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On the class of pointwise and integrally loaded differential equations

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We investigate a system of linear ordinary differential equations containing point and integral loadings with nonlocal boundary conditions. Boundary conditions include integral and point values of the unknown function. An essential feature of the problem is that the kernels of the integral terms in the differential equations depend only on the integration variable. It is shown that similar problems arise during feedback control of objects with both lumped and distributed parameters during point and integral measurements of the current state for the controllable object. The problem statement considered in the paper generalizes a lot of previously studied problems regarding loaded differential equations with nonlocal boundary conditions. By introducing auxiliary parameters, we obtain necessary conditions for the existence and uniqueness of a solution to the problem under consideration. To solve the problem numerically, we propose to use a representation of the solution to the original problem, which includes four matrix functions that are solutions to four auxiliary Cauchy problems. Using solutions to the auxiliary problems in boundary conditions, we obtain the values of the unknown function at the loading points. This is enough to get the desired solution. The paper describes the application of the method using the example of solving a test model problem.

Keywords: integro-differential equation, system of loaded equations, integral conditions, conditions of existence and uniqueness.

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Introduction

The paper studies the existence and uniqueness of the solution of nonlocal problems with respect to systems of linear ordinary differential equations, which are pointwise and integrally loaded, and the kernels of integral terms depend on one variable of integration. The nonlocal conditions are linear and contain point and integral values of the unknown function. Such problems are also called pointwise and integrally loaded and they arise in many practical applications [1–4]. The specific feature of the integral terms in the equations is important for the proposed approach to obtaining the existence and uniqueness conditions for the solution of the problem and for its both analytical and numerical solutions.

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The paper describes an example of an optimal feedback control problem for a heating process, which leads to the class of nonlocal problems considered in the paper. Feedback is carried out due to point and integral measurements of the rod's temperature, the results of which are used to form the current values of the control action [5, 6].

In the paper, it is shown that the considered class of nonlocal problems, by introducing new variables, can be reduced to well-studied pointwise loaded problems with separated boundary conditions [7–9]. But taking into account the significant increase in the dimension of the problem, such an approach to the study of the original problem is not recommended.

The approach to obtaining the existence and uniqueness conditions of a solution to the problem is used to a certain extent for the proposed method for solving the problem both in an analytical form in case of a constant matrix of a dynamical system, and for a numerical solution with a variable matrix of the system. We present a study and an analytical method for solving one illustrative problem using the proposed approach.

1 Problem statement and its analysis

We consider the following system of pointwise and integrally loaded differential equations:

$$\frac{du(x)}{dx} = A(x)u(x) + \sum_{i=1}^{l_1} B_i^1(x)u(x_i) +$$

$$+ \sum_{j=1}^{l_2} B_j^2(x) \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\xi)u(\xi)d\xi + D(x),\ x \in [x_0, x_f] \tag{1}$$

with non-local conditions

$$\sum_{i=1}^{l_3} \alpha_i u(x_{L_2+i}) + \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\xi)u(\xi)d\xi = \gamma. \tag{2}$$

Here $u(\cdot) \in \mathbb{R}^n$ is an unknown continuously differentiable function. There are: non-negative integers l_1, l_2, l_3, l_4; continuous n-dimensional square matrix functions $A(x)$, $B_i^1(x)$, $B_j^2(x)$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, at $x \in [x_0, x_f]$, $C_j(x)$ – at $x \in [x_{L_1+2j-1}, x_{L_1+2j}]$, $j = 1, 2, \ldots, l_2$, $\beta_j(x)$ – at $x \in [x_{L_3+2j-1}, x_{L_3+2j}]$; continuous n-dimensional vector function $D(x)$; n-dimensional vector γ; points x_i, $i = 1, 2, \ldots, L_4$, $L_1 = l_1$, $L_2 = L_1 + 2l_2$, $L_3 = L_2 + l_3$, $L_4 = L_3 + 2l_4$ from segment $[x_0, x_f]$ (some of the indicated points may coincide), and it is assumed that, without loss of generality, the following conditions are satisfied:

$$x_{L_1+2j} \geq x_{L_1+2j-1}, \quad j = 1, 2, \ldots, l_2, \quad x_{L_3+2j} \geq x_{L_3+2j-1}, \quad j = 1, 2, \ldots, l_4.$$

In the problem, it is required to find a continuously differentiable vector function $u(\cdot) \in \mathbb{R}^n$ for $x \in [x_0, x_f]$, satisfying the system of pointwise and integrally loaded differential equations (1) and nonlocal conditions (2), containing point and integral values of the unknown function.

An essential feature of problem (1) and (2) is the dependence of the integrands in equation (1) on one variable of integration. For example, optimal feedback control problems lead to such a problem [5,6]. In particular, the control synthesis problem for the heating process of a rod with the length d in the furnace, which can be described by the boundary value problem for the parabolic equation:

$$u_t'(x,t) = a^2 u_{x^2}''(x,t) + \mu(x)\left[\vartheta(t) - u(x,t)\right] \tag{3}$$

with some initial and boundary conditions

$$u(x,0) = \varphi(x),\ x \in [0,\ d]\,, \tag{4}$$

$$\frac{\partial u(0,t)}{\partial x} = \mu_1\left[\vartheta(t) - u(0,t)\right],\ t \in [0,T]\,, \tag{5}$$

$$\frac{\partial u(d,t)}{\partial x} = -\mu_1\left[\vartheta(t) - u(d,t)\right],\ t \in [0,T]\,. \tag{6}$$

Here $u(x,t)$ is the temperature of the rod at the point x at the moment t, $x \in [0,d]$, $t \in [0,T]$; $\mu(x)$; $\varphi(x)$, a, μ_1 are the specified functions and process parameters; $\vartheta(t)$ is a control function that determines the temperature inside the furnace. There is a certain optimality criterion characterizing the choice of control $\vartheta(t)$. Assume that the given points x_i and segments $[\tilde{x}_{2j-1}, \tilde{x}_{2j}]$ of the rod, we take the point $u(x_i,t)$, $i = 1,2,\ldots,l_1$ and integral $u(x,t), x \in [\tilde{x}_{2j-1}, \tilde{x}_{2j}], j = 1,2,\ldots,l_2$ measurements of the temperature. The measurement results are used to form the current temperature value in the furnace (feedback control) in the form of the following relationship

$$\vartheta(t) = \sum_{i=1}^{l_1} k_{1i} u(x_i,t) + \sum_{j=1}^{l_2} k_{2j} \int\limits_{\tilde{x}_{2j-1}}^{\tilde{x}_{2j}} \beta_j(\xi) u(\xi,t) d\xi. \tag{7}$$

Here the given functions $\beta_i(x), x \in [x_{2j-1}, x_{2j}]$ are weighted, the constant coefficients k_{1i}, k_{2j}, $i = 1,2,\ldots,l_1$, are the optimizable feedback parameters [5,6].

Substituting expression (7) into equation (3) and using the difference approximation of the derivatives with respect to t

$$\frac{\partial u(x,t_s)}{\partial t} = \frac{u(x,t_s) - u(x,t_{s-1})}{h_t} + O(h_t),$$

we obtain the following system of loaded differential equations:

$$a_0 \frac{d^2 u_s(x)}{dx^2} = a_1 u_s(x) + a_2(x) \left[\sum_{i=1}^{l_1} k_{1i} u_s(x_i) + \right.$$

$$\left. + \sum_{j=1}^{l_2} k_{2j} \int\limits_{\tilde{x}_{2j-1}}^{\tilde{x}_{2j}} \beta_j(\xi) u_s(\xi) d\xi \right] + f_s(x)\,, \quad x \in [0; d]\ \ ,\ \ s = 1,2,...,N_t. \tag{8}$$

Conditions (4)–(6) can be written in the form

$$\frac{du_s(0)}{dx} = \mu_1 \left[\sum_{i=1}^{l_1} k_{1i} u_s(x_i) + \sum_{j=1}^{l_2} k_{2j} \int\limits_{\tilde{x}_{2j-1}}^{\tilde{x}_{2j}} \beta_j(\xi) u_s(\xi) d\xi - u_s(0)\right], \tag{9}$$

$$\frac{du_s(d)}{dx} = -\mu_1 \left[\sum_{i=1}^{l_1} k_{1i} u_s(x_i) + \sum_{j=1}^{l_2} k_{2j} \int\limits_{\tilde{x}_{2j-1}}^{\tilde{x}_{2j}} \beta_j(\xi) u_s(\xi) d\xi - u_s(d)\right]. \tag{10}$$

In (8)–(10) the following notations are used: h_t is the discretization step, $u_0(x) = u(x,0) = \varphi(x)$, $x \in [0,d]$, $u_s(x) = u(x,t_s), t_s = sh_t$, $s = 1,2,\ldots,N_t$, $h_t = {}^{T}\!/_{N_t}$, $a_0 = a^2 h_t$, $a_1(x) = 1 + a_2(x)$, $a_2(x) = h_t \mu(x)$, $f_s(x) = u_{s-1}(x)$.

To determine the feedback parameters $k_{1i}, k_{2j}, i = 1, 2, \ldots, l_1,\ j = 1, 2, \ldots, l_2$ using any first-order numerical iterative optimization methods, first, it is required to construct formulas for the components of the gradient of the objective functional in terms of the optimizable parameters, and second, to numerically solve problem (8)–(10) for given current values of these parameters. It is clear that system (8) can be easily reduced to the considered system of first-order differential equations (1) and (2).

Note that some special cases of problem (1) and (2) were studied earlier. When $B_j^2(x) \equiv 0$, $j = 1, 2, \ldots, l_2$, we have a point-loaded system of differential equations, investigated in many papers, particularly, in [2–4, 7]. When $B_j^1(x) \equiv 0$, $j = 1, 2, \ldots, l_1$, we obtain an integro-differential system of equations whose kernels $C_j(x)$ depend only on the variable of integration [10–18]. Conditions (2) also generalize many other local and nonlocal conditions. Their particular cases are Cauchy conditions, two-point and multipoint conditions, conditions of an integral type [4, 19].

Problem (1) and (2), by introducing new unknowns, can be reduced to a two-point boundary value problem for a system of point-loaded differential equations. Let's show how it is done. We introduce new n-dimensional variables $\vartheta^j(x), j = 1, 2, \ldots, l_2$, satisfying the system of differential equations:

$$\begin{aligned} \frac{d\vartheta^j(x)}{dx} &= C_j(x)u(x), \quad x_{L_1+2j-1} < x \leq x_{L_1+2j},\ j = 1, 2, \ldots, l_2, \\ \vartheta^j(x) &= 0_n, \qquad\qquad\qquad x \leq x_{L_1+2j-1},\ \ j = 1, 2, \ldots, l_2, \end{aligned} \tag{11}$$

where 0_n is the n–dimensional zero vector. System (1) will have only point loading:

$$\frac{du(x)}{dt} = A(x)u(x) + \sum_{i=1}^{l_1} B_i^1(x)u(x_i) + \sum_{j=1}^{l_2} B_j^2(x)\vartheta^j(x_{L_1+2j}) + D(x), x \in [x_0, x_f]. \tag{12}$$

By introducing new n-dimensional vectors $w^j(x)$, $j = 1, 2, \ldots, l_4$, satisfying the system of differential equations

$$\begin{aligned} \frac{dw^j(x)}{dx} &= \beta_j(x)u(x), \quad x_{L_3+2j-1} < x \leq x_{L_3+2j},\ j = 1, 2, \ldots, l_4, \\ w^j(x) &= 0_n, \qquad\qquad\quad x \leq x_{L_3+2j-1},\ \ j = 1, 2, \ldots, l_4, \end{aligned} \tag{13}$$

conditions (2) are reduced to the multipoint conditions

$$\sum_{i=1}^{l_3} \alpha_i u(x_{L_2+i}) + \sum_{j=1}^{l_4} w^j(x_{L_3+2j}) = \gamma. \tag{14}$$

The order of the resulting linear system of loaded differential equations (11)–(13) is $(l_2 + l_4 + 1)n$. Using the approach proposed in [8,9], multipoint conditions (14) can be reduced to separated boundary conditions. To do this, each of the $2(l_3+l_4+1)$ segments between all the points x_0, x_{L_2+i}, $i = 1, 2, \ldots, l_3$, x_{L_3+2j}, $j = 1, 2, \ldots, l_4$, x_f after their ordering, is divided into two parts. For each of the halves of these segments between the points, systems of differential equations are introduced for new variables corresponding to $u(x)$, $\vartheta^i(x)$, $w^j(x)$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, but in different directions of change of the argument x. As a result, we obtain a system of differential equations of the order $2(l_3 + l_4 + 1)(l_2 + l_4 + 1)n$ with two-point boundary conditions of the form (after individual scaling for each segment and reducing them to segments of a unit length):

$$A_1 w(0) = A_2, \qquad A_3 w(1) = A_4,$$

where A_1, A_3 are the square matrices of size $2(l_3 + l_4 + 1)(l_2 + l_4 + 1)n$; A_2, A_4 are the vectors of the corresponding dimension.

Point-loaded equations with two-point and multipoint conditions have been studied well enough, necessary and sufficient existence and uniqueness conditions of a solution were obtained for them in [11] and [17], approaches to their numerical solution were proposed in [2, 7, 20]. Optimization and optimal control problems, inverse problems in various formulations described by point-loaded equations we investigated in [21–23], numerical methods for their solution are described in [21, 22].

Considering a significant increase in the dimension of the original problem (1) and (2), when it is reduced to a problem with point-loaded differential equations with separated boundary conditions, the use of the previously proposed methods both for study and their numerical solution is inappropriate. This is especially true for optimization and optimal control problems that require multiple solutions of problems of the kind (1) and (2).

Therefore, this paper studies the existence and uniqueness of solutions to problem (1), (2), and also proposes an approach to solving that does not require an increase in the dimension of the original problem.

2 Existence and uniqueness conditions for the solution of problem (1) and (2)

Consider the following auxiliary system of differential equations:

$$\frac{du(x)}{dx} = A(x)u(x) + \sum_{i=1}^{l_1} B_i^1(x)\overset{\smile}{\lambda}^{i} + \sum_{j=1}^{l_2} B_j^2(x)\overset{\frown}{\lambda}^{j} + D(x),\; x \in [x_0, x_f] \tag{15}$$

with conditions (2). Here, $\overset{\smile}{\lambda}^{i}$, $\overset{\frown}{\lambda}^{j}$, $i = 1, 2, \ldots, l_1$, $\quad j = 1, 2, \ldots, l_2$ are arbitrary n-dimensional vectors, the functions and parameters are the same as in equation (1).

Under the accepted assumptions on the functions involved in the problem, the solution to system (15) for an arbitrarily given initial condition

$$u(x_0) = u_0 \tag{16}$$

according to the Cauchy formula can be written as:

$$u(x) = F(x)u_0 + F(x)\int\limits_{x_0}^{x} F^{-1}(\xi)R(\xi)d\xi, \quad x \in [x_0, x_f], \tag{17}$$

$$R(\xi) = \sum_{i=1}^{l_1} B_i^1(\xi)\overset{\smile}{\lambda}^{i} + \sum_{j=1}^{l_2} B_j^2(\xi)\overset{\frown}{\lambda}^{j} + D(\xi). \tag{18}$$

Here, the n-dimensional square fundamental matrix $F(x)$ is a solution to the Cauchy problem

$$\frac{dF(x)}{dx} = A(x)F(x), \quad F(x_0) = I_n, \quad x \in [x_0, x_f], \tag{19}$$

where I_n is the n-dimensional identity matrix.

Let us introduce the notation:

$$\overset{\smile}{F}^{i}(x) = F(x)\int\limits_{x_0}^{x} F^{-1}(\xi)B_i^1(\xi)d\xi, \quad i = 1, 2, \ldots, l_1, \tag{20}$$

$$\overset{\frown}{F}^{j}(x) = F(x)\int\limits_{x_0}^{x} F^{-1}(\xi)B_j^2(\xi)d\xi, \quad j = 1, 2, \ldots, l_2, \tag{21}$$

$$F^1(x) = F(x) \int_{x_0}^{x} F^{-1}(\xi) D(\xi) d\xi. \quad (22)$$

Then solution (17)–(18) of the system of differential equations (15) with an arbitrary given initial condition u_0 and the vectors $\overset{\smile}{\lambda}^i$, $\overset{\frown}{\lambda}^j$, $i = 1, 2, \ldots, l_1$, $\quad j = 1, 2, \ldots, l_2$ can be written as:

$$u(x) = F(x) u_0 + \sum_{i=1}^{l_1} \overset{\smile}{F}^i(x) \overset{\smile}{\lambda}^i + \sum_{j=1}^{l_2} \overset{\frown}{F}^j(x) \overset{\frown}{\lambda}^j + F^1(x). \quad (23)$$

Considering an arbitrariness of the n-dimensional vectors u_0, $\overset{\smile}{\lambda}^i$, $\overset{\frown}{\lambda}^j$, $i = 1, 2, \ldots, l_1$, $\quad j = 1, 2, \ldots, l_2$, we require that they fulfill the following conditions:

$$\overset{\smile}{\lambda}^\nu = u(x_\nu), \quad \nu = 1, 2, \ldots, l_1, \quad (24)$$

$$\overset{\frown}{\lambda}^\mu = \int_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi) u(\xi) d\xi, \quad \mu = 1, 2, \ldots, l_2, \quad (25)$$

and conditions (2). It is clear that the total number of conditions in (2), (24) and (25) is equal and coincides with the total dimension of an arbitrary vector u_0, $\overset{\smile}{\lambda}^i$, $\overset{\frown}{\lambda}^j$, $i = 1, 2, \ldots, l_1$, $\quad j = 1, 2, \ldots, l_2$. Let us introduce the notation for vectors: $\overset{\smile}{\Lambda} = (\overset{\smile}{\lambda}^1, \overset{\smile}{\lambda}^2, ..., \overset{\smile}{\lambda}^{l_1})^T \in \mathbb{R}^{l_1 n}$, $\overset{\frown}{\Lambda} = (\overset{\frown}{\lambda}^1, \overset{\frown}{\lambda}^2, ..., \overset{\frown}{\lambda}^{l_2})^T \in \mathbb{R}^{l_2 n}$, $\Lambda = (\overset{\smile}{\Lambda}, \overset{\frown}{\Lambda}) \in \mathbb{R}^{(l_1+l_2)n}$.

"T" is the transposition sign.

From (24), taking into account (23), we obtain:

$$\overset{\smile}{\lambda}^\nu = F(x_\nu) u_0 + \sum_{i=1}^{l_1} \overset{\smile}{F}^i(x_\nu) \overset{\smile}{\lambda}^i + \sum_{j=1}^{l_2} \overset{\frown}{F}^j(x_\nu) \overset{\frown}{\lambda}^j + F^1(x_\nu). \quad \nu = 1, 2, ..., l_1. \quad (26)$$

From (25), taking into account (23), for $\mu = 1, 2, \ldots, l_2$, we obtain:

$$\overset{\frown}{\lambda}^\mu = \int_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\eta) \left[F(\eta) u_0 + \sum_{i=1}^{l_1} \overset{\smile}{F}^i(\eta) \overset{\smile}{\lambda}^i + \sum_{j=1}^{l_2} \overset{\frown}{F}^j(\eta) \overset{\frown}{\lambda}^j + F^1(\eta) \right] d\eta. \quad (27)$$

From conditions (2), taking into account (23), we obtain:

$$\sum_{i=1}^{l_3} \alpha_i \left[F(x_{L_2+i}) u_0 + \sum_{s=1}^{l_1} \overset{\smile}{F}^s(x_{L_2+i}) \overset{\smile}{\lambda}^s + \sum_{j=1}^{l_2} \overset{\frown}{F}^j(x_{L_2+i}) \overset{\frown}{\lambda}^j + F^1(x_{L_2+i}) \right] +$$

$$+ \sum_{j=1}^{l_4} \int_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta) \left[F(\eta) u_0 + \sum_{i=1}^{l_1} \overset{\smile}{F}^i(\eta) \overset{\smile}{\lambda}^i + \sum_{s=1}^{l_2} \overset{\frown}{F}^s(\eta) \overset{\frown}{\lambda}^s + F^1(\eta) \right] d\eta = \gamma. \quad (28)$$

Relations (26)–(28) are systems of linear algebraic equations of an $l_1 n$, $l_2 n$ and n-th order, respectively, with respect to the unknown n-dimensional vectors u_0, $\overset{\smile}{\lambda}^i$, $\overset{\frown}{\lambda}^j$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$. The total number of equations in these systems corresponds to the total number of unknowns: $(u_0, \overset{\smile}{\Lambda}, \overset{\frown}{\Lambda}) \in$

$\mathbb{R}^{(l_1+l_2+1)n}$. After simple transformations and grouping, the resulting algebraic system can be reduced to the form:

$$\begin{cases} G_{11}^{i}u_0 + G_{12}^{i}\overset{\smile}{\Lambda} + G_{13}^{i}\overset{\frown}{\Lambda} = G_{10}^{i}, \ i = 1, 2, ..., l_1, \\ G_{21}^{j}u_0 + G_{22}^{j}\overset{\smile}{\Lambda} + G_{23}^{j}\overset{\frown}{\Lambda} = G_{20}^{j}, \ j = 1, 2, ..., l_2, \\ G_{31}u_0 + G_{32}\overset{\smile}{\Lambda} + G_{33}\overset{\frown}{\Lambda} = G_{30}. \end{cases} \tag{29}$$

The matrix coefficients participating in (29) are determined from (26)–(28):

$$G_{11}^{i} = F(x_i) \in \mathbb{R}^{n\times n},$$

$$G_{12}^{i} = \left(\overset{\smile}{F}^{1}(x_i),\ ..., \overset{\smile}{F}^{i-1}(x_i), \overset{\smile}{F}^{i}(x_i) - \mathrm{I}_n, \overset{\smile}{F}^{i+1}(x_i), ..., \overset{\smile}{F}^{l_1}(x_i) \right) \in \mathbb{R}^{n\times l_1 n},$$

$$G_{13}^{i} = \left(\overset{\frown}{F}^{1}(x_i),\ \overset{\frown}{F}^{2}(x_i) ..., \overset{\frown}{F}^{l_2}(x_i) \right) \in \mathbb{R}^{n\times l_2 n},$$

$$G_{10}^{i} = -F^{1}(x_i) \in \mathbb{R}^{l_1 n}, \quad i = 1, 2, \ldots, l_1,$$

$$C_{21}^{j} = \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)F(\eta)d\eta \in \mathbb{R}^{n\times n},$$

$$G_{22}^{j} = \left(\int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\overset{\smile}{F}^{1}(\eta)d\eta,\ ..., \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\overset{\smile}{F}^{l_1}(\eta_i)d\eta \right) \in \mathbb{R}^{n\times l_1 n},$$

$$G_{23}^{j} = \left(\int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\overset{\frown}{F}^{1}(\eta)d\eta, ..., \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\overset{\frown}{F}^{j-1}(\eta)d\eta, \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\overset{\frown}{F}^{j}(\eta)d\eta - \mathrm{I}_n, \right.$$

$$\left. \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\overset{\frown}{F}^{j+1}(\eta)d\eta, ..., \int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j}} C_j(\eta)\overset{\frown}{F}^{l_2}(\eta)d\eta \right) \in \mathbb{R}^{n\times l_2 n},$$

$$G_{20}^{j} = -\int\limits_{x_{L_1+2j-1}}^{x_{L_1+2j-1}} C_j(\eta)F^{1}(\eta)d\eta \in \mathbb{R}^{l_2 n}, \quad j = 1, 2, \ldots, l_2,$$

$$G_{31} = \sum_{i=1}^{l_3} \alpha_i F(x_{L_2+i}) + \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta)F(\eta)d\eta \in \mathbb{R}^{n\times n},$$

$$G_{32} = \sum_{i=1}^{l_3} \alpha_i \sum_{s=1}^{l_1} \overset{\smile}{F}^{s}(x_{L_2+i}) + \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta) \sum_{i=1}^{l_1} \overset{\smile}{F}^{i}(\eta)d\eta \in \mathbb{R}^{n\times l_1 n},$$

$$G_{33} = \sum_{i=1}^{l_3} \alpha_i \sum_{j=1}^{l_2} \overset{\frown}{F}^{j}(x_{L_2+i}) + \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta) \sum_{s=1}^{l_2} \overset{\frown}{F}^{s}(\eta)d\eta \in \mathbb{R}^{n\times l_2 n},$$

$$G_{30} = \gamma - \sum_{i=1}^{l_3} \alpha_i F^1(x_{L_2+i}) - \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\eta) F^1(\eta) d\eta \in \mathbb{R}^n.$$

From the solution of algebraic system (29), we determine the initial value of the unknown function $u_0 = u(x_0)$, the point values

$$\overset{\smile}{\lambda}^{i} = u(x_i), i = 1, 2, \ldots, l_1,$$

and the integral values $\overset{\frown}{\lambda}^{j} = \int\limits_{L_1+2j-1}^{L_1+2j} C(\xi) u(\xi) d\xi, \quad j = 1, 2, \ldots, l_2$. This allows us to solve the Cauchy problem for the system of differential equations (15) instead of loaded system (1) with initial conditions (16) without using nonlocal conditions (2).

Thus, the existence and uniqueness of a solution to problem (1) and (2) depends on the existence and uniqueness of the vectors $u_0, \overset{\smile}{\lambda}^{i}, \overset{\frown}{\lambda}^{j}$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, which are solutions of algebraic system (29). This implies the following theorem.

Theorem 1. For the existence and uniqueness of a solution to problem (1) and (2), the rank of the $(l_1 + l_2 + 1)n$ -dimensional square matrix of algebraic system (29) must satisfy the condition:

$$rank \begin{bmatrix} G_{11}^1 & ... & G_{11}^{l_1} & G_{21}^1 & ... & G_{21}^{l_2} & G_{31} \\ G_{12}^1 & ... & G_{12}^{l_1} & G_{22}^1 & ... & G_{22}^{l_2} & G_{32} \\ G_{13}^1 & ... & G_{13}^{l_1} & G_{23}^1 & ... & G_{23}^{l_2} & G_{33} \end{bmatrix}^T = (l_1 + l_2 + 1)n. \tag{30}$$

It is clear that if the rank of the matrix in (30) is less than n, then algebraic system (29) may have no solutions or have an infinite number of solutions depending on the rank of augmented matrix (29). Consequently, original problem (1) and (2) may have an infinite number of solutions or not have them at all, respectively.

The above approach to studying the existence and uniqueness of a solution to problem (1) and (2) can also be used to solve the problem. But, as can be seen from the above formulas, to find the coefficients of the system of algebraic equations (29), it is necessary to have a fundamental matrix of solutions $F(x)$ and its inverse matrix $F^{-1}(x)$, $x \in [x_0, x_f]$. If the condition $A(x) \neq const$, $x \in [x_0, x_f]$ is met, the construction of these matrices in an analytical form is not possible in practice, and using numerical methods requires a large amount of computation and memory.

In the next section, we present an approach to the numerical solution of problem (1) and (2) is presented that does not require knowledge of the matrix $F^{-1}(x)$, $x \in [x_0, x_f]$.

3 Approach to the solution of the problem

Below, we propose an approach to solving problem (1) and (2) using auxiliary Cauchy problems for linear systems of differential equations. For the numerical solution of the auxiliary Cauchy problems, known methods and software packages can be used.

The proposed approach is based on the representation of solution (23) to auxiliary problem (15), (16) and the Cauchy problems given in the following theorem.

Theorem 2. The solution of the system of differential equations (15) for arbitrarily given independent initial condition (16) and n-dimensional vectors $\overset{\smile}{\lambda}^{i}, \overset{\frown}{\lambda}^{j}$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, can be uniquely represented as (23), if n-dimensional square matrix functions $F(x)$, $\overset{\smile}{F}^{i}(x)$, $\overset{\frown}{F}^{j}(x)$ and vector function

$F^1(x)$, at $x \in [x_0, x_f]$, are solutions of the corresponding Cauchy problems (19) and

$$\frac{d\breve{F}^{i}(x)}{dx} = A(x)\breve{F}^{i}(x) + B_i^1(x), \quad \breve{F}^{i}(x_0) = 0, \quad i = 1, 2, ..., l_1, \tag{31}$$

$$\frac{d\overset{\frown}{F}^{j}(x)}{dx} = A(x)\overset{\frown}{F}^{j}(x) + B_j^2(x), \ \overset{\frown}{F}^{j}(x_0) = 0, \ \ j = 1, 2, ..., l_2, \tag{32}$$

$$\frac{dF^1(x)}{dx} = A(x)F^1(x) + D(x), \ F^1(x_0) = 0. \tag{33}$$

Proof. According to Cauchy formula, the unique solutions to problems (31)–(33) are the functions $\breve{F}^{i}(x)$, $\overset{\frown}{F}^{j}(x)$, $F^1(x)$, respectively, defined by formulas (20)–(22). These formulas involve the matrix function $F(x)$, which is a fundamental solution to homogeneous systems with respect to (31)–(33) and the unique solution to Cauchy problem (19). It is clear that the functions $F(x)$, $\breve{F}^{i}(x)$, $\overset{\frown}{F}^{j}(x)$, $F^1(x)$ are independent of the initial condition u_0 and parameters $\breve{\lambda}^{i}$, $\overset{\frown}{\lambda}^{j}$. But the representation of the solution to the system of differential equations (15) in the form (23), by virtue of Cauchy formula (18), is unique for arbitrarily and independently given vectors u_0, $\breve{\lambda}^{i}$, $\overset{\frown}{\lambda}^{j}$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$.

From the above, we can formulate the following approach to solving original problem (1) and (2).

First, we solve auxiliary Cauchy problems (19), (31)–(33). After finding the functions $F(x)$, $\breve{F}^{i}(x)$, $\overset{\frown}{F}^{j}(x)$, $F^1(x)$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, further, taking into account the arbitrariness of the parameters $\breve{\lambda}^{i}$, $\overset{\frown}{\lambda}^{j}$, $i = 1, 2, \ldots, l_1$, $j = 1, 2, \ldots, l_2$, $u_0 \in \mathbb{R}^n$ in problem (15) and (2), we require that they fulfill conditions (24), (25) and (2). Then, from representation (23), we have:

$$\breve{\lambda}^{\nu} = u(x_\nu) = F(x_\nu)u_0 + \sum_{i=1}^{l_1} \breve{F}^{i}(x_\nu)\breve{\lambda}^{i} + \sum_{j=1}^{l_2} \overset{\frown}{F}^{j}(x_\nu)\overset{\frown}{\lambda}^{j} + F^1(x_\nu)\,, \quad \nu = 1, 2, ..., l_1, \tag{34}$$

$$\begin{aligned} \overset{\frown}{\lambda}^{\mu} = \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi)u(\xi)d\xi = \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi)F(\xi)\,u_0 d\xi + \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi) \sum_{i=1}^{l_1} \breve{F}^{i}(\xi)\,\breve{\lambda}^{i} d\xi + \\ + \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi) \sum_{j=1}^{l_2} \overset{\frown}{F}^{j}(\xi)\,\overset{\frown}{\lambda}^{j} d\xi + \int\limits_{x_{L_1+2\mu-1}}^{x_{L_1+2\mu}} C_\mu(\xi)F^1(\xi)d\xi\,, \ \mu = 1, 2, ..., l_2, \end{aligned} \tag{35}$$

$$\begin{aligned} \sum_{i=1}^{l_3} \alpha_i \left[F(x_{L_2+i})u_0 + \sum_{s=1}^{l_1} \breve{F}^{s}(x_{L_2+i})\breve{\lambda}^{s} + \sum_{j=1}^{l_2} \overset{\frown}{F}^{j}(x_{L_2+i})\overset{\frown}{\lambda}^{j} + F^1(x_{L_2+i}) \right] + \\ + \sum_{j=1}^{l_4} \int\limits_{x_{L_3+2j-1}}^{x_{L_3+2j}} \beta_j(\xi) \left[F(\xi)u_0 + \sum_{i=1}^{l_1} \breve{F}^{i}(\xi)\,\breve{\lambda}^{i} + \sum_{s=1}^{l_2} \overset{\frown}{F}^{s}(\xi)\overset{\frown}{\lambda}^{s} + F^1(\xi) \right] d\eta = \gamma. \end{aligned} \tag{36}$$

From (34)–(36) we get the system of $(l_1 + l_2 + 1)n$ linear equations with respect to unknowns n-dimensional vectors u_0, $\breve{\lambda}^{\nu}$, $\overset{\frown}{\lambda}^{\mu}$, $\nu = 1, 2, \ldots, l_1$, $\mu = 1, 2, \ldots, l_2$. Having determined these vectors, from representation (23), we find the desired solution to problem (1) and (2).

If among $B_i^1(x)$, $i = 1, 2, \ldots, l_1$, or $B_j^2(x)$, $j = 1, 2, \ldots, l_2$, there are functions having the same or different constant coefficients, then the number of auxiliary problems (31)–(33) can be reduced by the number of coinciding functions. For example, if $k_1 B_{i_1}^1(x) = k_2 B_{i_2}^1(x) = \ldots = k_s B_{i_s}^1(x)$, then instead of vectors $\breve{\lambda}^{i_1}, \ldots, \breve{\lambda}^{i_s}$ it suffices to introduce into (24) one vector $\breve{\lambda}^{i_*} = \sum\limits_{q=1}^{s} k_q u(x_{i_q})$.

Similarly, if $k_1 B_{j_1}^1(x) = k_2 B_{j_2}^2(x) = \ldots = k_s B_{j_s}^2(x)$, then instead of vectors $\widehat{\lambda}^{j_1}, \ldots, \widehat{\lambda}^{j_s}$ in (25) we introduce one vector $\widehat{\lambda}^{j_*} = \sum\limits_{q=1}^{s} k_q \int\limits_{x_{L_1+2j_q-1}}^{x_{L_1+2j_q}} C_{j_q}(\xi) u(\xi) d\xi$. One of such cases will be demonstrated by the example of an illustrative problem given in the next section.

4 Illustrative problem

Consider the following problem:

$$\frac{du(x)}{dx} = 3u(x) + 2u(1) + 3u(2) + 6\int\limits_2^3 u(\tau)d\tau - 6x^2 + 4x - 118, \; x \in [0, 4], \tag{37}$$

$$u(0) - 2u(3) + u(4) + 3\int\limits_1^2 u(\tau)d\tau = 13. \tag{38}$$

In equation (37) $A(x) = const = 3$, $x \in [0, 4]$; the functions $B_1^1(x) = 2$ and $B_2^1(x) = 3$ differ in constant coefficients; $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 2$, $x_4 = 3$, $x_5 = 0$, $x_6 = 3$, $x_7 = 4$, $x_8 = 1$, $x_9 = 2$, $l_1 = 2$, $l_2 = l_4 = 1$, $l_3 = 3$, $L_1 = 2$, $L_2 = 4$, $L_3 = 7$, $L_4 = 9$; $D(x) = -6x^2 + 4x - 118$; $C_1(x) = 1$; $\alpha_1 = 1$, $\alpha_2 = -2$, $\alpha_3 = 1$, $\beta_1 = 3$, $\gamma = 13$.

It is easy to verify that the solution to problem (37) and (38) is the function: $u(x) = 2x^2 + 1$.

Let us introduce the notation

$$\breve{\lambda}^1 = 2u(1) + 3u(2), \quad \widehat{\lambda}^1 = \int\limits_2^3 u(\xi)d\xi. \tag{39}$$

Let us construct auxiliary problems (31)–(33):

$$\frac{dF(x)}{dx} = 3F(x), \quad F(0) = 1, \tag{40}$$

$$\frac{d\breve{F}^1(x)}{dx} = 3\breve{F}^1(x) + 1, \quad \breve{F}^1(0) = 0, \tag{41}$$

$$\frac{d\widehat{F}^1(x)}{dx} = 3\widehat{F}^1(x) + 6, \quad \widehat{F}^1(0) = 0. \tag{42}$$

$$\frac{dF^1(x)}{dx} = 3F^1(x) - 6x^2 + 4x - 118, \quad F^1(0) = 0. \tag{43}$$

It is not difficult to determine solutions to Cauchy problems (40)–(43):

$$F(x) = e^{3x}, \quad \breve{F}^1(x) = \frac{1}{3}e^{3x} - \frac{1}{3},$$

$$\overset{\frown}{F}^{1}(x) = 2e^{3x} - 2, \quad F^1(x) = 2x^2 - \frac{118}{3}e^{3x} + \frac{118}{3}.$$

Using representation (23) and notation (39), we obtain:

$$\begin{aligned} \overset{\smile}{\lambda}^{1} &= \left(2F(1) + 3F^0(2)\right) u_0 + \left(2F^1(1) + 3F^1(2)\right) + \\ &+ \left(2\overset{\smile}{F}^{1}(1) + 3\overset{\smile}{F}^{1}(2)\right) \overset{\smile}{\lambda}^{1} + \left(2\overset{\frown}{F}^{1}(1) + 3\overset{\frown}{F}^{1}(2)\right) \lambda^1, \end{aligned} \tag{44}$$

$$\overset{\frown}{\lambda}^{1} = \int_2^3 \left[F(\xi)\, u_0 + F^1(\xi) + \overset{\smile}{F}^{1}(\xi)\, \overset{\smile}{\lambda}^{1} + \overset{\frown}{F}^{1}(\xi)\, \overset{\frown}{\lambda}^{1}\right] d\xi. \tag{45}$$

Substituting the found functions $F(x)$, $\overset{\smile}{F}^{1}(x)$, $\overset{\frown}{F}^{1}(x)$, $F^1(x)$ into (44), (45) and adding condition (38), we obtain the algebraic system (29):

$$\begin{cases} 3\left(3e^6 + 2e^3\right) u_0 + \left((3e^6 + 2e^3) - 8\right) \overset{\smile}{\lambda}^{1} + \left(6(3e^6 + 2e^3) - 30\right) \overset{\frown}{\lambda}^{1} = \\ \qquad = 118(3e^6 + 2e^3) - 674, \\ 3\left(e^9 - e^6\right) u_0 + \left((e^9 - e^6) - 3\right) \overset{\smile}{\lambda}^{1} + \left(6(e^9 - e^6) - 27\right) \overset{\frown}{\lambda}^{1} = \\ \qquad = 118(e^9 - e^6) - 468\,, \\ 3\left((e^{12} - 2e^9 + e^6 - e^3) + 1\right) u_0 + \left((e^{12} - 2e^9 + e^6 - e^3) - 2\right) \overset{\smile}{\lambda}^{1} + \\ + \left(6(e^{12} - 2e^9 + e^6 - e^3) - 12\right) \overset{\frown}{\lambda}^{1} = 118(e^{12} - 2e^9 + e^6 - e^3) - 227. \end{cases}$$

Direct computation shows that the rank of the matrix of this system is equal to 3, and its only solution is:

$$u_0 = 1, \quad \overset{\smile}{\lambda}^{1} = 33, \quad \overset{\frown}{\lambda}^{1} = \frac{41}{3}.$$

Then from representation (23) we obtain the required solution:

$$u(x) = F(x)u_0 + \overset{\smile}{F}^{1}(x) \cdot \overset{\smile}{\lambda}^{1} + \overset{\frown}{F}^{1}(x) \cdot \overset{\frown}{\lambda}^{1} + F^1(x) = 2x^2 + 1, \quad x \in [0,\, 4].$$

Conclusion

We have proposed an approach to the study and solving a class of nonlocal problems with respect to linear ordinary pointwise and integrally loaded differential equations. The main specificity of integral loadings is that the kernels of the integral terms depend on only one variable of integration. This made it possible to reduce solving the original problem to solving auxiliary Cauchy problems with respect to ordinary differential equations.

The considered problem is of independent interest. But as shown in the paper, the optimal control problems for objects with feedback are reduced to it, in which the current state measurements of an object can be of a point and interval nature.

We have obtained existence and uniqueness conditions of the solution for the considered class of problems, and provided study and solution of one illustrative problem.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Нүктелік және интегралдық жүктелген дифференциалдық теңдеулер класы жайында

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Бейлокал шекаралық шарттары бар нүктелік және интегралдық жүктемелерден тұратын сызықтық қарапайым дифференциалдық теңдеулер жүйесі зерттелген. Шекаралық шарттарға белгісіз функцияның интегралдық және нүктелік мәндері жатады. Есептің маңызды шарты дифференциалдық теңдеулердегі интегралдық мүшелердің ядролары тек интеграциялық айнымалыға тәуелділігінде. Ұқсас есептердің басқарылатын объектінің ағымдағы күйін нүктелік және интегралдық өлшеулер кезінде біріктірілген және бөлінген параметрлері бар екі объектінің де кері байланысын бақылау кезінде туындайтыны көрсетілген. Мақалада қарастырылған есептің қойылуы бейлокал шекаралық шарттармен жүктелген дифференциалдық теңдеулер бойынша бұрын зерттелген көптеген есептерді жалпылайды. Көмекші параметрлерді енгізу арқылы қарастырылатын есептің шешімінің бар және жалғыз болуының қажетті шарттары алынды. Есепті сандық түрде шешу үшін төрт көмекші Коши есебінің шешімі болып табылатын төрт матрицалық функцияны қамтитын бастапқы есептің шешімін пайдалану ұсынылады. Шекаралық жағдайларда көмекші есептердің шешімдерін пайдалана отырып, жүктеу нүктелеріндегі белгісіз функцияның мәндері алынды. Бұл қажетті шешімді алу үшін жеткілікті. Мақалада модельдік есепті шешудің мысалы арқылы әдісті қолдану көрсетілген.

Кілт сөздер: интегралдық-дифференциалдық теңдеу, жүктелген теңдеулер жүйесі, интегралдық шарттар, бейлокал шарттар, бар және жалғыз болу шарттары.

О классе точечно и интегрально нагруженных дифференциальных уравнений

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Исследована система линейных обыкновенных дифференциальных уравнений, содержащая точечные и интегральные нагружения, с нелокальными краевыми условиями. Краевые условия включают интегральные и точечные значения неизвестной функции. Существенным условием в задаче является то, что ядра интегральных слагаемых в дифференциальных уравнениях зависят лишь от переменной интегрирования. Показано, что подобные задачи возникают при управлении с обратной связью как объектами с сосредоточенными, так и распределенными параметрами при точечных и интегральных замерах текущего состояния управляемого объекта. Указанная в статье постановка задачи обобщает многие исследованные ранее задачи относительно нагруженных дифференциальных уравнений с нелокальными краевыми условиями. Введением вспомогательных параметров получены необходимые условия существования и единственности решения рассматриваемой задачи. Для численного решения задачи предложено использовать представление решения исходной задачи, включающее четыре матричные функции, являющиеся решениями четырех вспомогательных задач Коши. Используя решения вспомогательных задач в краевых условиях, получены значения неизвестной функции в точках нагружения. Это достаточно, чтобы получить искомое решение. В статье приведено изложение применения метода на примере решения модельной задачи.

Ключевые слова: интегро-дифференциальное уравнение, система нагруженных уравнений, интегральные условия, нелокальные условия, условия существования и единственности.

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