the other sets $e \subset \{1, \dots, m\}$, the proof of inequality (9) is similar.

In the definition of the function $\overline{f}$, the measures $|E_j|$ satisfy $|E_j| \ge t_j$, $j = 1, \dots, m$. Therefore, from inequalities (7)–(9) we obtain

$$\prod_{j=1}^{m} \frac{1}{|E_j|} \int_{E_m} \cdots \int_{E_1} |f(\overline{x})|\, d\overline{x} \le C \cdot \Big\{ \prod_{j=1}^{m} t_j^{-\frac{1}{p_j}} \|f - U_{\overline{n}}(f)\|_{\overline{p}} + \sum_{e\subset\{1,\dots,m\}} \prod_{j\notin e} t_j^{-\frac{1}{p_j}} \sum_{\overline{s}\in G_e(\overline{n})} \prod_{j\in e} 2^{\frac{s_j}{p_j}} \|\delta_{\overline{s}}(f)\|_{\overline{p}} \Big\}.$$

Hence, by the definition of the least upper bound of a set, we obtain the assertion of the theorem. □

Theorem 2. *Suppose that* $1 \le p_j < q_j < +\infty$, $1 \le \theta_j < +\infty$. *If* $f \in L_{\overline{p}}(I^m)$ *and the quantity*

$$\sigma(f) \equiv \Big\{ \sum_{s_m=1}^{\infty} 2^{s_m\theta_m\left(\frac{1}{p_m} - \frac{1}{q_m}\right)} \Big[\cdots \Big[\sum_{s_1=1}^{\infty} 2^{s_1\theta_1\left(\frac{1}{p_1} - \frac{1}{q_1}\right)} \|\delta_{\overline{s}}(f)\|_{\overline{p}}^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} \cdots \Big]^{\frac{\theta_m}{\theta_{m-1}}} \Big\}^{\frac{1}{\theta_m}}$$

is finite, then the function f belongs to $L_{\overline{q},\overline{\theta}}(I^m)$ and the following inequality holds:

$$\|f\|_{\overline{q},\overline{\theta}} \le C(\overline{p},\overline{q},\overline{\theta})\cdot\sigma(f).$$

Proof. Blozinski [1, p. 161] proved the relation

$$\|f\|_{\overline{q},\overline{\theta}} \asymp \|\overline{f}\|_{\overline{q},\overline{\theta}}, \qquad 1\le q_j<+\infty,\ \ 1\le\theta_j<+\infty, \quad j=1,\dots,m.$$

Using this relation, we find

$$\|f\|_{\overline{q},\overline{\theta}} \le C\|\overline{f}\|_{\overline{q},\overline{\theta}} = C\biggl\{\sum_{n_m=0}^{\infty}\int_{2^{-n_m-1}}^{2^{-n_m}} t_m^{\frac{\theta_m}{q_m}-1}\biggl[\cdots\biggl[\sum_{n_1=0}^{\infty}\int_{2^{-n_1-1}}^{2^{-n_1}} t_1^{\frac{\theta_1}{q_1}-1} \times \bigl(\overline{f}(t_1,\dots,t_m)\bigr)^{\theta_1}dt_1\biggr]^{\frac{\theta_2}{\theta_1}}\cdots\biggr]^{\frac{\theta_m}{\theta_{m-1}}}dt_m\biggr\}^{\frac{1}{\theta_m}}. \tag{10}$$

Further, applying Theorem 1 and taking into account the relation

$$\int_{2^{-n-1}}^{2^{-n}} t^{\beta-1}dt \asymp 2^{-n\beta}, \tag{11}$$

from (10) we obtain

$$\|f\|_{\overline{q},\overline{\theta}} \le C\Biggl[\biggl\{\sum_{n_m=0}^{\infty}2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)}\biggl[\cdots\biggl[\sum_{n_1=0}^{\infty}2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)}\|f-U_{\overline{n}}(f)\|_{\overline{p}}^{\theta_1}\biggr]^{\frac{\theta_2}{\theta_1}}\cdots\biggr]^{\frac{\theta_m}{\theta_{m-1}}}\biggr\}^{\frac{1}{\theta_m}}$$
$$+\biggl\{\sum_{n_m=0}^{\infty}\int_{2^{-n_m-1}}^{2^{-n_m}} t_m^{\frac{\theta_m}{q_m}-1}\biggl[\cdots\biggl[\sum_{n_1=0}^{\infty}\int_{2^{-n_1-1}}^{2^{-n_1}} t_1^{\frac{\theta_1}{q_1}-1}\biggl(\sum_{e\subset\{1,\dots,m\}}\prod_{j\notin e}t_j^{-\frac{1}{p_j}}$$
$$\times\sum_{\overline{s}\in G_e(\overline{n})}\prod_{j\in e}2^{\frac{s_j}{p_j}}\|\delta_{\overline{s}}(f)\|_{\overline{p}}\biggr)^{\theta_1}\biggr]^{\frac{\theta_2}{\theta_1}}\cdots\biggr]^{\frac{\theta_m}{\theta_{m-1}}}dt_m\biggr\}^{\frac{1}{\theta_m}}\Biggr]. \tag{12}$$

Suppose that $e=\{1,\dots,i\}$, $i\le m$. Then, using relation (11) and successively applying the triangle inequality, we obtain

$$\biggl\{\sum_{n_m=0}^{\infty}\int_{2^{-n_m-1}}^{2^{-n_m}} t_m^{\frac{\theta_m}{q_m}-1}\biggl[\cdots\biggl[\sum_{n_1=0}^{\infty}\int_{2^{-n_1-1}}^{2^{-n_1}} t_1^{\frac{\theta_1}{q_1}-1}\biggl(\prod_{j=i+1}^{m}t_j^{-\frac{1}{p_j}}\sum_{s_m=n_m+1}^{\infty}\cdots\sum_{s_{i+1}=n_{i+1}+1}^{\infty}$$
$$\times\sum_{s_i=1}^{n_i}\cdots\sum_{s_1=1}^{n_1}\prod_{j=1}^{i}2^{\frac{s_j}{p_j}}\|\delta_{\overline{s}}(f)\|_{\overline{p}}\biggr)^{\theta_1}\biggr]^{\frac{\theta_2}{\theta_1}}\cdots\biggr]^{\frac{\theta_m}{\theta_{m-1}}}dt_m\biggr\}^{\frac{1}{\theta_m}}$$
$$\le C\biggl\{\sum_{n_m=0}^{\infty}2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)}\biggl[\cdots\sum_{n_{i+1}=0}^{\infty}2^{n_{i+1}\theta_{i+1}\left(\frac{1}{p_{i+1}}-\frac{1}{q_{i+1}}\right)}$$
$$\times\biggl[\sum_{n_i=0}^{\infty}2^{-n_i\frac{\theta_i}{q_i}}\biggl[\cdots\biggl[\sum_{n_1=0}^{\infty}2^{-n_1\frac{\theta_1}{q_1}}\biggl(\sum_{s_m=n_m+1}^{\infty}\cdots\sum_{s_{i+1}=n_{i+1}+1}^{\infty}\sum_{s_i=1}^{n_i}\cdots\sum_{s_1=1}^{n_1}$$
$$\times\prod_{j=1}^{i}2^{\frac{s_j}{p_j}}\|\delta_{\overline{s}}(f)\|_{\overline{p}}\biggr)^{\theta_1}\biggr]^{\frac{\theta_2}{\theta_1}}\cdots\biggr]^{\frac{\theta_i}{\theta_{i-1}}}\biggr]^{\frac{\theta_{i+1}}{\theta_i}}\cdots\biggr]^{\frac{\theta_m}{\theta_{m-1}}}\biggr\}^{\frac{1}{\theta_m}}$$
$$\le C\biggl\{\sum_{n_m=0}^{\infty}2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)}\biggl(\sum_{s_m=n_m+1}^{\infty}\biggl[\sum_{n_{m-1}=0}^{\infty}2^{n_{m-1}\theta_{m-1}\left(\frac{1}{p_{m-1}}-\frac{1}{q_{m-1}}\right)}$$

$$\times \Big(\sum_{s_{m-1}=n_{m-1}+1}^{\infty} \Big[\cdots \Big[\sum_{n_{i+1}=0}^{\infty} 2^{n_{i+1}\theta_{i+1}\left(\frac{1}{p_{i+1}}-\frac{1}{q_{i+1}}\right)} \Big(\sum_{s_{i+1}=n_{i+1}+1}^{\infty} \Big[\sum_{n_i=0}^{\infty} 2^{-n_i\frac{\theta_i}{q_i}}$$

$$\times \Big(\sum_{s_i=1}^{n_i} 2^{\frac{s_i}{p_i}} \Big[\sum_{n_{i-1}=0}^{\infty} 2^{-n_{i-1}\frac{\theta_{i-1}}{q_{i-1}}} \Big(\sum_{s_{i-1}=1}^{n_{i-1}} 2^{\frac{s_{i-1}}{p_{i-1}}} \cdots \Big[\sum_{n_1=0}^{\infty} 2^{-n_1\frac{\theta_1}{q_1}}$$

$$\times \Big(\sum_{s_1=1}^{n_1} 2^{\frac{\theta_1}{p_1}} \|\delta_{\overline{s}}(f)\|_{\overline{p}} \Big)^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} \cdots \Big)^{\theta_{i-1}} \Big\} \Big]^{\frac{\theta_i}{\theta_{i-1}}} \Big)^{\theta_i} \Big]^{\frac{\theta_{i+1}}{\theta_i}} \Big)^{\theta_{i+1}} \Big]^{\frac{\theta_{i+2}}{\theta_{i+1}}} \cdots \Big]^{\frac{\theta_m}{\theta_{m-1}}} \Big)^{\theta_m} \Big\}^{\frac{1}{\theta_m}}$$

(we apply part a) of Lemma A to the sum over the indices n_j, $j = 1, \dots, i$, and part b) to the sum over n_j, $j = i+1, \dots, m$)

$$\le C\Big\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \Big[\cdots \Big[\sum_{n_{i+1}=0}^{\infty} 2^{n_{i+1}\theta_{i+1}\left(\frac{1}{p_{i+1}}-\frac{1}{q_{i+1}}\right)} \Big[\sum_{n_i=0}^{\infty} 2^{-n_i\frac{\theta_i}{q_i}} \Big(\sum_{s_i=1}^{n_i} 2^{\frac{s_i}{p_i}}$$

$$\times \Big[\sum_{n_{i-1}=0}^{\infty} 2^{-n_{i-1}\frac{\theta_{i-1}}{q_{i-1}}} \Big(\sum_{s_{i-1}=1}^{n_{i-1}} 2^{\frac{s_{i-1}}{p_{i-1}}} \cdots \Big[\sum_{n_1=0}^{\infty} 2^{-n_1\frac{\theta_1}{q_1}} \Big(\sum_{s_1=1}^{n_1} 2^{\frac{s_1}{p_1}} \|\delta_{\overline{s}}(f)\|_{\overline{p}} \Big)^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} \cdots$$

$$\le C\Big\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \Big[\sum_{n_{m-1}=0}^{\infty} 2^{n_{m-1}\theta_{m-1}\left(\frac{1}{p_{m-1}}-\frac{1}{q_{m-1}}\right)} \cdots$$

$$\times \Big[\sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \|\delta_{\overline{s}}(f)\|_{\overline{p}}^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} \cdots \Big]^{\frac{\theta_m}{\theta_{m-1}}} \Big\}^{\frac{1}{\theta_m}}. \qquad (13)$$

Inequalities (12) and (13) imply

$$\|f\|_{\overline{q},\overline{\theta}} \le C\Big[\Big\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \Big[\cdots \Big[\sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)}$$

$$\times \|f - U_{\overline{n}}(f)\|_{\overline{p}}^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} \cdots \Big]^{\theta_m\theta_{m-1}} \Big\}^{\frac{1}{\theta_m}}$$

$$+ \Big\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \Big[\sum_{n_{m-1}=0}^{\infty} 2^{n_{m-1}\theta_{m-1}\left(\frac{1}{p_{m-1}}-\frac{1}{q_{m-1}}\right)} \cdots$$

$$\times \Big[\sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \|\delta_{\overline{s}}(f)\|_{\overline{p}}^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} \cdots \Big]^{\frac{\theta_m}{\theta_{m-1}}} \Big\} \Big]. \qquad (14)$$

Since

$$f(\overline{x}) - U_{\overline{n}}(f, \overline{x}) = \sum_{s_m=n_m+1}^{\infty} \cdots \sum_{s_1=n_1+1}^{\infty} \delta_{\overline{s}}(f, \overline{x}),$$

using the property of the norm and Lemma A, we obtain

$$\Big\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \Big[\cdots \Big[\sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \|f - U_{\overline{n}}(f)\|_{\overline{p}}^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} \cdots \Big]^{\frac{\theta_m}{\theta_{m-1}}} \Big\}^{\frac{1}{\theta_m}}$$

$$\le \Big\{ \sum_{n_m=0}^{\infty} 2^{n_m\theta_m\left(\frac{1}{p_m}-\frac{1}{q_m}\right)} \Big[\cdots \Big[\sum_{n_1=0}^{\infty} 2^{n_1\theta_1\left(\frac{1}{p_1}-\frac{1}{q_1}\right)} \|\delta_{\overline{n}}(f)\|_{\overline{p}}^{\theta_1} \Big]^{\frac{\theta_2}{\theta_1}} \cdots \Big]^{\frac{\theta_m}{\theta_{m-1}}} \Big\}^{\frac{1}{\theta_m}}. \qquad (15)$$

Inequalities (14) and (15) imply the assertion of the theorem. □

Remark 2. It is well known that, in the case $1 < q_j < +\infty$, $1 < \theta_j < +\infty$, $j = 1, \dots, m$, the space $L_{\overline{q},\overline{\theta}}(I^m)$ has absolutely continuous norms. Therefore, the adjoint space $L^*_{\overline{q},\overline{\theta}}(I^m)$ coincides with the dual space $L_{\overline{q}',\overline{\theta}'}(I^m)$, where

$$\overline{q}' = (q'_1, \dots, q'_m), \quad \overline{\theta}' = (\theta'_1, \dots, \theta'_m), \quad q'_j = \frac{q_j}{q_j - 1}, \quad \theta'_j = \frac{\theta_j}{\theta_j - 1}, j = 1, \dots, m$$

(see [1, p. 161]).

Now, using the duality principle, just as in [7], and Theorem 2, we can verify the validity of the following assertion.

Theorem 3. *Suppose that* $1 < q_j < \tau_j < +\infty$, $1 < \theta_j < +\infty$, $j = 1, \dots, m$. *If* $f \in L_{\overline{q},\overline{\theta}}(I^m)$,

$$f(\overline{x}) \sim \sum_{\overline{s} \in \mathbb{Z}^m_+} b_{\overline{s}} \sum_{\overline{k} \in \rho(\overline{s})} e^{i\langle \overline{k}, \overline{x} \rangle},$$

then the following inequality holds:

$$\|f\|_{\overline{q},\overline{\theta}} \geq C \cdot \left\{ \sum_{s_m=1}^{\infty} 2^{s_m \theta_m \left(\frac{1}{\tau_m} - \frac{1}{q_m}\right)} \left[\cdots \left[\sum_{s_1=1}^{\infty} 2^{s_1 \theta_1 \left(\frac{1}{\tau_1} - \frac{1}{q_1}\right)} \|\delta_{\overline{s}}(f)\|^{\theta_1}_{\overline{\tau}} \right]^{\frac{\theta_2}{\theta_1}} \cdots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}.$$

Further, let us consider the order of approximation of the functional class $H^{\overline{r}}_{\overline{p}}$.

First, note that the partial sum operator

$$S_{Q^{\overline{\gamma}}_n}(f, \overline{x}) = \sum_{\overline{k} \in Q^{\overline{\gamma}}_n} a_{\overline{k}}(f) \cdot e^{i\langle \overline{k}, \overline{x} \rangle}$$

is bounded in the space $L_{\overline{q},\overline{\theta}}(I^m)$, $1 < q_j < +\infty$, $1 < \theta_j < +\infty$, $j = 1, \dots, m$, i.e.,

$$\|S_{Q^{\overline{\gamma}}_n}(f)\|_{\overline{q},\overline{\theta}} \leq C \cdot \|f\|_{\overline{q},\overline{\theta}}, \qquad f \in L_{\overline{q},\overline{\theta}}(I^m). \tag{16}$$

This assertion follows from the Marcinkiewicz theorem on multipliers (see [17, p. 239] and [2, Corollary 1, p. 106]).

From inequality (16), we obtain

$$\|f - S_{Q^{\overline{\gamma}}_n}(f)\|_{\overline{q},\overline{\theta}} \leq C(q, \theta) \cdot E^{(\overline{\gamma})}_n(f)_{\overline{q},\overline{\theta}}, \quad f \in L_{\overline{q},\overline{\theta}}(I^m). \tag{17}$$

Theorem 4. *Suppose that*

$$1 \leq p_j < q_j < +\infty, \quad 1 \leq \theta_j < +\infty, \quad \frac{1}{p_j} - \frac{1}{q_j} < r_j, \qquad j = 1, \dots, m,$$

$$r_1 = \cdots = r_\nu < r_{\nu+1} \leq \cdots \leq r_m, \qquad \gamma_j = \frac{r_j}{r_1}, \quad j = 1, \dots, m,$$

$$\frac{1}{p_j} - \frac{1}{q_j} = \frac{1}{p_1} - \frac{1}{q_1}, \;\; j = 1, \dots, \nu, \quad r_1\left(\frac{1}{p_j} - \frac{1}{q_j}\right) < r_j\left(\frac{1}{p_1} - \frac{1}{q_1}\right), \;\; j = \nu + 1, \dots, m.$$

Then the following relation holds:

$$\sup_{f \in H^{\overline{r}}_{\overline{p}}} E^{(\overline{\gamma})}_n(f)_{\overline{q},\overline{\theta}} \asymp 2^{-n\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

Proof. Let us prove that, for any $f \in H_{\overline{p}}^{\overline{r}}$, we have the following inequality:

$$\|f - S_{Q_n^{\overline{\gamma}}}(f)\|_{\overline{q},\overline{\theta}} \le C \cdot 2^{-n\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}. \tag{18}$$

Applying Theorem 2 to the function $f - S_{Q_n^{\overline{\gamma}}}(f) \in L_{\overline{q},\overline{\theta}}(I^m)$, we see that

$$\|f - S_{Q_n^{\overline{\gamma}}}(f)\|_{\overline{q},\overline{\theta}} \le C \cdot \left\{ \sum_{s_m=0}^{\infty} 2^{s_m\left(\frac{1}{p_m} - \frac{1}{q_m}\right)\theta_m} \left[\cdots \left[\sum_{s_1=0}^{\infty} 2^{s_1\left(\frac{1}{p_1} - \frac{1}{q_1}\right)\theta_1} \times \|\delta_{\overline{s}}(f - S_{Q_n^{\overline{\gamma}}}(f))\|_{\overline{p}}^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \cdots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}. \tag{19}$$

Since

$$\delta_{\overline{s}}(f - S_{Q_n^{\overline{\gamma}}}(f)) = \begin{cases} 0 & \text{for any } \overline{s} \in Y^m(\overline{\gamma}, n), \\ \delta_{\overline{s}}(f) & \text{for any } \overline{s} \notin Y^m(\overline{\gamma}, n), \end{cases}$$

from inequality (19) we obtain

$$\begin{aligned} \|f - S_{Q_n^{\overline{\gamma}}}(f)\|_{\overline{q},\overline{\theta}} &\le C \cdot \left\| \left\{ \prod_{j=1}^{m} 2^{s_j\left(\frac{1}{p_j} - \frac{1}{q_j}\right)} \|\delta_{\overline{s}}(f)\|_{\overline{p}} \right\}_{\overline{s} \in Y^m(\overline{\gamma}, n)} \right\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^m)} \\ &\le C \cdot \left\| \left\{ \prod_{j=1}^{m} 2^{-s_j\left(r_j + \frac{1}{q_j} - \frac{1}{p_j}\right)} \right\}_{\overline{s} \in Y^m(\overline{\gamma}, n)} \right\|_{\ell_{\overline{\theta}}(\mathbb{Z}_+^m)}. \end{aligned} \tag{20}$$

In Lemma 2, set

$$\alpha = r_1 + \frac{1}{q_1} - \frac{1}{p_1} > 0, \qquad \gamma_j = \frac{r_j + 1/q_j - 1/p_j}{r_1 + 1/q_1 - 1/p_1}, \ \ \gamma_j' = \frac{r_j}{r_1}, \quad j = 1, \dots, m.$$

Then the condition $r_1 = \dots = r_\nu$ implies $\gamma_j' = 1$, $j = 1, \dots, \nu$, and

$$\frac{1}{p_1} - \frac{1}{q_1} = \dots = \frac{1}{p_\nu} - \frac{1}{q_\nu} \implies \gamma_j = 1, \qquad j = 1, \dots, \nu,$$

$$r_1\left(\frac{1}{p_j} - \frac{1}{q_j}\right) < r_j\left(\frac{1}{p_1} - \frac{1}{q_1}\right) \implies \gamma_j' < \gamma_j, \qquad j = \nu+1, \dots, m.$$

In view of Lemma 2, it follows from (20) that

$$\|f - S_{Q_n^{\overline{\gamma}}}(f)\|_{\overline{q},\overline{\theta}} \le C \cdot 2^{-n\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} \cdot n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}$$

for any function $f \in H_{\overline{p}}^{\overline{r}}$. Estimate (18) is proved.

Let us prove the lower bound. To do this, consider the function

$$f_0(\overline{x}) = \sum_{\overline{s} \in \mathbb{Z}_+^m} \prod_{j=1}^{m} 2^{-s_j\left(r_j + 1 - \frac{1}{p_j}\right)} \sum_{\overline{k} \in \rho(\overline{s})} e^{i\langle \overline{k}, \overline{x} \rangle}.$$

It is well known that $(1/C_0) f_0 \in H_{\overline{p}}^{\overline{r}}$, $1 < p_j < +\infty$, $j = 1, \dots, m$, where C_0 is a positive constant.

Let us choose a number $\tau \in (1, p_j)$ for all $j = 1, \dots, m$ and apply Theorem 3 to the function $f_0 - S_{Q_n^{\overline{\gamma}}}(f_0)$. Then

$$\|f_0 - S_{Q_n^{\overline{\gamma}}}(f_0)\|_{\overline{q},\overline{\theta}} \ge C \cdot \left\{ \sum_{s_m=0}^{\infty} 2^{s_m\theta_m\left(\frac{1}{\tau} - \frac{1}{q_m}\right)} \left[\cdots \left[\sum_{s_1=0}^{\infty} 2^{s_1\theta_1\left(\frac{1}{\tau} - \frac{1}{q_1}\right)} \|\delta_{\overline{s}}(f_0 - S_{Q_n^{\overline{\gamma}}}(f_0))\|_{\tau}^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \cdots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}$$

$$\geq C \cdot \Bigg\{ \sum_{s_m \leq \frac{n}{\gamma_m}} 2^{s_m\theta_m\left(\frac{1}{\tau}-\frac{1}{q_m}\right)} \Bigg[\sum_{s_{m-1} \leq \frac{1}{\gamma_{m-1}}(n-s_m\gamma_m)} 2^{s_{m-1}\theta_{m-1}\left(\frac{1}{\tau}-\frac{1}{q_{m-1}}\right)} \dots$$

$$\times \Bigg[\sum_{s_1 \leq (n-\sum_{j=2}^m s_j\gamma_j)} 2^{s_1\theta_1\left(\frac{1}{\tau}-\frac{1}{q_1}\right)} \|\delta_{\overline{s}}(f_0)\|_\tau^{\theta_1} \Bigg]^{\frac{\theta_2}{\theta_1}} \dots \Bigg]^{\frac{\theta_m}{\theta_{m-1}}} \Bigg\}^{\frac{1}{\theta_m}}. \tag{21}$$

Taking into account the relation

$$\|\delta_{\overline{s}}(f_0)\|_\tau \asymp \prod_{j=1}^m 2^{s_j\left(\frac{1}{p_j}-\frac{1}{\tau}-r_j\right)},$$

from (21) and (17) we obtain

$$E_n^{(\overline{\gamma})}(f_0)_{\overline{q},\overline{\theta}} \geq C \cdot \Bigg\{ \sum_{s_m \leq \frac{n}{\gamma_m}} 2^{-s_m\theta_m\left(r_m+\frac{1}{q_m}-\frac{1}{p_m}\right)}$$

$$\times \Bigg[\sum_{s_{m-1} \leq \frac{1}{\gamma_{m-1}}(n-s_m\gamma_m)} 2^{-s_{m-1}\theta_{m-1}\left(r_{m-1}+\frac{1}{q_{m-1}}-\frac{1}{p_{m-1}}\right)} \dots$$

$$\times \Bigg[\sum_{s_1 \leq n-\sum_{j=2}^m s_j\gamma_j} 2^{-s_1\theta_1\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)} \Bigg]^{\frac{\theta_2}{\theta_1}} \dots \Bigg]^{\frac{\theta_m}{\theta_{m-1}}} \Bigg\}^{\frac{1}{\theta_m}}$$

$$\geq C \cdot 2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)} \cdot n^{\sum_{j=2}^{\nu}\frac{1}{\theta_j}}.$$

The theorem is proved. □

Theorem 5. *Suppose that*

$$1 \leq p_j < q_j < +\infty, \quad j = 1,\dots,m, \quad \overline{\gamma} = \frac{\overline{r}}{r_1}, \quad r_1 = \dots = r_\nu < r_{\nu+1} \leq \dots \leq r_m,$$

$$\frac{1}{p_1} - \frac{1}{q_1} = \dots = \frac{1}{p_\nu} - \frac{1}{q_\nu}, \qquad r_1\left(\frac{1}{p_j}-\frac{1}{q_j}\right) < r_j\left(\frac{1}{p_1}-\frac{1}{q_1}\right), \quad j = \nu+1,\dots,m.$$

Then the following estimate holds:

$$\sup_{f \in H_{\overline{p}}^{\overline{r}}} E_n^{\overline{\gamma}}(f)_{\overline{q},\infty} \leq C(p,r,q) \cdot 2^{-n\left(r_1+\frac{1}{q_1}-\frac{1}{p_1}\right)} \cdot n^{\nu-1}.$$

Proof. Using the definition of the Marcinkiewicz space, Theorem 1 and taking the condition $1/p_j - 1/q_j > 0$, $j = 1,\dots,m$, into account, we obtain

$$\|f\|_{\overline{q},\infty} \leq C(p,q) \cdot \sum_{\overline{s} \in \mathbb{Z}_+^m} \prod_{j=1}^m 2^{s_j\left(\frac{1}{p_j}-\frac{1}{q_j}\right)} \|\delta_{\overline{s}}(f)\|_{\overline{p}}.$$

This inequality, which holds for any function $f \in H_{\overline{p}}^{\overline{r}}$, yields

$$\|f - S_{Q_n^{\overline{\gamma}}}(f)\|_{\overline{q},\infty} \leq C(p,q) \cdot \sum_{\langle \overline{s},\overline{\gamma}\rangle > n} \prod_{j=1}^m 2^{-s_j\left(r_j+\frac{1}{q_j}-\frac{1}{p_j}\right)}. \tag{22}$$

Set

$$\gamma_j' = \frac{r_j + 1/q_j - 1/p_j}{r_1 + 1/q_1 - 1/p_1}.$$

Then $\gamma_j < \gamma'_j$ for all $j = \nu+1, \dots, m$, because, by the assumption of the theorem,

$$r_1\left(\frac{1}{p_j} - \frac{1}{q_j}\right) < r_j\left(\frac{1}{p_1} - \frac{1}{q_1}\right), \qquad j = \nu+1, \dots, m.$$

Now, using the estimate (see [7, Lemma C])

$$\sum_{\langle \overline{s}, \overline{\gamma} \rangle > n} 2^{-\alpha \langle \overline{s}, \overline{\gamma}' \rangle} \le C(\gamma) \cdot 2^{-n\alpha} \cdot n^{\nu-1},$$

from inequality (22) we obtain

$$\|f - S_{Q_n^{\overline{\gamma}}}(f)\|_{\overline{q},\infty} \le C(p,q) \cdot 2^{-n\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} n^{\nu-1}.$$

The theorem is proved. □

Remark 3. Note that, in the case $q_1 = \dots = q_m$, $\theta_1 = \dots = \theta_m$, $p_1 = \dots = p_m$, Theorems 2–4 imply certain results due to Temlyakov [7].

Remark 4. Estimates of the value of a nonincreasing rearrangement of a function of one variable via its best approximation by trigonometric polynomials were established by É. A. Storozhenko [18].

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