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THE EFFECT OF TOPOLOGICAL DEFECT ON THE MASS SPECTRA OF HEAVY AND HEAVY-LIGHT QUARKONIA

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In this present study, the effect of Topological Defect on the mass spectra of heavy and heavy-light mesons such as charmonium, bottomonium, and charm-strange $(c\bar{s})$ *,bottom-charm* $(b\bar{c})$ *respectively are studied with the Hulthen plus Yukawa potential. The Schrödinger equation is solved analytically using the Nikiforov-Uvarov method. The approximate solutions of the energy spectrum and un-normalized wave function were obtained. We applied the present results to predict the mass spectra of heavy and heavy-light mesons in the presence and absence of a topological defect for different quantum states. We noticed that when the topological defect increases the mass spectra are shifted and move closer to the experimental data. However, when compared to the work of other researchers, the results established an improvement.*

Keywords: Schrödinger equation; Nikiforov-Uvarov method; Hulthen-Yukawa Potential; Mass Spectra; Topological Defect

Introduction

The existence of heavy quark-bound states was discovered independently at the Stanford Linear Accelerator Center (SLAC) [1] and Brookhaven National Laboratory (BNL) [2] in 1974. Since then, the study of the quarkonia system has been investigated both experimentally and theoretically by particle physicists [3,4]. Studies on heavy quark systems are crucial because they provide knowledge on interaction potential, confinement, quantum chromodynamics (QCD) coupling constant, and various other inputs to the standard model [5]. The solutions of the Schrodinger equation (SE) with potential models are used in describing the energy spectra of diatomic molecules, theoretical measures, and mass spectra (MS) of the heavy and heavy-light mesons [6-8]. In the study of mass spectra of the heavy and heavy-light mesons, confining-type potentials are generally used. The widely used potential is the Cornell potential (CP) which contains a Coulomb interaction term and a linear confinement term [9]. Analytically, different methods have been proposed and employed in solving the SE with a chosen potential model of interest, such as, the Asymptotic iteration method AIM [10], the Nikiforov-Uvarov (NU) method [11,12], the NU Functional Analysis (NUFA) method [13,14], the series expansion method (SEM) [15], WKB approximation method [16], exact quantization rule [17], and so on. The analytical study of heavy quark with CP has gained remarkable attention from scholars [18,19]. For instance, Vega and Flores, [20] obtained the analytical solutions of the SE with CP using the vibrational method (VM) and super symmetric quantum mechanics. The eigenvalues were used to calculate the MS of the mesons. Also, Kumar et al. [21] used the NUFA method to solved the SE with generalized CP. The result was used to determine the MS of the heavy quarks. Furthermore, Hassanabadi, et al. [22], used the VM to solve the SE with CP. The mesonic wave function was computed using the eigenvalues. To examine the MS of mesons, researchers have recently modeled exponential-type potentials to study the MS of quarkonia [23]. The prediction of the MS of the heavy mesons (HMs), potential models such as Varshni [24], Hulthen plus Hellmann potential [25], and others have been used. For instance, Purohit et al. [26] used the solutions of the Klein-Gordon equation potential (KGE) to predict the HMs by combining linear plus modified Yukawa potentials.

Hulthen potential [27] and Yukawa potential [28] are used in different branches of physics, including nuclear and particle physics, among others. It has been noted that more experimental data tends to fit a mixture of at least two potential models than a single potential [27]. The Hulthen plus Yukawa potential

(HYP) was proposed by Li and Chang [29] to investigate the nonlinear optical characteristics of the GaAs/AlnGa1-n as quantum dots system. The HYP takes the form [29],

$$V(r) = -\frac{R_0 e^{-\vartheta r}}{1-e^{-\vartheta r}} - \frac{b e^{-\vartheta r}}{r} \ , \tag{1}$$

where R_0, and b are the strength of the potential, ϑ is the screening parameter and r is inter-nuclear distance.

Scholars have found the impact of a topological defect (TD) with a single particle in a particular potential to be an intriguing topic [30]. Its genesis is thought to have taken place during a phase transition in the early universe. Researchers have recently become interested in how TD affects the dynamics of both relativistic and non-relativistic systems, including screw dislocation [31], bound electron eigenstates, and holes to a declination. Furtado et al. [32] examined the Landau levels in the presence of a TD in light of these. Additionally, Hassanabadi and Hosseinpour [33] looked into how TD affected hydrogen atoms in curve-space time. Topological defects have long been a hot topic in domains like condensed matter and gravitational Physics [34]. They play a significant role in altering the physical characteristics of many quantum systems. A linear defect in an elastic medium, such as a dislocation or desperation, causes a change in the topology of the medium, which has an effect on the medium's physical characteristics [34]. Ahmed [35] investigated how TD and external fields affected diatomic molecules. It was demonstrated that the TD and external fields cause the energy levels to change. No researchers are yet to report the impact of TD on the mass spectra of heavy and heavy-light mesons in light of these observations. Therefore, the purpose of this study is to use the NU approach to solve SE with HYP in order to examine the effect of TD on the mass spectra of heavy and heavy-light mesons. For convenience, we have assumed that our mesons are spinless particles [36].

Equation 1 is modeled to include the Coulomb and confinement terms using a series of powers up to order three to expand the exponential terms, and we get

$$V(r) = -\frac{B}{r} + Ar + Dr^2 + \mathrm{E} \ , \tag{2}$$

where

$$\left.\begin{aligned} B &= \frac{R_0}{\vartheta} + 2b, \ \mathrm{A} = -\frac{\vartheta R_0}{12} - \frac{b\vartheta^2}{2} \\ D &= \frac{b\vartheta^2}{6}, \ \mathrm{E} = \frac{R_0}{2} + b\vartheta \end{aligned}\right\} \tag{3}$$

2. Formalism

The line element that explains spacetime with a point-like global monopole (PGM) takes the form: [30]

$$ds^2 = -c^2 dt^2 + \frac{dr^2}{\alpha^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{4}$$

where $0 \prec \alpha = 1 - 8\pi G\eta_0^2 \prec 1$ is the parameter related to the PGM which depends on the energy scale η_0. Furthermore, Eq. (4) portrays a space time with scalar curvature

$$R = R_\mu^\mu = \frac{2\left(1-\alpha^2\right)}{r^2} \tag{5}$$

In this way, the SE takes the form

$$-\frac{\hbar^2}{2\mu}\nabla_{LB}^2 \psi(\vec{r},t) + V(r,t)\psi(\vec{r},t) = i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} \tag{6}$$

where μ is the particle's mass, $\nabla_{LB}^2 = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g} g^{ij}\partial_j\right)$ with $g = \det(g_{ij})$, is the Laplace-Beltrami operator and $V(r,t) = V(r)$ is GMP(1).Thereby, the SE for the GMP in a medium with the presence of the PGM(1) is

$$-\frac{\hbar^2}{2\mu r^2}\left[\begin{array}{l}\alpha^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\\+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\end{array}\right]\psi(r,\theta,\varphi,t)+V\psi(r,\theta,\varphi,t)=i\hbar\frac{\partial\psi(r,\theta,\varphi,t)}{\partial t} \tag{7}$$

Here, let us consider a particular solution to Eq.(5) given in terms of the eigenvalues of the angular momentum operator $\hat{L}^2$ as

$$\psi(r,\theta,\varphi,t)=e^{-\frac{E_{nl}t}{\hbar}}\frac{U(r)}{r}Y_{l,m}(\theta,\varphi) \tag{8}$$

where $Y_{l,m}(\theta,\varphi)$ are spherical harmonics and $R(r)$ is the radial wave function.

3.The solutions of the Schrödinger equation with Hulthen-Yukawa Potential11

The NU technique is used in this investigation. The details are provided in Ref. [11]. Then, we substitute Eq. (2) into Eq.(7), the radial wave equation is obtain as

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu E_{nl}}{\alpha^2\hbar^2}+\frac{2\mu B}{\alpha^2\hbar^2 r}-\frac{2\mu Ar}{\alpha^2\hbar^2}-\frac{2\mu Dr^2}{\alpha^2\hbar^2}-\frac{2\mu E}{\alpha^2\hbar^2}-\frac{l(l+1)}{\alpha^2 r^2}\right]R(r)=0 \tag{9}$$

Let,

$$x=\frac{1}{r} \tag{10}$$

Putting Eq. (10) into Eq. (9), gives

$$\frac{d^2R(\mathrm{x})}{dx^2}+\frac{2x}{x^2}\frac{dR}{dx}+\frac{1}{x^4}\left[\frac{2\mu E_{nl}}{\alpha^2\hbar^2}+\frac{2\mu Bx}{\alpha^2\hbar^2}-\frac{2\mu A}{\alpha^2\hbar^2 x}-\frac{2\mu D}{\alpha^2\hbar^2 x^2}-\frac{2\mu E}{\alpha^2\hbar^2}-\frac{l(l+1)x^2}{\alpha^2}\right]R(\mathrm{x})=0 \tag{11}$$

The approximation scheme (AS) on the terms $\frac{A}{x}$ and $\frac{D}{x^2}$ is introduced by assuming that there is a characteristic radius r_0 of the meson. The AS is achieved by the expansion in a power series around r_0 ; i.e. around $\delta\equiv\frac{1}{r_0}$, up to the second order [24]. By setting $y=x-\delta$ and around $y=0$ we have;

$$\frac{A}{x}=\frac{\alpha_1}{y+\delta}=\frac{\alpha_1}{\delta\left(1+\frac{y}{\delta}\right)}=\frac{\alpha_1}{\delta}\left(1+\frac{y}{\delta}\right)^{-1} \tag{12}$$

Equation (12) yields

$$\frac{A}{x}=A\left(\frac{3}{\delta}-\frac{3x}{\delta^2}+\frac{x^2}{\delta^3}\right), \tag{13}$$

Similarly,

$$\frac{D}{x^2}=D\left(\frac{6}{\delta^2}-\frac{8x}{\delta^3}+\frac{3x^2}{\delta^4}\right) \tag{14}$$

Equations (13) and (14) are Plugged into Eq. (11) and yields,

$$\frac{d^2R(x)}{dx^2}+\frac{2x}{x^2}\frac{dR(x)}{dx}+\frac{1}{x^4}\left[-\varepsilon+\beta_1 x-\beta_2 x^2\right]R(x)=0 \tag{15}$$

where

$$\left.\begin{array}{l}-\varepsilon=\left(\frac{2\mu E_{nl}}{\alpha^2\hbar^2}-\frac{6\mu A}{\alpha^2\hbar^2\delta}-\frac{12\mu D}{\alpha^2\hbar^2\delta^2}-\frac{6\mu E}{\alpha^2\hbar^2}\right),\ \beta_1=\left(\frac{2\mu B}{\alpha^2\hbar^2}+\frac{6\mu A}{\alpha^2\hbar^2\delta^2}+\frac{16\mu D}{\alpha^2\hbar^2\delta^3}\right)\\ \beta_2=\left(\frac{2\mu A}{\alpha^2\hbar^2\delta^3}+\frac{6\mu D}{\alpha^2\hbar^2\delta^4}+\frac{l(l+1)}{\alpha^2}\right)\end{array}\right\} \tag{16}$$

Linking Eq.(15) and Eq. (1) of Ref. [11], we obtain

$$\left.\begin{aligned} &\tilde{\tau}(x)=2x,\ \sigma(x)=x^2 \\ &\tilde{\sigma}(x)=-\varepsilon+\beta_1 x-\beta_2 x^2 \\ &\sigma'(x)=2x,\ \sigma''(x)=2 \end{aligned}\right\} \tag{17}$$

Equation (17) is substituted into Ref. [11]'s Eq. (11) to produce

$$\pi(x)=\pm\sqrt{\varepsilon-\beta_1 x+\left(\beta_2+k\right)x^2} \tag{18}$$

The value of k is obtained by taking the discriminant of the function under the square root.

$$k=\frac{\beta_1^2-4\beta_2\varepsilon}{4\varepsilon} \tag{19}$$

Plugging Eq. (19) into Eq. (18) gives

$$\pi(x)=\pm\left(\frac{\beta_1 x}{2\sqrt{\varepsilon}}-\frac{\varepsilon}{\sqrt{\varepsilon}}\right) \tag{20}$$

From Eq. (20), we have

$$\pi'_-(x)=-\frac{\beta_1}{2\sqrt{\varepsilon}} \tag{21}$$

Putting Eqs. (17) and (20) into Eq.(6) of Ref. [11] gives

$$\tau(x)=2x-\frac{\beta_1 x}{\sqrt{\varepsilon}}+\frac{2\varepsilon}{\sqrt{\varepsilon}} \tag{22}$$

From Eq. (22) we get

$$\tau'(x)=2-\frac{\beta_1}{\sqrt{\varepsilon}} \tag{23}$$

Using Eqs. (10) and (13) from Ref. [11], we have the following,

$$\lambda=\frac{\beta_1^2-4\beta_2\varepsilon}{4\varepsilon}-\frac{\beta_1}{2\sqrt{\varepsilon}} \tag{24}$$

$$\lambda_n=\frac{n\beta_1}{\sqrt{\varepsilon}}-n^2-n \tag{25}$$

Equating Eqs. (24) and (25), followed by substitution of Eqs. (3) and (16) yield the energy spectrum of the HYP

$$E_{nl}=\frac{a}{2}+b\vartheta-\frac{3}{2\delta}\left(\frac{a\vartheta}{6}+b\vartheta^2\right)+\frac{b\vartheta^2}{\delta^2}-\frac{\alpha^2\hbar^2}{8\mu}\left[\frac{\frac{2\mu}{\alpha^2\hbar^2}\left(b+\frac{a}{\vartheta}\right)-\frac{3\mu}{\alpha^2\hbar^2\delta^2}\left(\frac{a\vartheta}{6}+b\vartheta^2\right)+\frac{8\mu b\vartheta^2}{3\alpha^2\hbar^2\delta^3}}{n+\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{\mu W_1}{\alpha^2\hbar^2\delta^3}\left(\frac{a\vartheta}{6}+b\vartheta^2\right)+\frac{\mu b\vartheta^2}{\alpha^2\hbar^2\delta^4}+\frac{l(l+1)}{\alpha^2}}}\right]^2 \tag{26}$$

In terms of related Laguerre polynomials, the un-normalized wave function is given as

$$\psi(\mathrm{z})=N_{nl}z^{-\frac{\alpha}{2\sqrt{\varepsilon}}}e^{-\frac{\varepsilon}{z\sqrt{\varepsilon}}}L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{z\sqrt{\varepsilon}}\right) \tag{27}$$

where N_{nl} is normalization constant, which can be obtained from

$$\int_0^\infty|\psi_{nl}(r)|^2dr=1 \tag{28}$$

4. Results and discussion

Using the following equation [37], the MS of the heavy and heavy-light mesons are predicted

$$\mathrm{M}=m_x+m_{\bar{x}}+E_{nl} \tag{29}$$

where $m_{\bar{x}}$ is quarkonia mass and E_{nl} is eigenvalues.

Substituting Eq. (26) into Eq. (29) we have,

$$\mathrm{M} = m_x + m_{\bar{x}} + \frac{a}{2} + b\vartheta - \frac{3}{2\delta}\left(\frac{a\vartheta}{6} + b\vartheta^2\right) + \frac{b\vartheta^2}{\delta^2} - \frac{\alpha^2\hbar^2}{8\mu}\left[\frac{\frac{2\mu}{\alpha^2\hbar^2}\left(b + \frac{a}{\vartheta}\right) - \frac{3\mu}{\alpha^2\hbar^2\delta^2}\left(\frac{a\vartheta}{6} + b\vartheta^2\right) + \frac{8\mu b\vartheta^2}{3\alpha^2\hbar^2\delta^3}}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\mu}{\alpha^2\hbar^2\delta^3}\left(\frac{a\vartheta}{6} + b\vartheta^2\right) + \frac{\mu b\vartheta^2}{\alpha^2\hbar^2\delta^4} + \frac{l(l+1)}{\alpha^2}}}\right]^2 \quad (30)$$

where μ is the reduced mass and α is the TD.

The numerical values of the heavy and heavy-light meson masses are taken from [38]. The potential parameters were fitted with experimental data (ED). Experimental data are taken from [39, 40].

The MS of the heavy and heavy-light mesons were predicted in the absence and presence of TD for different quantum states. Absence and presence corresponds to when α=1 and $\alpha = 2 = 3$ respectively. The mass spectra of charmonium in the presence of TD is seen to be exactly as the ED for 1S and 2S quantum state, but in other states we noticed that they predicted results are close to the ED. As the TD was introduced by setting TD = 2 and 3, the values of the predicted masses were seen to be approaching the ED and was seen to be improved from works reported by [15,37] as shown in Table 1.

Table 1: Mass spectra of charmonium in (GeV)
$(m_c = 1.209\ GeV, \mu = 0.6045\ GeV, \alpha = 1.0, a = -3.5647\ GeV, b = 352.7375\ GeV, \delta = 1.65\ GeV, \theta = 0.01, \hbar = 1)$
$(m_c = 1.209\ GeV, \mu = 0.6045\ GeV, \alpha = 2.0, a = -3.64024\ GeV, b = 356.5334\ GeV, \delta = 1.65\ GeV, \theta = 0.01, \hbar = 1)$
$(m_c = 1.209\ GeV, \mu = 0.6045\ GeV, \alpha = 3.0, a = -3.7153\ GeV, b = 360.2911\ GeV, \delta = 1.65\ GeV, \theta = 0.01, \hbar = 1)$

State	α	Our Result	AIM[37]	SEM[15]	Experiment [39]
1S	1.0	3.096000024	3.096	3.095922	3.096
	2.0	3.095999987	3.096	3.095922	3.096
	3.0	3.096000033	3.096	3.095922	3.096
2S	1.0	3.686000010	3.686	3.685893	3.686
	2.0	3.685999995	3.686	3.685893	3.686
	3.0	3.686000014	3.686	3.685893	3.686
3S	1.0	4.012881463	4.275	4.322881	4.040
	2.0	4.022871465	4.275	4.322881	4.040
	3.0	4.039981463	4.275	4.322881	4.040
4S	1.0	4.133343194	4.865	4.989406	4.263
	2.0	4.163343195	4.865	4.989406	4.263
	3.0	4.196335319	4.865	4.989406	4.263
1P	1.0	3.23224764	3.214	-	3.525
	2.0	3.399250489	3.214	-	3.525
	3.0	3.511700284	3.214	-	3.525
2P	1.0	3.568719439	3.773	3.756506	3.773
	2.0	.763087336	3.773	3.756506	3.773
	3.0	3.771820015	3.773	3.756506	3.773
1D	1.0	3.454603197	3.412	-	3.770
	2.0	3.624976340	3.412	-	3.770
	3.0	3.892528368	3.412	-	3.770
2D	1.0	3.802639037	-	-	4.159
	2.0	3.867334734	-	-	4.159
	3.0	3.987992543	-	-	4.159

For bottomonium, the mass spectra for 1S and 2S states are seen to be equal to ED. But when the TD was introduced the mass spectra increases and tends to approach the ED and was improved in comparison to the report by [15,37] as shown in Table 2. In the case of bottom-charm $(b\bar{c})$,when TD was absence, the values of the calculated masses for 1S and 2S agreed with the ED as well as the when the TD was set to 2 and 3.

We noticed that 3S, 1P and 2P do not have ED for comparison. The predicted values for 3S,1P and 2P increase as the TD is increased and was seen to be improved from the works reported by [41,43,44] as shown in Table 3. The prediction of the mass spectra of charm-strange $(c\bar{s})$ meson shows that when TD = 1,1S and 2S states agreed with ED. Also, when TD = 2 = 3,1S and 2S also agreed with the ED. We noticed that as the TD increases the values of the calculated values increases and approaches the ED and was seen to be improved from the works reported by [41,42] as shown in Table 4.

The mass spectra of heavy and heavy-light mesons are plotted against the principal quantum number (n) as shown in Figs.1-4. In Fig.1, the MS of charmonium is plotted against n, it is observed that as TD increases from 1 to 3, the MS increases. In Fig.2, the MS of bottomonium is plotted against n.

Table 2: Mass spectra of bottomonium in (GeV)

$(m_b = 4.823\ GeV, \mu = 2.4115\ GeV,\ a = -1.6952\ GeV,\ b = 167.7012\ GeV, \theta = 0.01, \delta = 1.65\ GeV, \hbar = 1, \alpha = 1)$

$(m_b = 4.823\ GeV, \mu = 2.4115\ GeV,\ a = -1.73281\ GeV,\ b = 169.61855\ GeV, \theta = 0.01, \delta = 1.65\ GeV, \hbar = 1, \alpha = 2)$

$(m_b = 4.823\ GeV, \mu = 2.4115\ GeV,\ a = -1.7696\ GeV,\ b = 171.4677\ GeV, \theta = 0.01, \delta = 1.65\ GeV, \hbar = 1, \alpha = 3)$

State	α	Our Result	AIM[37]	SEM[15]	Experiment[39]
1S	1.0	9.460000022	9.460	9.515194	9.460
	2.0	9.459999989	9.460	9.515194	9.460
	3.0	9.460000048	9.460	9.515194	9.460
2S	1.0	10.02300001	10.023	10.01801	10.023
	2.0	10.02299999	10.023	10.01801	10.023
	3.0	10.02300003	10.023	10.01801	10.023
3S	1.0	10.30289537	10.585	10.44142	10.355
	2.0	10.31001127	10.585	10.44142	10.355
	3.0	10.34493277	10.585	10.44142	10.355
4S	1.0	10.38099212	11.148	10.85777	10.580
	2.0	10.47097214	11.148	10.85777	10.580
	3.0	10.56709741	11.148	10.85777	10.580
1P	1.0	9.626090124	9.492	-	9.899
	2.0	9.799443695	9.492	-	9.899
	3.0	10.02340701	9.492	-	9.899
2P	1.0	10.22006701	10.038	10.09446	10.260
	2.0	10.24115658	10.038	10.09446	10.260
	3.0	10.25907436	10.038	10.09446	10.260
1D	1.0	9.802224210	9.551	-	10.164
	2.0	9.964826266	9.551	-	10.164
	3.0	10.22010124	9.551	-	10.164

Table 3: Mass spectra of $b\bar{c}$ meson in (GeV)

$(m_c = 1.209\ GeV, m_b = 4.823\ GeV, \mu = 0.967\ GeV,\ a = -2.7002\ GeV,\ b = 267.0454\ GeV, \theta = 0.01, \delta = 1.6\ GeV, \hbar = 1, \alpha = 1.0)$

$(m_c = 1.209\ GeV, m_b = 4.823\ GeV, \mu = 0.967\ GeV,\ a = -2.7604\ GeV,\ b = 270.0844\ GeV, \theta = 0.01, \delta = 1.6\ GeV, \hbar = 1, \alpha = 2.0)$

$(m_c = 1.209\ GeV, m_b = 4.823\ GeV, \mu = 0.967\ GeV,\ a = -2.8202\ GeV,\ b = 273.0768\ GeV, \theta = 0.01, \delta = 1.6\ GeV, \hbar = 1, \alpha = 3.0)$

State	α	Our Result	[43]	[44]	[41]	Experiment [40]
1S	1.0	6.273999877	6.349	6.264	6.268	6.274
	2.0	6.273999987	6.349	6.264	6.268	6.274
	3.0	6.273999990	6.349	6.264	6.268	6.274
2S	1.0	6.870999945	6.821	6.856	6.895	6.871
	2.0	6.870999994	6.821	6.856	6.895	6.871
	3.0	6.870999996	6.821	6.856	6.895	6.871
3S	1.0	7.079844893	7.175	7.244	7.522	-
	2.0	7.079923693	7.175	7.244	7.522	-
	3.0	7.079938283	7.175	7.244	7.522	-
1P	1.0	6.450099644	6.715	6.700	6.529	-
	2.0	6.580866397	6.715	6.700	6.529	-

	3.0	6.871276565	6.715	6.700	6.529	-
2P	1.0	6.925459986	7.156	7.108	7.102	-
	2.0	6.969242914	7.156	7.108	7.102	-
	3.0	7.079961566	7.156	7.108	7.102	-

Table 4: Mass spectra of $c\bar{s}$ meson in (GeV)
$(m_c = 1.209\,GeV, m_s = 0.419\,GeV, \mu = 0.3111\,GeV, \theta = 0.01, a = -2.0227\,GeV, b = 198.2519\,GeV, \delta = 1.6\,GeV, \hbar = 1, \alpha = 1.0)$
$(m_c = 1.209\,GeV, m_s = 0.419\,GeV, \mu = 0.3111\,GeV, \theta = 0.01, a = -2.1031\,GeV, b = 202.2735\,GeV, \delta = 1.6\,GeV, \hbar = 1, \alpha = 2.0)$
$(m_c = 1.209\,GeV, m_s = 0.419\,GeV, \mu = 0.3111\,GeV, \theta = 0.01, a = -2.0227\,GeV, b = 206.3001\,GeV, \delta = 1.6\,GeV, \hbar = 1, \alpha = 3.0)$

State	α	Our Result	[41]	AIM[42]	Exp.[40]
1S	1.0	1.967999992	1.969	2.512	1.968
	2.0	1.968000003	1.969	2.512	1.968
	3.0	1.968000009	1.969	2.512	1.968
2S	1.0	2.316999996	2.318	2.709	2.317
	2.0	2.317000002	2.318	2.709	2.317
	3.0	2.317000004	2.318	2.709	2.317
3S	1.0	2.569135452	2.667	2.906	2.700
	2.0	2.655146351	2.667	2.906	2.700
	3.0	2.709948372	2.667	2.906	2.700
1P	1.0	2.070929595	2.126	2.649	2.112
	2.0	2.147345620	2.126	2.649	2.112
	3.0	2.157038303	2.126	2.649	2.112
1D	1.0	2.180107355	2.374	2.859	2.318
	2.0	2.28087479	2.374	2.859	2.318
	3.0	2.319154842	2.374	2.859	2.318

A similar trend is observed. The variation of MS of charm-strange $(c\bar{s})$ against n is plotted as shown in Fig.3. It is observed that the mass spectra, increases as TD is increased In Fig.4, the MS of bottom-charm $(b\bar{c})$, is plotted against the principal quantum number. The MS is seen to increase when the values of TD is increased, which shows the effect of TD on the mass spectra.

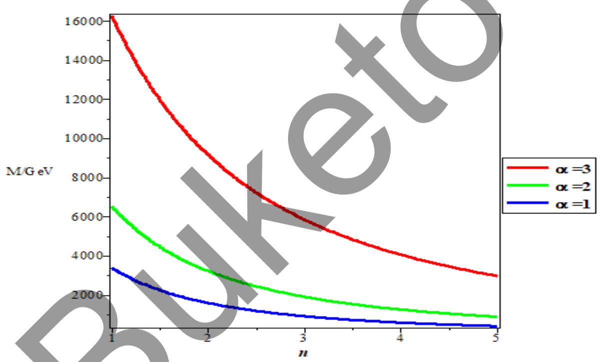

Fig.1. Variation of the mass spectra of charmonium with principal quantum number for different values of α

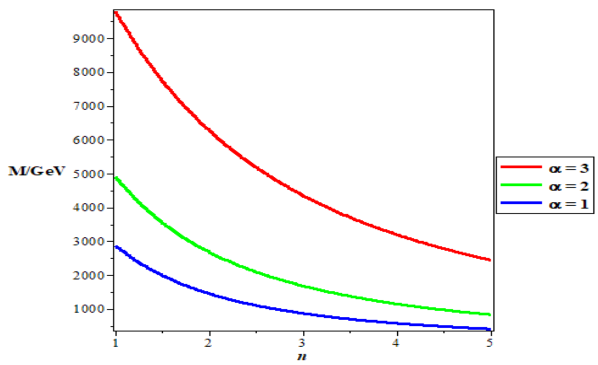

Fig.2. Variation of the mass spectra of bottomonium with principal quantum number for different values of α

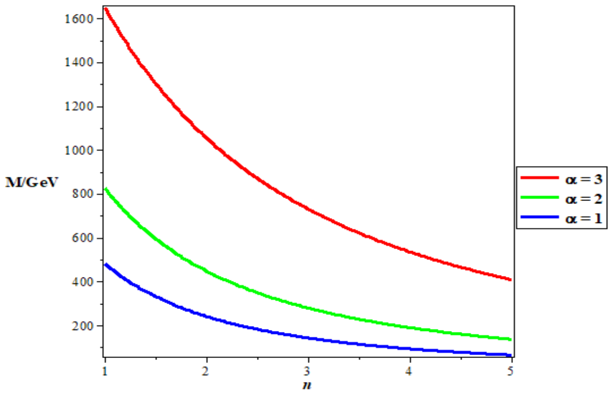

Fig.3. Variation of the mass spectra of $c\bar{s}$ meson with principal quantum number for different values of α

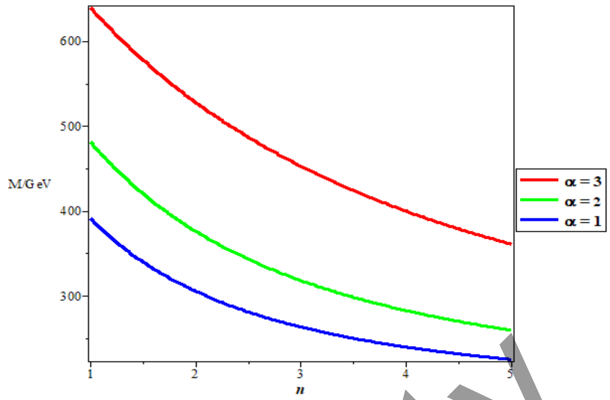

Fig.4. Variation of the mass spectra of $b\bar{c}$ meson with principal quantum number for different values of α

Conclusion

In the current study, the Hulthen-Yukawa Potential is used to examine the effect of Topological Defect on the mass spectra of heavy and heavy-light mesons such as charmonium, bottomonium, charm-strange, and bottom-charm respectively. Analytically, the SE was solved using the NU approach. The un-normalized wave function and the energy spectrum were found. The mass spectra of heavy and heavy-light mesons in the presence and absence of TD for various quantum states were predicted. We observed that the mass spectra rise and shift closer to the ED as the TD increases. However, when compared to the work of other researchers, the results obtained demonstrated an improvement. This research could be expanded to examine the thermal properties of heavy and heavy-light mesons.

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