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The theorems about traces and extensions for functions from Nikolsky-Besov spaces with generalized mixed smoothness

The theory of embedding of spaces of differentiable functions studies important relations of differential (smoothness) properties of functions in various metrics and has wide application in the theory of boundary value problems of mathematical physics, approximation theory and other fields of mathematics.
In this article, we prove the theorems about traces and extensions for functions from Nikolsky-Besov spaces with generalized mixed smoothness and mixed metrics. The proofs of the obtained results is based on the inequality of different dimensions for trigonometric polynomials in Lebesgue spaces with mixed metrics and the embedding theorem of classical Nikolsky-Besov spaces in the space of continuous functions.

Keywords: Nikolsky-Besov spaces, generalized mixed smoothness, mixed metrics, a trace of function, an extension of function.

Introduction

One of the first results related to the theory of embedding of spaces of differentiable functions was a result of S.L. Sobolev [1]. This theory studies important relations of differential (smoothness) properties of functions in various metrics. Further development of this theory is associated with new classes of function spaces defined and studied in the works of S.M. Nikolsky [2, 3], O.V. Besov [4, 5], P.I. Lizorkin [6], H. Triebel [7, 8] and many others. The development of this research was determined both by its internal problems and by its applications in the theory of boundary value problems of mathematical physics and approximation theory ([9]–[17]).

This paper continues our investigations of Nikolsky-Besov spaces with generalized mixed smoothness and mixed metrics, which began in the works [18, 19]. In this article, we prove theorems on traces and continuations for functions from the above-mentioned spaces. The proof of these results is based on applying the inequality of different dimensions for trigonometric polynomials in mixed-metric Lebesgue spaces and the Nikolsky-Besov classical spaces embedding theorem into the space of continuous functions.

1 Definitions and auxiliary results

Let $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, $\mathbb{T}^{\mathbf{d}} = \{\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) : \mathbf{x}_i \in \mathbb{T}^{d_i} = [0, 2\pi)^{d_i}, i = 1, \ldots, n\}$ and $f(\mathbf{x}) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be measurable function on $\mathbb{T}^{\mathbf{d}}$.

Let $\mathbf{1} \le \mathbf{p} = (p_1, \ldots, p_n) \le \infty$. We say that the function f belongs to the Lebesgue space with mixed metrics $L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})$ if

$$\|f\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} = \left(\int_{\mathbb{T}^{d_n}} \left(\ldots \left(\int_{\mathbb{T}^{d_1}} |f(\mathbf{x}_1, \ldots, \mathbf{x}_n|^{p_1} d\mathbf{x}_1\right)^{p_2/p_1} \ldots\right)^{p_n/p_{n-1}} d\mathbf{x}_n\right)^{1/p_n} < \infty.$$

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In a case when $p_i = \infty$ the expression $\left(\int_{\mathbb{T}^{d_i}} |f(\mathbf{x}_i)|^{p_i}\, d\mathbf{x}_i\right)^{1/p_i}$ means that $\operatorname{ess\,sup}_{\mathbf{x}_i \in \mathbb{T}^{d_i}} |f(\mathbf{x}_i)|$.

Let us denote by

$$\Delta_{\mathbf{s}}(f, \mathbf{x}) = \sum_{\mathbf{k} \in \rho(\mathbf{s})} a_{\mathbf{k}} e^{i\langle \mathbf{k}, \mathbf{x}\rangle_{\mathbf{d}}}$$

the trigonometric series of $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle \mathbf{k}, \mathbf{x}\rangle_{\mathbf{d}}}$, where $\langle \mathbf{k}, \mathbf{x}\rangle_{\mathbf{d}} = \sum_{i=1}^{n}\sum_{j=1}^{d_i} k_j^i x_j^i$ is the (modified) inner product, $\rho(\mathbf{s}) = \{\mathbf{k} = (\mathbf{k}_1, \ldots, \mathbf{k}_n) \in \mathbb{Z}^{\mathbf{d}} : \left[2^{s_i - 1}\right] \leq \max_{j=1,\ldots,d_i} |k_j^i| < 2^{s_i}, i = 1, \ldots, n\}$ and $[a]$ is the integer part of the number a.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, $\mathbf{1} \leq \mathbf{q} = (\mathbf{q_1}, \ldots, \mathbf{q_n}) \leq \infty$ and $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_n) < \infty$.

The anisotropic Nikolsky-Besov space with generalized mixed smoothness and mixed metrics $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ is a set of the series $f \sim \sum_{\mathbf{k}\in\mathbb{Z}^{\mathbf{d}}} a_{\mathbf{k}} e^{i\langle \mathbf{k}, \mathbf{x}\rangle_{\mathbf{d}}}$ such as

$$\|f\|_{B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \left\| \left\{ 2^{(\alpha,\mathbf{s})} \|\Delta_{\mathbf{s}}(f)\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \right\} \right\|_{l_{\mathbf{q}}} < \infty,$$

where $(\alpha, \mathbf{s}) = \sum_{i=1}^{n} \alpha_i s_i$ is the inner product and $\|\cdot\|_{l_{\mathbf{q}}}$ is the norm of a discrete Lebesgue space with mixed metrics $l_{\mathbf{q}}$.

Here $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ is a version of spaces, which was introduced and studied in [20].

Remark 1. The anisotropic Nikolsky-Besov space with generalized mixed smoothness $B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}})$ mentioned above is a hybrid structure of Nikolsky-Besov space (concerning to variables included in one multi-variable) [2, 4] and spaces with dominant mixed derivative (concerning to variables included in different multi-variables) [21, 22]. In the isotropic case, when p and q are scalars, analogs of these spaces were studied by D.B. Bazarkhanov [23].

Let us denote by $\bar{\mathbf{p}} = (p_1, \ldots, p_m)$ for the multi-index $\mathbf{p} = (p_1, \ldots, p_m, p_{m+1}, \ldots, p_n)$.

Lemma 1 (Inequality of different dimensions, [3]). Let $T_{\mathbf{s}}(\mathbf{x})$ be a trigonometric polynomial of order not higher than $\mathbf{s} = (\mathbf{s_1}, \ldots, \mathbf{s_m}, \mathbf{s_{m+1}}, \ldots, \mathbf{s_n})$ by variables $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_m}, \mathbf{x_{m+1}}, \ldots, \mathbf{x_n})$ and $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_m, p_{m+1}, \ldots, p_n) < \infty$, then for an arbitrary fixed point $(\mathbf{x_{m+1}}, \ldots, \mathbf{x_n}) \in \mathbb{T}^{d_{m+1}} \times \ldots \times \mathbb{T}^{d_n}$ the following inequality holds

$$\|T_{\mathbf{s}}(\cdot, \ldots, \cdot, \mathbf{x_{m+1}}, \ldots, \mathbf{x_n})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \leq C \prod_{i=m+1}^{n} s_i^{d_i/p_i} \|T_{\mathbf{s}}\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})},$$

where C is a positive constant independent on $\mathbf{s}$.

2 Main results

In this section, we prove the trace and continuation theorems for functions from Nikolsky-Besov spaces with generalized mixed smoothness and anisotropic Lorentz spaces are proved.

Theorem 1. Let $\mathbf{0} < \alpha = (\alpha_1, \ldots, \alpha_m, \alpha_{\mathbf{m+1}}, \ldots, \alpha_{\mathbf{n}}) < \infty$, $\mathbf{1} \leq \mathbf{q} = (q_1, \ldots, q_m, q_{m+1}, \ldots, q_n) \leq \infty$, $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_m, p_{m+1}, \ldots, p_n) < \infty$, then for $\alpha_i = d_i/p_i$, $q_i = 1$ where $i = m+1, \ldots, n$ the following embedding holds

$$B_{\mathbf{p}}^{\alpha\mathbf{q}}(\mathbb{T}^{\mathbf{d}}) \hookrightarrow B_{\bar{\mathbf{p}}}^{\bar{\alpha}\bar{\mathbf{q}}}(\mathbb{T}^{\bar{\mathbf{d}}}).$$

Proof. According to the inequality of different dimensions (Lemma 1) and Minkowski inequality, we obtain

$$\|f(\cdot,\ldots,\cdot,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)\|_{B^{\bar{\alpha}\bar{\mathbf{q}}}_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$

$$= \left\|\left\{2^{(\bar{\alpha},\bar{\mathbf{s}})}\left\|\Delta_{\bar{\mathbf{s}}}\left(f(\cdot,\ldots,\cdot,\mathbf{x}_{m+1},\ldots,\mathbf{x}_n)\right)\right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})}\right\}\right\|_{l_{\bar{\mathbf{q}}}} \le$$

$$\le \left\|\left\{2^{(\bar{\alpha},\bar{\mathbf{s}})}\left\|\sum_{s_n=0}^{\infty}\cdots\sum_{s_{m+1}=0}^{\infty}\Delta_{s_{m+1}}\left(\ldots\Delta_{s_n}\left(\Delta_{\bar{\mathbf{s}}}\left(f\right)\right)\right)\right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})}\right\}\right\|_{l_{\bar{\mathbf{q}}}} \le$$

$$\le \left\|\left\{2^{(\bar{\alpha},\bar{\mathbf{s}})}\sum_{s_n=0}^{\infty}\cdots\sum_{s_{m+1}=0}^{\infty}\left\|\Delta_{s_{m+1}}\left(\ldots\Delta_{s_n}\left(\Delta_{\bar{\mathbf{s}}}\left(f\right)\right)\right)\right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})}\right\}\right\|_{l_{\bar{\mathbf{q}}}} =$$

$$= \left\|\left\{2^{(\bar{\alpha},\bar{\mathbf{s}})}\sum_{s_n=0}^{\infty}\cdots\sum_{s_{m+1}=0}^{\infty}\left\|\Delta_{\mathbf{s}}\left(f\right)\right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})}\right\}\right\|_{l_{\bar{\mathbf{q}}}} \le$$

$$\le C_1\left\|\left\{2^{(\bar{\alpha},\bar{\mathbf{s}})}\sum_{s_n=0}^{\infty}2^{(s_nd_n)/p_n}\cdots\sum_{s_{m+1}=0}^{\infty}2^{(s_{m+1}d_{m+1})/p_{m+1}}\left\|\Delta_{\mathbf{s}}\left(f\right)\right\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})}\right\}\right\|_{l_{\bar{\mathbf{q}}}} \le$$

$$\le C_1\sum_{s_n=0}^{\infty}\cdots\sum_{s_{m+1}=0}^{\infty}\left\|\left\{2^{(\alpha,\mathbf{s})}\left\|\Delta_{\mathbf{s}}\left(f\right)\right\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})}\right\}\right\|_{l_{\bar{\mathbf{q}}}} = C_1\left\|\left\{2^{(\alpha,\mathbf{s})}\left\|\Delta_{\mathbf{s}}\left(f\right)\right\|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})}\right\}\right\|_{l_{\mathbf{q}}} = C_1\|f\|_{B^{\alpha\mathbf{q}}_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})},$$

here $\alpha_i = d_i/p_i$, $q_i = 1$ where $i = m+1,\ldots,n$.

Let us show the conditions $\alpha_i = d_i/p_i$, $q_i = 1$ where $i = m+1,\ldots,n$ ensure that the following property holds

$$\|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - \varphi(\mathbf{x}_1,\ldots,\mathbf{x}_m)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \to 0$$

for $\max\limits_{i=m+1,\ldots,n}|\mathbf{h}_i| \to \mathbf{0}$, here $|\mathbf{h}_i| = \sqrt{\sum\limits_{j=1}^{d_i}(h^i_j)^2}$, $(i = m+1,\ldots,n)$.

Indeed, let $N \in \mathbb{N}$ and

$$\Gamma_N = \left\{\mathbf{s} = (\mathbf{s_1},\ldots,\mathbf{s_n}) \in \mathbb{Z}^n : \prod_{i=1}^{n}\max(1, \max_{j=1,\ldots,d_i}|s^i_j|) \le N\right\},$$

then

$$\|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - \varphi(\mathbf{x}_1,\ldots,\mathbf{x}_m)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$
$$= \|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$
$$= \|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) +$$
$$+ S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0}) +$$
$$+ S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0}) - f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \le$$
$$\le \|f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} +$$
$$+ \|S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{h}_{m+1},\ldots,\mathbf{h}_n) - S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} +$$
$$+ \|S_{\Gamma_N}(f;\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0}) - f(\mathbf{x}_1,\ldots,\mathbf{x}_m,\mathbf{0},\ldots,\mathbf{0})\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} = I_1 + I_2 + I_3,$$

where $S_{\Gamma_N}(f; \mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_{m+1}, \ldots, \mathbf{x}_n)$ is the partial sum of the Fourier series of the function $f(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_{m+1}, \ldots, \mathbf{x}_n)$, corresponding to the hyperbolic cross Γ_N.

We will use Minkowski, different dimensions (Lemma 1) and Hölder inequalities to estimate I_1 and I_3. For $k = 1$ or $k = 3$ we have

$$I_k \leq \sup_{\mathbf{x}_{m+1},\ldots,\mathbf{x}_n} \| f(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_{m+1}, \ldots, \mathbf{x}_n) -$$

$$- S_{\Gamma_N}(f; \mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_{m+1}, \ldots, \mathbf{x}_n) \|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$

$$= \sup_{\mathbf{x}_{m+1},\ldots,\mathbf{x}_n} \left\| \sum_{\mathbf{s} \notin \Gamma_N} \Delta_{\mathbf{s}}(f; \mathbf{x}) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \leq \sum_{\mathbf{s} \notin \Gamma_N} \sup_{x_{m+1},\ldots,x_n} \| \Delta_{\mathbf{s}}(f; \mathbf{x}) \|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \leq$$

$$\leq C_1 \sum_{\mathbf{s} \notin \Gamma_N} 2^{-\langle \bar{\alpha}, \bar{\mathbf{s}} \rangle} 2^{\langle \alpha, \mathbf{s} \rangle} \| \Delta_{\mathbf{s}}(f; \mathbf{x}) \|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \leq$$

$$\leq C_1 \left\| \left\{ 2^{\langle \alpha, \mathbf{s} \rangle} \| \Delta_{\mathbf{s}}(f; \mathbf{x}) \|_{L_{\mathbf{p}}(\mathbb{T}^{\mathbf{d}})} \right\}_{\mathbf{s} \notin \Gamma_N} \right\|_{l_{\mathbf{q}}} \cdot \left\| \left\{ 2^{-\langle \bar{\alpha}, \bar{\mathbf{s}} \rangle} \right\}_{\mathbf{s} \notin \Gamma_N} \right\|_{l_{\mathbf{q}'}} \leq$$

$$\leq C_2 \| f - S_{\Gamma_N}(f) \|_{B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}})} \to 0 \text{ при } N \to \infty.$$

Moreover, to estimate I_2, we use the fact that the trigonometric polynomial $S_{\Gamma_N}(f; \mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_{m+1}, \ldots, \mathbf{x}_n)$ is a continuous function, then we receive

$$\| S_{\Gamma_N}(f; \mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{h}_{m+1}, \ldots, \mathbf{h}_n) - S_{\Gamma_N}(f; \mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{0}, \ldots, \mathbf{0}) \|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \to 0$$

for $\max\limits_{i=m+1,\ldots,n} |\mathbf{h}_i| \to 0$.

This completes the proof.

Remark 2. In contrast to the trace theorem for functions from Nikolsky-Besov spaces with dominating mixed derivative [21, 22], proved for $\alpha_i > d_i/p_i$ where $i = m+1, \ldots, n$, in Theorem 1, the limiting case $\alpha_i = d_i/p_i$ is considered under the condition $q_i = 1$ for $i = m+1, \ldots, n$ (this effect was previously seen for example in [24, 25]).

Theorem 2. Let $\mathbf{0} < \alpha = (\alpha_1, \ldots, \alpha_m, \alpha_{m+1}, \ldots, \alpha_n) < \infty$, $\mathbf{1} \leq \mathbf{q} = (q_1, \ldots, q_m, q_{m+1}, \ldots, q_n) \leq \infty$, $\mathbf{1} < \mathbf{p} = (p_1, \ldots, p_m, p_{m+1}, \ldots, p_n) < \infty$. Then for $\alpha_i = d_i/p_i$ and $q_i = 1$, $i = m+1, \ldots, n$, for the function $\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_m) \in B_{\bar{\mathbf{p}}}^{\bar{\alpha} \bar{\mathbf{q}}}(\mathbb{T}^{\bar{\mathbf{d}}})$ it is possible to construct a function $f(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{x}_{m+1}, \ldots, \mathbf{x}_n)$ having the following properties

$$f \in B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}});$$

$$\| f \|_{B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}})} \leq C \| \varphi \|_{B_{\bar{\mathbf{p}}}^{\bar{\alpha} \bar{\mathbf{q}}}(\mathbb{T}^{\bar{\mathbf{d}}})};$$

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{0}, \ldots, \mathbf{0}) = \varphi(\mathbf{x}_1, \ldots, \mathbf{x}_m).$$

Proof. Let $\varphi \in B_{\bar{\mathbf{p}}}^{\bar{\alpha} \bar{\mathbf{q}}}(\mathbb{T}^{\bar{\mathbf{d}}})$. This function can be represented as a series converging to it in the sense of $L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})$

$$\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \sum_{\bar{\mathbf{s}} = \mathbf{0}}^{\infty} \Delta_{\bar{\mathbf{s}}}(\varphi(\mathbf{x}_1, \ldots, \mathbf{x}_m))$$

and

$$\| \varphi \|_{B_{\bar{\mathbf{p}}}^{\bar{\alpha} \bar{\mathbf{q}}}(\mathbb{T}^{\bar{\mathbf{d}}})} = \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \| \Delta_{\bar{\mathbf{s}}}(\varphi) \|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}}.$$

Let us choose the functions $f_i(\mathbf{x}_i)$ from the $B_{p_i}^{\alpha_i 1}(\mathbb{T}^{d_i})$, where $\alpha_i = d_i/p_i$, such as $f_i(\mathbf{0}) = 1$ for $i = m+1, \dots, n$. Let us introduce a new function

$$f(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{x}_{m+1}, \dots, \mathbf{x}_n) = \sum_{\mathbf{s}=0}^{\infty} \Delta_{\bar{\mathbf{s}}} \left(\varphi(\mathbf{x}_1, \dots, \mathbf{x}_m)\right) \prod_{i=m+1}^{n} \Delta_{s_i} \left(f_i(\mathbf{x}_i)\right).$$

Consequently, for this function, we get

$$\|f\|_{B_{\mathbf{p}}^{\alpha \mathbf{q}}(\mathbb{T}^{\mathbf{d}})} = \left\| \left\{ 2^{(\alpha, \mathbf{s})} \left\| \Delta_{\mathbf{s}} \left(\varphi\right) \right\|_{L_{\mathbf{p}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\mathbf{q}}} =$$

$$= \left\| \left\{ 2^{(\bar{\alpha}, \bar{\mathbf{s}})} \left\| \Delta_{\bar{\mathbf{s}}} \left(\varphi\right) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \right\} \right\|_{l_{\bar{\mathbf{q}}}} \prod_{i=m+1}^{n} \|f_i\|_{B_{p_i}^{\alpha_i 1}(\mathbb{T}^{d_i})} = C_6 \|\varphi\|_{B_{\bar{\mathbf{p}}}^{\bar{\alpha} \bar{\mathbf{q}}}(\mathbb{T}^{\bar{\mathbf{d}}})}.$$

According to the condition $f_i(\mathbf{0}) = 1$ for $i = m+1, \dots, n$ we have

$$f(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{0}, \dots, \mathbf{0}) = \varphi(\mathbf{x}_1, \dots, \mathbf{x}_m).$$

Note that the conditions $\alpha_i = d_i/p_i$ and $q_i = 1$ for $i = m+1, \dots, n$ ensure the continuity of the functions $f_i(\mathbf{x}_i)$. Therefore we obtain

$$\|f(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{x}_{m+1}, \dots, \mathbf{x}_n) - \varphi(\mathbf{x}_1, \dots, \mathbf{x}_m)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} =$$

$$= \left\| \varphi(\mathbf{x}_1, \dots, \mathbf{x}_m) \left(\prod_{i=m+1}^{n} f_i(\mathbf{x}_i) - 1 \right) \right\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \leq$$

$$\leq \|\varphi(\mathbf{x}_1, \dots, \mathbf{x}_m)\|_{L_{\bar{\mathbf{p}}}(\mathbb{T}^{\bar{\mathbf{d}}})} \left| \prod_{i=m+1}^{n} f_i(\mathbf{x}_i) - 1 \right| \to 0$$

here $\max\limits_{i=m+1,\dots,n} |\mathbf{x}_i| \to 0$.

These arguments show that φ is the trace of the function f.

The proof is complete.

Remark 3. Note that the continuation operator constructed in the proof of Theorem 2 is linear. We should note here that in the work of V.I. Burenkov and M.L. Goldman [26], where it is shown that in the limiting case for classical anisotropic Nikolsky-Besov spaces it is possible to construct only a nonlinear continuation operator, but this effect is not observed for Nikolsky-Besov spaces with a dominant mixed derivative.

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Жалпыланған аралас тегістілігі бар Никольский-Бесов кеңістіктеріндегі функциялар үшін іздер және жалғасулар теоремалары

Дифференциалданатын функциялар кеңістіктерінің енгізу теориясы әртүрлі метрикаларда функциялардың маңызды байланыстары мен функцияның дифференциалдық (тегістілік) қасиеттерінің қатынастарын зерттейді және де математикалық физиканың шектік есептер теориясында, жуықтау теориясында және математиканың басқа да салаларында кеңінен қолданысқа ие. Мақалада жалпыланған аралас тегістігі және аралас метрикасы бар Никольский-Бесов кеңістіктері функцияларының іздері мен жалғасы туралы теоремалар дәлелденген. Алынған нәтижелердің дәлелдеуі аралас метрикасы бар Лебег кеңістіктеріндегі тригонометриялық полиномдар үшін әр түрлі өлшемді теңсіздіктерін және классикалық Никольский-Бесов кеңістіктерінің үзіліссіз функциялар кеңістігіне ену теоремасын қолдануға негізделген.

Кілт сөздер: Никольский-Бесовтың кеңістігі, жалпыланған аралас тегістік, аралас метрика, функцияның ізі, функцияның жалғасы.

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Теоремы о следах и продолжениях для функций из пространств Никольского-Бесова с обобщенной смешанной гладкостью

Теория вложения пространств дифференцируемых функций изучает важные связи и соотношения дифференциальных (гладкостных) свойств функций в различных метриках и имеет широкое применение в теории краевых задач математической физики, теории приближений и других разделах математики. В данной статье мы доказываем теоремы о следах и продолжениях для функций из пространств Никольского-Бесова с обобщенной смешанной гладкостью и со смешанной метрикой. Доказательства полученных результатов основаны на использовании неравенства разных измерений для тригонометрических полиномов в пространствах Лебега со смешанной метрикой и теореме вложения классических пространств Никольского-Бесова в пространство непрерывных функций.

Ключевые слова: пространства Никольского-Бесова, обобщенная смешанная гладкость, смешанная метрика, след функции, продолжение функции.

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