https://doi.org/10.31489/2025M3/184-199 *Research article*

The representation theorem of the Robinson hybrid

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This research lies within the domain of model theory, which investigates the properties of, broadly speaking, incomplete theories. The article introduces novel methods for classifying classes of structures whose associated theories are Jonssonian, forming a distinct subclass within the broader category of inductive theories. This subclass is characterized by satisfying the standard model-theoretic properties of joint embedding and amalgamation. The focus is placed specifically on the second kind of hybrids, those involving theories with different signatures. As a representative case of such hybrids among Jonsson theories, we examine the classical examples of the theory of unars and the theory of undirected graphs. The study proposes and formalizes several new notions, including the perfect Robinson hybrid, the center of a Robinson hybrid, the Kaiser class of a theory, and the concept of triple factorization. Within the framework of these definitions, we establish new results, among them a theorem confirming the existence of a unique countably categorical theory of S-acts, which is syntactically equivalent to the Robinson hybrid formed by the aforementioned classes.

Keywords: Jonsson theory, Robinson theory, hybrid, perfect Robinson hybrid, similarity, K_T-equivalence, ω-categorical, cosemanticness relation, S-act, triple factorization.

2020 Mathematics Subject Classification: 03C35, 03C48, 03C52, 03C65.

Introduction

This work is part of the field of model theory, which examines the model-theoretic properties of, more generally, incomplete theories. It is widely recognized that modern model theory is a fast-evolving branch of mathematics with numerous significant topics. However, this framework is mainly developed for and tailored to the analysis of complete theories. The domain of incomplete theories is extensive, and within it, one can identify the subclass of inductive theories. This classification can be supported by at least the following reasoning. Specifically, a theory is considered inductive if every increasing chain of models remains a model of the theory itself. In other words, a theory is inductive when it is closed under chains of its models. On the other hand, it is a well-known result that such theories can be axiomatized by universal-existential sentences. It can also be observed that the main classical examples from algebra correspond to inductive theories. The most characteristic example of an inductive theory is group theory. Notably, this is also an example of an incomplete theory.

Within inductive theories, one can distinguish the well-studied subclass of Jonsson theories. For an introduction to this subclass, the reader may refer to the following literature: [1–3].

Among Jonsson theories, perfect Jonsson theories hold a particularly significant position. The study of this subclass has been the subject of several works, including [4–6].

The investigation of Jonsson theories is also valuable in the context of contemporary applications in information technology. This is not coincidental, as Jonsson theories, due to their general incompleteness, admit finite models. The identification and analysis of the relationship between infinite and finite

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This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP22686796 “Hybrids of fragments and their small models”).

Received: 16 January 2025; *Accepted:* 4 June 2025.

models of Jonsson theories generates particular interest in this topic. This is because, unlike complete theories, which do not consider finite models, Jonsson theories examine the interplay of many classical concepts associated with complete theories within the framework of finite models. In particular, works such as [4, 7, 8] study such properties as categoricity, stability, various companions, axiomatizability, model completeness, atomic and prime models.

This paper explores two well-known examples of theories: the theory of all unars and the theory of undirected graphs. The study of elementary theories related to the structure of these signatures is widely recognized in the work of many researchers. These works contain many classical results describing various first-order properties related to the complete theories of these structures. Jonsson theories corresponding to these examples were studied in [4, 9, 10]. In the present work, we investigate hybrids of Jonsson theories, where the theories forming the hybrid are the theory of unars and the theory of undirected graphs. It should be noted that studies related to hybrids of Jonsson theories have been considered in [11, 12].

A notable development in the study of both Jonsson theories and inductive theories in general is the exploration of a distinguished subclass of models, referred to in this work as the Kaiser class. This class represents a natural extension of the class of existentially closed models associated with any inductive theory. Since it is well established that inductive theories possess a nonempty set of existentially closed models, the investigation of the Kaiser class introduces a novel and significant problem within the realms of classical model theory and universal algebra. When we refer to classical model theory, we mean problems related not only to incomplete theories but also to complete theories. Thus, in our view, the range of questions considered in this article is of particular interest in relation to topics that arise in classical model theory concerning the concept of hybrid of Jonsson theories.

1 Essential concepts of Jonsson's model theory

This section provides the foundational groundwork necessary for the further development of results concerning Jonsson theories and the corresponding classes of their models. The notions discussed here form the conceptual core of the model-theoretic framework within which the subsequent results are formulated and proved.

Jonsson model theory provides a natural semantic setting for analyzing algebraic structures such as unars and undirected graphs, which are known to satisfy the defining conditions of this class of theories. In particular, key properties such as universality and homogeneity serve as central invariants that characterize the semantic behavior of Jonsson theories and are tightly connected to the concept of saturation in models.

The notion of saturation, especially within universally homogeneous models, leads to the identification of a distinguished subclass of Jonsson theories, known as perfect Jonsson theories. These theories are of particular interest due to their stable semantic properties and the behavior of their existentially closed models.

An important feature of this subclass is that perfection is preserved under passage to the center of the theory. That is, if a Jonsson theory is perfect, then its center retains this property as well. This relationship reflects a deep structural symmetry within the semantic layers of Jonsson frameworks.

This section will focus primarily on universal Jonsson theories that describe two major classes of structures: unars with a single unary function symbol and undirected graphs formulated in a signature with one binary relation. To this end, the definition of universality is recalled, together with a formal introduction of the notion of κ-categoricity, which plays a central role in the classification of models in this context.

In what follows, we introduce the definitions and principal results required for the study of existentially closed models and the analysis of perfectness and categoricity within the Jonsson framework.

These notions play a crucial role in the formulation and proof of the main theorems presented in this paper.

Let's outline the key concepts and statements of model-theoretical constructs essential for understanding and working within the framework of Jonsson theories and their associated classes of models.

It has been established that many classical algebraic structures, such as unars and graphs, satisfy the conditions of Jonsson theories [4].

The notions of universality and homogeneity in a model emphasize the semantic invariant characteristic of any Jonsson theory, that is, its semantic model. Moreover, it has been demonstrated that whether this model is saturated or not has a profound impact on the structural features of both the Jonsson theory itself and its corresponding class of models.

The saturation of universally homogeneous models, in the sense defined by Jonsson, leads to the identification of a distinguished subclass of Jonsson theories, whose elements are termed perfect Jonsson theories.

It can be observed that if a Jonsson theory T is perfect, then its center T^*, i.e., the elementary theory of its semantic model $\mathfrak{C}_T$, is also a perfect Jonsson theory [4].

A characterization of perfect Jonsson theories was formulated in [4].

As the focus will be on universal Jonsson theories of all unars of the signature with one unary functional symbol and the theory of undirected graphs in a signature with one binary relation symbol, it is useful to recall the definition of universality. A theory T is called *universal* if it is equivalent to a set of universal sentences [1].

In order to establish the main results of this paper, it is necessary to introduce the framework of κ-categorical Jonsson theories, along with a characterization of existentially closed models within the theory T.

Definition 1. [4] A Jonsson theory T is said to be *κ-categorical* for some cardinal $\kappa \geq \omega$ if any two models of T with cardinality κ are isomorphic.

The following result, originally proven in [4], establishes the equivalence of ω-categoricity for a Jonsson theory and its center, provided that the theory is complete to $\forall\exists$-sentences.

Theorem 1. [4] Let T be $\forall\exists$-complete Jonsson theory. Then the following statements are equivalent:

1) T is ω-categorical.
2) The center T^* of T is ω-categorical.

The following theorem plays a central role in establishing one of the main results of this article. It provides a sufficient condition for a Jonsson theory to be perfect.

Theorem 2. [4] If a Jonsson theory T is ω-categorical, then T is perfect.

Definition 2. [1] A model A of theory T is said to be an *existentially closed model* of T, if for any extension $B \models T$ with $A \subseteq B$, and for any existential formula $\exists x\varphi(x, \bar{y})$, if $B \models \exists x\varphi(x, \bar{a})$ for some tuple $\bar{a} \in A$, then $A \models \exists x\varphi(x, \bar{a})$.

The class E_T, consisting of all existentially closed models of a Jonsson theory T, is guaranteed to be non-empty, due to the inductiveness of T. Clearly, $E_T \subseteq Mod(T)$, so E_T forms a natural subclass of the class of models of T.

Proposition 1. [1] Let T be an inductive theory. Then T has a model companion T' if and only if the class E_T of existentially closed models of T is elementary.

This criterion provides a useful tool for verifying the existence of model companions in the context of inductive theories.

In particular, if a Jonsson theory is perfect, then the class of its existentially closed models is known to be elementary.

The relationship between the two universal Jonsson theories, in terms of their centers and corresponding semantic models, is captured by the following proposition:

Proposition 2. [4] Let T_1 and T_2 be universal Jonsson theories. Then the following conditions are equivalent:

1) The theories T_1 and T_2 are equal; that is, they consist of exactly the same set of first-order sentences.
2) The semantic models $\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ of the Jonsson theories T_1 and T_2, respectively, are isomorphic.
3) The centers of the theories, denoted T_1^* and T_2^*, are equal; that is, the elementary theories of their corresponding semantic models coincide.

2 Exploring the Robinson Spectrum in the Context of Jonsson Theories

The study of model-theoretic spectra associated with classes of first-order structures offers a rich framework for understanding the logical and semantic properties of these classes. Among such spectra, the Jonsson spectrum and its special case, the Robinson spectrum, serve as key tools in analyzing how certain theories interact with structural features of models.

Let L be a first-order language with signature σ, and let K denote a class of L-structures. In this context, we are interested in the collection of all Jonsson theories whose models include all elements of K. This leads naturally to the notion of the Jonsson spectrum of the class K, which captures the diversity of Jonsson axiomatizations that are valid across all structures in K.

Particularly notable is the subclass of Jonsson theories axiomatizable purely by universal sentences; these correspond precisely to the classical Robinson theories. Accordingly, the Robinson spectrum of K can be seen as a refined instance of the broader Jonsson spectrum, restricted to theories of a specific syntactic form. This interrelation allows for a layered approach: by first investigating the more general Jonsson setting, one can then derive meaningful insights into Robinson spectra and their applications.

An essential component in the structural analysis of these spectra is the concept of cosemanticness, which relates theories via their shared semantic core, or center. This equivalence relation partitions spectra into classes of semantically indistinguishable (though potentially syntactically distinct) theories, offering a deeper lens into the interplay between logic and model theory.

The present section introduces and develops the formal machinery underlying both Jonsson and Robinson spectra. We examine how these constructs are defined, how they behave under equivalence by cosemanticness, and how they manifest in concrete algebraic settings such as unars and undirected graphs. Through this analysis, we highlight fundamental differences between the two spectra, particularly in terms of the uniqueness of theories within equivalence classes, and trace the implications for broader concepts such as existential closure and categoricity.

This discussion culminates in a generalization of classical quasivarieties to what we term semantic Jonsson quasivarieties, which serve as a natural setting for interpreting Robinson spectra. These semantic structures, grounded in model-theoretic extensions of elementary theories, provide a fertile ground for exploring categorical properties and model completeness in enriched logical frameworks.

Let L be a first-order language with a signature σ, and let K be a class of L-structures. We consider a particular set of theories associated with K, known as the Jonsson spectrum of the class K. This concept is formally defined as follows:

Definition 3. [4] The *Jonsson spectrum* of the class K, denoted by $JSp(K)$, is the set of all Jonsson theories with signature σ such that every structure in K is a model of the theory. Formally,

$$JSp(K) = \{T \mid T \text{ is a Jonsson theory and } \forall \mathfrak{A} \in K,\ \mathfrak{A} \models T\}.$$

A detailed treatment of the structure and characteristics of Jonsson spectra can be found in [4].

In the special case where a Jonsson theory is axiomatized solely by universal sentences, one recovers the classical notion of a Robinson theory. Thus, the Jonsson spectrum framework naturally extends to encompass the Robinson spectrum as a specific instance, providing a natural generalization of this concept.

Definition 4. The *Robinson spectrum* of the class K, denoted $RSp(K)$, consists of all Robinson theories with signature σ that are satisfied by every structure in K. Formally,

$$RSp(K) = \{T \mid T \text{ is a Robinson theory and } \forall \mathfrak{A} \in K, \mathfrak{A} \models T\}.$$

Within the framework of Jonsson theories, the notion of the cosemanticness relation plays a central role. Let T_1 and T_2 be Jonsson theories with centers T_1^* and T_2^*, respectively.

The following concept was originally formulated by Professor T.G. Mustafin:

Definition 5. [4] Two Jonsson theories T_1 and T_2 are said to be *cosemantic* (denoted $T_1 \bowtie T_2$) if their centers coincide, i.e., $T_1^* = T_2^*$.

It was established in [4] that this cosemanticness relation defines an equivalence relation on the class of Jonsson theories. Consequently, when this relation is applied to the Jonsson spectrum $JSp(K)$, the set is naturally partitioned into equivalence classes, referred to as cosemantic classes. The corresponding quotient set is denoted by $JSp(K)/_{\bowtie}$. This quotient set provides a useful framework for extending classical results and formulating broader generalizations within the theory. In an analogous manner, the quotient set $RSp(K)/_{\bowtie}$ can be introduced for the Robinson spectrum.

An essential result in the context of Robinsonian theories and the Robinson spectrum is the following proposition:

Proposition 3. [13] Let K be an arbitrary class of L-structures (possibly consisting of a single structure), and let $RSp(K)/_{\bowtie}$ be the quotient set of the Robinson spectrum of K with respect to cosemanticness. Then every cosemanticness class $[\Delta]$ contains exactly one theory. In other words, for any two Robinsonian L-theories T and T', the relation of cosemanticness is equivalent to the equality (logical equivalence) of theories; that is, $T \bowtie T' \Leftrightarrow T = T'$.

In the Robinson spectrum, when factorized by cosemanticness, each cosemanticness class is a singleton.

This proposition highlights a fundamental distinction between the Jonsson and Robinson spectra under the cosemanticness relation. In the case of the Robinson spectrum $RSp(K)$, factorization by cosemanticness yields a discrete partition: each equivalence class contains exactly one theory. This reflects the fact that for Robinsonian theories, semantic identity is equivalent to syntactic identity.

By contrast, for the Jonsson spectrum $JSp(K)$, the situation is more intricate. The equivalence relation of cosemanticness does not, in general, reduce to syntactic equality. That is, distinct Jonsson theories can share the same center and thus belong to the same cosemanticness class. Consequently, the quotient set $JSp(K)/_{\bowtie}$ may contain nontrivial equivalence classes, each consisting of multiple syntactically distinct yet semantically related theories.

This structural divergence between the two spectra is crucial for understanding the role of centers in classification problems and reflects deeper differences in the expressiveness and axiomatizability of Robinson versus Jonsson theories.

We now proceed to the formulation of the concept known as a semantic Jonsson quasivariety.

Let K be a class of quasivarieties of the first-order language L, as defined in [14], and let $L_0 \subset L$, where L_0 is the set of sentences of language L. Consider the elementary theory $Th(K)$ of this class K. By adding $\forall\exists$-sentences of language L, denoted by $\forall\exists(L_0)$, which are not contained in $Th(K)$, we can define the set of Jonsson theories $J(Th(K))$ as follows.

Denotation 1. $J(Th(K)) = \{\Delta \mid \Delta = Th(K) \cup \{\varphi^i\}\}$, where Δ is a Jonsson theory, φ^i denotes either a formula from $\forall\exists(L_0)$ or its negation, $i \in \{0, 1\}$, and $Th(K)$ is the elementary theory of the class of quasivarieties K.

Every theory $\Delta \in J(Th(K))$ is associated with a semantic model, denoted $\mathfrak{C}_\Delta$. We now define the set of all such models:

Denotation 2. $J\mathbb{C} = \{\mathfrak{C}_\Delta \mid \Delta \in J(Th(K)),\ \mathfrak{C}_\Delta \text{ is a semantic model of } \Delta\}$.

The set $J\mathbb{C}$ is referred to as a *semantic Jonsson quasivariety* associated with the class K, provided that its elementary theory $Th(J\mathbb{C})$ itself forms a Jonsson theory.

This construction generalizes the traditional notion of a quasivariety by integrating semantic properties tied specifically to Jonsson type extensions. Unlike standard quasivarieties, which are defined purely syntactically (e.g., by quasi-identities or Horn sentences), a semantic Jonsson quasivariety is formed by considering model-theoretic extensions of a given elementary theory $Th(K)$ via additional $\forall\exists$-sentences. These extensions do not necessarily follow from $Th(K)$ and may vary across different Jonsson theories $\Delta \in J(Th(K))$.

This concept differs substantially from the notion of a classical quasivariety. It is well known that if a quasivariety is countably categorical, then it is also uncountably categorical. However, this does not hold for a semantic Jonsson quasivariety. A counterexample is given by the theory of the semantic Jonsson quasivariety of abelian groups.

The Robinson spectra associated with universal unars and undirected graphs have been investigated within the framework of semantic Jonsson quasivarieties.

Let us consider an unar structure $\mathfrak{U}$, which is a model over the signature $\sigma_{\mathfrak{U}} = \langle f \rangle$, where f is a unary functional symbol. Define the sequence of iterated applications of f recursively as follows: $f_0(x) = x$, $f_{n+1}(x) = f(f_n(x))$, $n \in \omega$. Given elements $a, b \in U$ are called *U-connected* in X if there exist natural numbers m and n such that $f_m(a) = f_n(b)$ and $f_0(a) = f_m(a),\ f_0(b), \ldots, f_n(b) \in X$.

A subset $X \subseteq U$ is said to be *U-connected* if every pair of elements from X is U-connected. A subsystem $B \subseteq U$ whose universe forms is the maximal U-connected subset of carrier U is referred to as a *component* in the structure $\mathfrak{U}$. Furthermore, if B is a component, then the set $\{a \in B : \exists n \in \omega \text{ such that } \mathfrak{U} \models f_n(a) = a\}$ is called a *cycle* of the component.

Now consider a graph structure $\mathfrak{G}$, which is modeled as an algebraic system with signature $\sigma_{\mathfrak{G}} = \langle R \rangle$, where R is a binary symmetric relation. In this setting, elements of the universe are referred to as *vertices*, and a pair $\langle x, y \rangle$ forms an *edge* if $R(x, y)$ holds. A graph in which the relation R is empty, that is, contains no edges, is called a *totally disconnected graph*.

Based on the foundational results established in [4], it follows that the universal parts of the elementary theories of these structures denoted $Th_\forall(\mathfrak{U})$ and $Th_\forall(\mathfrak{G})$ for unars and undirected graphs, respectively, constitute their corresponding Robinson theories. Hence, these theories provide canonical examples of Robinson spectra for algebraic systems within the domain of semantic Jonsson quasivarieties.

Thus, we define the set

$$J\mathbb{C}_{\mathfrak{U}} = \{\mathfrak{C}_{\Delta_{\mathfrak{U}}} \mid \Delta_{\mathfrak{U}} \in J(Th(K_{\mathfrak{U}})),\ \mathfrak{C}_{\Delta_{\mathfrak{U}}} \models \Delta_{\mathfrak{U}}\}\ ,$$

where the signature $\sigma_{\mathfrak{U}} = \langle f \rangle$, and f is unary functional symbol. Here $\Delta_{\mathfrak{U}}$ denotes a Robinson theory of unars. The set $J\mathbb{C}_{\mathfrak{U}}$ is referred to as the *semantic Jonsson quasivariety of Robinson unars*, as introduced in [4].

Following [4], we define the Robinson spectrum of the set $J\mathbb{C}_{\mathfrak{U}}$ as follows:

Definition 6. Let $RSp(J\mathbb{C}_{\mathfrak{U}})$ denote the set of all Robinson theories $\Delta_{\mathfrak{U}}$ in the signature $\sigma_{\mathfrak{U}}$ such that every model $\mathfrak{C}_{\Delta_{\mathfrak{U}}} \in J\mathbb{C}_{\mathfrak{U}}$ satisfies the theory $\Delta_{\mathfrak{U}}$. That is,

$$RSp(J\mathbb{C}_{\mathfrak{U}}) = \{\Delta_{\mathfrak{U}} \mid \Delta_{\mathfrak{U}} \text{ is a Robinson theory of unars, and } \forall \mathfrak{C}_{\Delta_{\mathfrak{U}}} \in J\mathbb{C}_{\mathfrak{U}},\ \mathfrak{C}_{\Delta_{\mathfrak{U}}} \models \Delta_{\mathfrak{U}}\}.$$

This set is called the *Robinson spectrum* of the semantic Jonsson quasivariety $J\mathbb{C}_{\mathfrak{U}}$.

The quotient set of this spectrum is denoted by $RSp(J\mathbb{C}_{\mathfrak{U}})_{/\bowtie}$, which consists of equivalence classes $[\Delta_{\mathfrak{U}}]$ determined by the cosemanticness relation (that is, theories that share the same center).

Similarly, we can define a corresponding structure for undirected graphs. Consider the set

$$J\mathbb{C}_{\mathfrak{G}} = \{\mathfrak{C}_{\Delta_{\mathfrak{G}}} \mid \Delta_{\mathfrak{G}} \in J(Th(K_{\mathfrak{G}})),\ \mathfrak{C}_{\Delta_{\mathfrak{G}}} \models \Delta_{\mathfrak{G}}\},$$

where $\Delta_{\mathfrak{G}}$ is a Robinson theory formulated over the signature $\langle R\rangle$ of undirected graphs, R is a binary symmetric relation, i.e., the standard signature of undirected graphs. The set $J\mathbb{C}_{\mathfrak{G}}$ is thus interpreted as the *semantic Jonsson quasivariety of Robinson undirected graphs.*

Definition 7. Let $\sigma_{\mathfrak{G}}$ be the signature $\langle R\rangle$, where R is a binary symmetric relation. The set of all Robinson theories $\Delta_{\mathfrak{G}}$ such that every semantic model $\mathfrak{C}_{\Delta_{\mathfrak{G}}} \in J\mathbb{C}_{\mathfrak{G}}$ satisfies $\Delta_{\mathfrak{G}}$, that is,

$$RSp(J\mathbb{C}_{\mathfrak{G}}) = \{\Delta_{\mathfrak{G}} \mid \Delta_{\mathfrak{G}} \text{ is a Robinson theory of undirected graphs, and } \forall \mathfrak{C}_{\Delta_{\mathfrak{G}}} \in J\mathbb{C}_{\mathfrak{G}},\ \mathfrak{C}_{\Delta_{\mathfrak{G}}} \models \Delta_{\mathfrak{G}}\},$$

is called the *Robinson spectrum* of the semantic Jonsson quasivariety $J\mathbb{C}_{\mathfrak{G}}$ of Robinson undirected graphs.

As in previous constructions, one can define the corresponding *cosemantic quotient set*, denoted by $RSp(J\mathbb{C}_{\mathfrak{G}})/_{\bowtie}$, which consists of equivalence classes $[\Delta_{\mathfrak{G}}]$ under the cosemanticness relation, that is, theories whose centers coincide.

In the ω-categorical setting, a model-theoretic characterization of existentially closed models has been established for both unars and undirected graphs. The corresponding results are presented in the following theorems.

Theorem 3. Let $[\Delta_{\mathfrak{U}}]$ be a class of ω-categorical Robinson theories of unars. Then the following statements are equivalent:

1) $\mathfrak{A} \in E_{[\Delta_{\mathfrak{U}}]}$; that is, $\mathfrak{A}$ is an existentially closed model of the class $[\Delta_{\mathfrak{U}}]$;
2) $\mathfrak{A}$ is a disjoint union of components, each of which contains a cycle of the same length.

Theorem 4. Let $[\Delta_{\mathfrak{G}}]$ be a class of ω-categorical Robinson theories of undirected graphs, and let $E_{[\Delta_{\mathfrak{G}}]}$ denote the class of existentially closed models for this class. Then the following are equivalent:

1) $\mathfrak{B} \in E_{[\Delta_{\mathfrak{G}}]}$, i.e., $\mathfrak{B}$ is an existentially closed model of $[\Delta_{\mathfrak{G}}]$;
2) $\mathfrak{B}$ is an infinite totally disconnected graph.

Here, $E_{[\Delta_{\mathfrak{U}}]}$ and $E_{[\Delta_{\mathfrak{G}}]}$ denote the sets of existentially closed models corresponding to the cosemantic classes $[\Delta_{\mathfrak{U}}]$ and $[\Delta_{\mathfrak{G}}]$, respectively.

3 Jonsson theories similarity

The concept of similarity between first-order theories plays a central role in modern model theory, particularly in the classification and comparison of theories with respect to both syntactic and semantic characteristics. In this section, we focus on a specific class of theories – namely, Jonsson theories – and explore various notions of similarity that arise within this framework.

Our exposition begins with a foundation in generalized Jonsson theories, also known as α-Jonsson theories, which extend the classical definition by parameterizing inductiveness, amalgamation, and joint embedding properties via an ordinal index α. These properties ensure that models of the theory behave coherently when considered in chains, embeddings, or pushouts, and are crucial in establishing a robust structural framework for such theories.

To deepen the analysis of similarity, this section introduces two primary dimensions of comparison: syntactic similarity, based on mappings between formula algebras or existential lattices, and semantic similarity, defined via isomorphisms between so-called pure triples associated with models or semantic universes. These notions were initially developed for complete theories in the foundational work of

Professor T.G. Mustafin [15] and subsequently generalized to the Jonsson context by Professor A.R. Yeshkeyev.

The treatment of similarity culminates in precise criteria — such as bijective correspondences between existential lattices or structural isomorphism of model-theoretic automorphism groups — that allow us to relate two theories at a deep logical and algebraic level. Furthermore, the section highlights the critical insight that syntactic similarity always implies semantic similarity, whereas the converse does not necessarily hold.

The theoretical apparatus is complemented by illustrative examples and algebraic constructions, including S-acts (algebraic systems over a monoid), which serve as canonical models used to construct envelopes of arbitrary theories. These models provide a concrete setting for understanding how one theory can simulate or encapsulate the expressive power of another through inessential extensions.

Finally, this section culminates in the formalization of similarity at the level of Jonsson spectrum classes, offering an even broader perspective on how entire families of theories can be compared via their syntactic and semantic cores. The results obtained herein lay the groundwork for the subsequent sections, where the equivalence of centers, perfectness, and existential completeness play a decisive role in characterizing such similarities.

The following examples illustrate key concepts related to Γ-embeddings, Γ-chains, and model-theoretic properties of theories such as α-inductiveness, the α-joint embedding property (α-JEP), and the α-amalgamation property (α-AP). They help clarify how formulas from a given set Γ are preserved under various model-theoretic constructions and how theories behave with respect to chains and embeddings of varying levels of complexity [4].

Example 1 (On Γ-embeddings). Let Γ be the set of all quantifier-free formulas in the language $L = \{<\}$, and consider two structures $\mathcal{A} = (\mathbb{N}, <)$ and $\mathcal{B} = (\mathbb{Z}, <)$. Let $f\colon \mathbb{N} \to \mathbb{Z}$ be the inclusion map defined by $f(n) = n$. Since the order $<$ on $\mathbb{N}$ is preserved in $\mathbb{Z}$, and all quantifier-free formulas true in $\mathcal{A}$ remain true under f in $\mathcal{B}$, the map f is a Γ-embedding.

Example 2 (On Γ-chains). Consider a sequence of structures $\mathcal{A}_i = (\mathbb{Q}_i, <)$, where $\mathbb{Q}_i$ denotes the set of rational numbers with denominators at most 2^i. Then for each $i < j$, the inclusion $\mathcal{A}_i \subseteq_\Gamma \mathcal{A}_j$ holds with respect to $\Gamma = \{<\}$, since the order is preserved and extended. The sequence $\{\mathcal{A}_i\}_{i<\omega}$ thus forms a Γ-chain.

Example 3 (On α-inductiveness). Let T be the theory of linear orders. Consider a chain of countable models $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots$, where each $\mathcal{A}_i$ is a copy of $(\mathbb{N}, <)$ extended by adding isolated elements. The union of this Π_1-chain is again a model of T; hence, T is 1-inductive.

Example 4 (On α-joint embedding property). Let T be the theory of undirected graphs without additional properties. Any two graphs $\mathcal{A}$ and $\mathcal{B}$ can be jointly embedded into their disjoint union $\mathcal{M} = \mathcal{A} \sqcup \mathcal{B}$. The natural inclusion maps are Π_0-embeddings; thus, T satisfies 0-JEP.

Example 5 (On α-amalgamation property). Let T be the theory of vector spaces over a fixed field. Given three vector spaces $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2$ and linear embeddings $f_1\colon \mathcal{A} \to \mathcal{B}_1$, $f_2\colon \mathcal{A} \to \mathcal{B}_2$, the pushout (amalgam) exists and is also a vector space. Therefore, T satisfies 0-AP.

The concept of generalized Jonsson theories, also referred to as α-Jonsson theories, extends the classical notion of Jonsson theories by incorporating ordinal-indexed structural conditions. The following definition is based on the formulation presented in [2].

Consider the following definition, which introduces the notion of an α-Jonsson theory — a type of first-order theory characterized by specific model-theoretic properties.

Definition 8. [2] A theory T is called *α-Jonsson* (for ordinals $0 \leq \alpha \leq \omega$) if it has an infinite model and satisfies three key structural properties: closure under unions of Π_α chains (that is, α-inductiveness); the ability to jointly embed any two of its models into a common extension (α-JEP);

and the possibility of amalgamating models over a common substructure (α-AP). These conditions ensure that the theory possesses a well-behaved and robust class of models, suitable for advanced structural analysis.

By comparing this definition with that of a Jonsson theory, we observe a key difference: the latter is specialized to the case $\alpha = 0$, which yields the classical Jonsson theories. When $\alpha = \omega$ are referred to as complete Jonsson theories. In practice, the index $\alpha = 0$ is often omitted when referring to ordinary Jonsson theories. It is worth noting that, under this generalized framework, Jonsson theories are not necessarily complete.

As demonstrated in [2], Professor T.G. Mustafin established syntactic counterparts of the α-JEP and α-AP properties. These criteria provide an equivalent, formula-based perspective on the corresponding semantic conditions.

Proposition 4. [2] The following statements are equivalent:
1) The theory T satisfies the α-joint embedding property.
2) The α-JEP holds for all countable models of T.
3) For any disjoint tuples of variables $\overline{x}$ and $\overline{y}$, and any consistent sets of formulas $p(\overline{x})$ and $q(\overline{y})$ from $\Sigma_{\alpha+1}$, the union $T \cup p(\overline{x}) \cup q(\overline{y})$ is consistent, provided that both $T \cup p(\overline{x})$ and $T \cup q(\overline{y})$ are consistent separately.

Proposition 5. [2] The following conditions are equivalent:
1) The theory T satisfies the α-amalgamation property.
2) T satisfies the α-AP for countable structures.
3) For any two consistent sets of formulas $p(\overline{x})$ and $q(\overline{x})$ from $\Sigma_{\alpha+1}$ such that the following three sets are all consistent:

$$T \cup p(\overline{x}), \quad T \cup q(\overline{x}), \quad \text{and} \quad T \cup \{\neg\varphi(\overline{x}) \mid \varphi(\overline{x}) \in \Sigma_{\alpha+1},\, \varphi(\overline{x}) \notin p(\overline{x}) \cap q(\overline{x})\},$$

the union $T \cup p(\overline{x}) \cup q(\overline{x})$ is also consistent.
4) For every model $\mathcal{A} \models T$ and tuple $\overline{a} \in \mathcal{A}$, the set $\mathrm{Th}_{\Sigma_{\alpha+1}}(\mathcal{A}, \overline{a})$ can be extended to a unique maximal $\Sigma_{\alpha+1}$-type over T in the expanded language $L(\overline{a})$.

In the study of model theory, an important distinction is drawn between semantic and syntactic properties of theories. Semantic properties concern the behavior and structure of models, while syntactic properties are tied to the formal deductive system. The following propositions illustrate this distinction by clarifying the relationship between completeness and semantic similarity, and by enumerating key semantic notions that play a central role in classification theory.

Proposition 6. [15] If two theories T_1 and T_2 are complete, then they are necessarily semantically similar. However, the converse does not hold: semantically similar theories need not be syntactically similar.

Proposition 7. [15] The following concepts are classified as *semantic* in nature: type, forking, λ-stability, Lascar rank, strong type, Morley sequence, orthogonality, regularity of types, and $I(\aleph_\alpha, T)$ — the spectrum function.

We now turn our attention to a particular class of algebraic structures that will serve as the context for applying the main results established earlier. In the English-language model-theoretic literature, structures known as polygons over a monoid S are commonly referred to as S-acts [16]. Below, we provide a formal definition of this class.

Definition 9. [16] An S-act is a structure of the form $\langle A; f_\alpha : \alpha \in S\rangle$, where each f_α is a unary function on A, and the following axioms hold:
1) *Identity preservation:* $f_e(a) = a$ for all $a \in A$, where $e \in S$ is the identity element of the monoid.

2) *Compatibility with monoid operation:* $f_{\alpha\beta}(a) = f_\alpha(f_\beta(a))$ for all $\alpha, \beta \in S$ and for all $a \in A$.

The results that follow will demonstrate that for every complete theory, there exists another theory that is syntactically similar to it.

Theorem 5. [15] For every theory T_2 in a finite signature, there exists a theory T_1 of S-acts such that some inessential extension of T_1 is an almost envelope of T_2.

Theorem 6. [15] For every theory T_2 in an infinite signature, there exists a theory T_1 of S-acts such that some inessential extension of T_1 is an envelope of T_2.

This section presents a series of known results concerning syntactic and semantic similarities between Jonsson theories, as well as their extensions to classes of such theories. These notions generalize analogous concepts from the theory of complete first-order theories, as previously studied in works such as [13, 15], and have been systematically developed in [4].

In particular, the definitions of Jonsson syntactic similarity and Jonsson semantic similarity aim to capture structural equivalences between the existential fragments and semantic models of Jonsson theories. The formalization of these similarities relies on isomorphisms between lattices of existential formulas and between so-called semantic triples associated with the theories. The notion of the center of a Jonsson theory, denoted T^*, also plays a key role in transferring results from Jonsson theories to their complete analogues. Illustrative examples of Jonsson syntactic similarity between theories can be found in [4].

Analogously to the case of complete theories, Professor A.R. Yeshkeyev introduced the notion of Jonsson semantic similarity between two Jonsson theories [4]. The following result, which is similar to Proposition 6 but formulated in the context of Jonsson theories, was also established in [4].

Theorem 7. [4] Suppose that T_1 and T_2 are Jonsson theories that are syntactically similar in the Jonsson framework. Then they are also semantically similar within the same context.

By extending certain definitions from [15] and applying methods for working with Jonsson theories, it has been shown that, within the class of perfect existentially complete Jonsson theories, the introduced notions of syntactic and semantic similarity coincide with their counterparts in the class of complete theories, as defined in [13].

Theorem 8. [4] Let T_1 and T_2 be two existentially complete perfect Jonsson theories. Then the following statements are logically equivalent — that is, each holds if and only if the other does:

1) T_1 and T_2 are syntactically similar in the sense of Jonsson theories; that is, there exists a structure-preserving correspondence between their existential formulas that respects logical operations such as conjunction and existential quantification.

2) Their centers, T_1^* and T_2^*, are syntactically similar as complete theories; that is, the corresponding complete theories (obtained as the elementary theories of their respective semantic models) are related by a syntactic similarity that aligns their lattices of formulas.

To ensure precision in the subsequent exposition, we adopt the following designation. The syntactic and semantic similarities between two complete theories T_1 and T_2 will be denoted by $T_1 \overset{S}{\bowtie} T_2$ and $T_1 \underset{S}{\bowtie} T_2$, respectively. When dealing specifically with Jonsson theories, we will write $T_1 \overset{S}{\rtimes} T_2$ to indicate syntactic similarity in the Jonsson context, and $T_1 \underset{S}{\rtimes} T_2$ to denote their semantic similarity.

The following corollary for two Jonsson theories T_1 and T_2 in the language L was obtained in [4].

Corollary 1. [4] If the theories T_1 and T_2 are Jonsson syntactically similar ($T_1 \overset{S}{\rtimes} T_2$), then they are also Jonsson semantically similar ($T_1 \underset{S}{\rtimes} T_2$). Moreover, this is equivalent to the theories T_1 and T_2 being cosemantic, expressed as $T_1 \bowtie T_2$.

The notions of Jonsson semantic and syntactic similarity were further generalized to classes of Jonsson theories in [4]. As a result, a generalization of Theorem 7 was obtained for two classes from the Jonsson spectrum. This generalized result plays a crucial role in the proof of Theorem 11.

Lemma 1. [4] Let $\mathfrak{A} \in Mod(\sigma_1)$, $\mathfrak{B} \in Mod(\sigma_2)$, $[T_1] \in JSp(\mathfrak{A})/_{\bowtie}$, $[T_2] \in JSp(\mathfrak{B})/_{\bowtie}$ be perfect $\exists$-complete classes, then

$$[T_1] \overset{S}{\rtimes} [T_2] \Leftrightarrow [T_1^*] \overset{S}{\bowtie} [T_2^*].$$

4 Countable categoricity of Robinson hybrid and its similarity

In model theory, the notion of hybrid offers a constructive means of generating new theories by combining existing ones. Within the framework of Jonsson and Robinson theories, this operation enables the formation of syntactically or semantically enriched theories that retain key properties of their components. This section is devoted to the study of such hybrids, particularly their structure, categoricity, and the relations that govern their similarities.

The central object of analysis is the hybrid of Jonsson theories — a concept that allows two theories (with either identical or distinct signatures) to be combined via algebraic operations such as the Cartesian product, sum, or direct sum. These hybrid constructions fall into two main types, depending on whether the signatures of the input theories coincide. When extended to Robinson theories, those axiomatized by universal sentences, the same hybrid framework leads to the definition of Robinson hybrids, which inherit the logical rigor and syntactic simplicity characteristic of this subclass.

To support the analysis of such hybrids, we further examine the notions of perfectness, semantic models, and theoretical centers, particularly in the context of countable languages. A hybrid theory is said to be perfect if it coincides with the elementary theory of its saturated model; in such cases, its model-theoretic center plays a crucial role in determining categoricity and logical equivalence.

A key part of this section is the development of Kaiser equivalence, a newly introduced equivalence relation between Jonsson theories. This relation compares theories by their associated Kaiser classes, which capture the semantic behavior of existential fragments of models. Alongside this, we examine additional equivalence relations syntactic similarity and cosemanticness, that further refine the classification of theories within Robinson spectra.

The main results presented in this section show that, under certain conditions, the hybrid of two ω-categorical Robinson theories remains ω-categorical. Moreover, by applying triple factorization over Robinson spectra of semantic Jonsson quasivarieties (such as unars and undirected graphs), we construct a unique countably categorical theory of S-acts that is syntactically similar to a Robinson hybrid. This demonstrates not only the internal coherence of hybrid constructions but also the robustness of syntactic similarity in preserving key model-theoretic properties.

We begin by introducing the necessary definitions and preliminary results required to formulate the main theorems of this paper.

The concept of a hybrid of Jonsson theories was considered in [12]. By analogy, in the context of studying the Robinson spectra of semantic Jonsson quasivarieties for Robinson unars and undirected graphs, we introduce the notion of a Robinson hybrid corresponding to two Robinson theories.

Definition 10. 1) Let T_1 and T_2 be Robinson theories in a countable language L with the same signature σ, and let $\mathfrak{C}_{T_1}$ and $\mathfrak{C}_{T_2}$ be their semantic models, respectively. In the case where the Robinson theories T_1 and T_2 have a common signature, we define a hybrid of the first type of these Robinson theories as the theory $Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$, provided that this theory is Robinson in the language of signature σ. We denote this hybrid as $HR(T_1, T_2)$, where the operation $\diamond \in \{\times, +, \oplus\}$ and $\mathfrak{C}_1 \diamond \mathfrak{C}_2 \in Mod\,\sigma$. Here, $\times$ represents the Cartesian product, $+$ denotes the sum, and $\oplus$ indicates the direct sum. Thus, the algebraic construction $\mathfrak{C}_1 \diamond \mathfrak{C}_2$ is referred to as the semantic hybrid of the theories T_1 and T_2.

2) If T_1 and T_2 are Robinson theories with different signatures σ_1 and σ_2, respectively, then the theory $HR(T_1, T_2) = Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$ is called a hybrid of the second type, provided that this theory is Robinson in the language with the signature $\sigma = \sigma_1 \cup \sigma_2$ where $\mathfrak{C}_1 \diamond \mathfrak{C}_2 \in Mod\,\sigma$.

Clearly, 1) is a special case of 2).

Since Robinson theories are special cases of Jonsson theories, we can further use the notion of a perfect Robinson hybrid and also consider the concept of the center of a Robinson hybrid, which we denote by $HR^*(T_1, T_2)$, where $HR^*(T_1, T_2)$ is the center of the Robinson theory $Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$.

Based on the definition of hybrids of Robinson theories, it is also possible to define hybrids corresponding to two classes of Robinson theories.

Definition 11. 1) Let K be an axiomatizable class of models in a countable language L with signature σ, and let $[T_1]$, $[T_2] \in RSp(K)/_{\bowtie}$. The hybrid of the first type $HR([T_1], [T_2])$ of the classes $[T_1]$ and $[T_2]$ is the theory $Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$ provided that this theory is Robinson in the language with signature σ, where $\mathfrak{C}_i$ are semantic models of the classes $[T_i]$ for $i = 1, 2$, and $\diamond \in \{\times, +, \oplus\}$, where $\times$ denotes the Cartesian product, $+$ denotes the sum, and $\oplus$ denotes the direct sum of models.

2) Let K_1 and K_2 be axiomatizable classes of models of a countable language with different signatures σ_1 and σ_2, respectively, and let $[T_1] \in RSp(K_1)/_{\bowtie}$ and $[T_2] \in RSp(K_2)/_{\bowtie}$. Then the theory $HR([T_1], [T_2]) = Th_{\forall}(\mathfrak{C}_1 \diamond \mathfrak{C}_2)$ is called the hybrid of the second type of the classes $[T]_1$ and $[T_2]$, provided that this theory is Robinson in the language with signature $\sigma = \sigma_1 \cup \sigma_2$, where $\mathfrak{C}_1 \diamond \mathfrak{C}_2 \in Mod\sigma$.

To prove our result, we need a classical theorem on the characterization of countably categorical theories.

Theorem 9. [1] Let T be a complete theory. Then the following are equivalent:

a) T is ω-categorical;

b) for each $n < \omega$, T has only finitely many types in the variables $x_1, \ldots, x_n$.

In this article, we introduce a new concept, called Kaiser equivalence, between two Jonsson theories. As a starting point, we consider the definition of the Kaiser class of a theory.

Definition 12. A class K_T=$\{\mathfrak{A} \in Mod(T) : T^0(\mathfrak{A})$ is a Jonsson theory$\}$ is called a *Kaiser class* of the theory T, where $T^0(\mathfrak{A}) = Th_{\forall\exists}(\mathfrak{A})$.

Next, we consider a binary relation between the Kaiser classes of two Jonsson theories, T_1 and T_2.

Definition 13. Let T_1 and T_2 be Jonsson theories. We say that T_1 and T_2 are *K_T-equivalent* if $K_{T_1} = K_{T_2}$.

It is clear that the defined relation between two Jonsson theories is an equivalence relation.

Let $J\mathbb{C}\mathfrak{U}$ and $J\mathbb{C}\mathfrak{G}$ be the semantic Jonsson quasivarieties of Robinson unars and undirected graphs, respectively. Let $RSp(J\mathbb{C}\mathfrak{U})$ and $RSp(J\mathbb{C}\mathfrak{G})$ denote their corresponding Robinson spectra.

In addition, we define the following types of relations on these spectra:

1) syntactic similarity in the sense of Jonsson;

2) equivalence with respect to the class K_T;

3) the relation of cosemantic equivalence.

It is important to emphasize that, according to Proposition 3, each of these equivalence classes contains exactly one element.

It is straightforward to verify that each of the defined relations constitutes an equivalence relation. As a result, we can consider the corresponding quotient sets of the Robinson spectra of the classes $J\mathbb{C}_{\mathfrak{U}}$ and $J\mathbb{C}_{\mathfrak{G}}$ under these relations. This construction, which we refer to as triple factorization, is denoted by $RSp(J\mathbb{C}_{\mathfrak{U}})/_{\substack{S\\ \bowtie\\ K}}$ and $RSp(J\mathbb{C}_{\mathfrak{G}})/_{\substack{S\\ \bowtie\\ K}}$. Here, $[\dddot{\Delta}_{\mathfrak{U}}]$ denotes the equivalence class containing the theory $\Delta_{\mathfrak{U}}$ from $RSp(J\mathbb{C}_{\mathfrak{U}})/_{\substack{S\\ \bowtie\\ K}}$, and similarly, $[\dddot{\Delta}_{\mathfrak{G}}]$ corresponds to the class of the theory $\Delta_{\mathfrak{G}}$ from $RSp(J\mathbb{C}_{\mathfrak{G}})/_{\substack{S\\ \bowtie\\ K}}$. Each such equivalence class consists of a single theory of unars or undirected graphs.

We now proceed to the key findings of this article. It is important to note that in the results that follow, we consider only the Cartesian product as the operation $\diamond$.

Theorem 10. Let $[\ddot{\Delta}_{\mathfrak{U}}]$ and $[\ddot{\Delta}_{\mathfrak{G}}]$ denote the equivalence classes of ω-categorical Robinson theories corresponding to unars and undirected graphs, respectively. Then their Robinson hybrid $HR([\ddot{\Delta}_{\mathfrak{U}}],[\ddot{\Delta}_{\mathfrak{G}}])$ is also an ω-categorical Robinson theory.

Proof. Since, as stated in Proposition 3, these classes consist of a single element, we can further work only with theories. Also, by the definition of a Kaiser class of the theory, these theories are complete for universal (existential) sentences. Then, by Theorem 1, we obtain that the centers of these theories, denoted by $\dddot{\Delta}^*_{\mathfrak{U}}$ and $\dddot{\Delta}^*_{\mathfrak{G}}$, are also complete, countably categorical Robinson theories. Therefore, by Theorem 9, we have that for each $n<\omega$, $\dddot{\Delta}^*_{\mathfrak{U}}$ and $\dddot{\Delta}^*_{\mathfrak{G}}$ have only finitely many types in the variables $x_1,\ldots,x_n$.

As we know $\dddot{\Delta}^*_{\mathfrak{U}}$ and $\dddot{\Delta}^*_{\mathfrak{G}}$ are Robinson theories, then they have existentially closed semantic models $\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{U}}}$ and $\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{G}}}$, respectively, each of which realizes a finite number of types. Let us now consider a Cartesian product of their semantic models, $\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{U}}}\times\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{G}}}\in E_{HR(\dddot{\Delta}^*_{\mathfrak{U}},\dddot{\Delta}^*_{\mathfrak{G}})}$. By definition of the Cartesian product, this model also realizes a finite number of types. Therefore, the Robinson hybrid of second type of $\dddot{\Delta}^*_{\mathfrak{U}}$ and $\dddot{\Delta}^*_{\mathfrak{G}}$, denoted by $HR(\dddot{\Delta}^*_{\mathfrak{U}},\dddot{\Delta}^*_{\mathfrak{G}})=Th_{\forall}(\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{U}}}\times\mathfrak{C}_{\dddot{\Delta}^*_{\mathfrak{G}}})$ is ω-categorical Robinson theory.

Note that, according to Theorem 2, $\ddot{\Delta}_{\mathfrak{U}}$ and $\ddot{\Delta}_{\mathfrak{G}}$ are perfect Robinson theories. Consequently, the classes of existentially closed models of $\ddot{\Delta}_{\mathfrak{U}}$ and $\ddot{\Delta}_{\mathfrak{G}}$ coincide with the classes of models of their centers. Since the Robinson hybrid of these theories is a universally (existentially) complete theory, it follows that this Robinson hybrid is countably categorical.

We can also extend one of the results from [12] by applying triple factorization to the Robinson spectra of the semantic Jonsson quasivarieties of unars and undirected graphs. As a result, we obtain a countably categorical theory of S-acts that is syntactically similar to the Robinson hybrid of these classes.

Theorem 11. Let $[\ddot{\Delta}_{\mathfrak{U}}]$ and $[\ddot{\Delta}_{\mathfrak{G}}]$ be the equivalence classes of ω-categorical Robinson theories of unars of the signature with one unary functional symbol and the theory of undirected graphs that is considered in the signature containing one binary relation symbol, respectively. Then there exists a ω-categorical class of Robinson theories of S-acts, that is Jonsson syntactically similar to the Robinson hybrid $HR([\ddot{\Delta}_{\mathfrak{U}}],[\ddot{\Delta}_{\mathfrak{G}}])$ of these classes, where each class is a single-element class.

Proof. Since, by Proposition 3, these classes are singletons, we can further work directly with the corresponding theories. By Theorem 2, the countably categorical hybrid $HR(\ddot{\Delta}_{\mathfrak{U}},\ddot{\Delta}_{\mathfrak{G}})$ is a perfect Robinson theory. Since its center, denoted by $HR^*(\ddot{\Delta}_{\mathfrak{U}},\ddot{\Delta}_{\mathfrak{G}})$, is complete, it follows from Theorem 5 that there exists a complete theory of the S-acts, denoted by $T_{S_{act}}$, such that $H^*(\ddot{\Delta}_{\mathfrak{U}},\ddot{\Delta}_{\mathfrak{G}})\overset{S}{\bowtie}T_{S_{act}}$. Then, by Proposition 6, we also have $HR^*(\ddot{\Delta}_{\mathfrak{U}},\ddot{\Delta}_{\mathfrak{G}})\underset{S}{\bowtie}T_{S_{act}}$.

Since the notion of a type is semantic according to Proposition 7, the notion of a formula is also semantic. Furthermore, since both JEP and AP are semantic concepts, the properties JEP and AP are equivalent to the consistency of certain formulas, which follows from Propositions 4 and 5.

As all axioms hold in the semantic model, $\forall$-axiomatizability is a semantic property. This, in turn, implies that the property of being a Robinson theory is also a semantic concept. Therefore, the theory $T_{S_{act}}$ qualifies as a Robinson theory as well.

Given that $HR^*(\ddot{\Delta}_{\mathfrak{U}},\ddot{\Delta}_{\mathfrak{G}})$ is a perfect hybrid, the semantic model $\mathfrak{C}_{HR(\ddot{\Delta}_{\mathfrak{U}},\ddot{\Delta}_{\mathfrak{G}})}$ of the hybrid $HR(\ddot{\Delta}_{\mathfrak{U}},\ddot{\Delta}_{\mathfrak{G}})$ is saturated. Moreover, since $HR^*(\ddot{\Delta}_{\mathfrak{U}},\ddot{\Delta}_{\mathfrak{G}})\underset{S}{\bowtie}T_{S_{act}}$ it follows from Definition 18 that the semantic triples of these theories are isomorphic. Hence, $\mathfrak{C}_{HR(\dddot{\Delta}_{\mathfrak{U}},\dddot{\Delta}_{\mathfrak{G}})} \cong \mathfrak{C}_{T_{S_{act}}}$. Therefore $\mathfrak{C}_{T_{S_{act}}}$ is also saturated, and thus $T_{S_{act}}$ is a perfect Robinson theory.

Consider $RSp(\mathfrak{C}_{T_{S_{act}}})$. Since the theory $T_{S_{act}}$ is perfect, we have that $|RSp(\mathfrak{C}_{T_{S_{act}}})_{/\underset{K}{\overset{S}{\bowtie}}}| = 1$. Let $\Delta \in RSp(\mathfrak{C}_{T_{S_{act}}})$, meaning Δ is Robinson theory and $\Delta^* = T_{S_{act}}$. We will show that Δ is a perfect $\exists$-complete Robinson theory.

Given that $HR^*(\dddot{\Delta}_{\mathfrak{U}}, \dddot{\Delta}_{\mathfrak{G}}) \underset{S}{\bowtie} \Delta^*$, it follows from the definition of semantic similarity for complete theories that Δ is a perfect Robinson theory. If, in addition, Δ is $\exists$-complete, then we may replace $T'_{S_{act}}$ with Δ. By Lemma 1, we then conclude that $HR(\dddot{\Delta}_{\mathfrak{U}}, \dddot{\Delta}_{\mathfrak{G}}) \overset{S}{\rtimes} \Delta = T'_{S_{act}}$. If Δ is not $\exists$-complete, we apply the following procedure to complete the theory. Since $\Delta \subset T_{S_{act}}$, for any existential sentence φ in the signature language of Δ such that $\Delta \not\vdash \varphi$ and $\Delta \not\vdash \neg\varphi$, but $\varphi \in T_{S_{act}}$, we define the theory $\Delta' = \Delta \cup \{\varphi\}$.

Since $\Delta \subset \Delta' \subset T_{S_{act}}$, and both Δ and $T_{S_{act}}$ are Robinson theories, it follows from Proposition 7 that Δ' is also a Robinson theory. If Δ' is not $\exists$-complete, we continue the process by successively adding existential sentences $\varphi \in T_{S_{act}}$ until Δ' becomes $\exists$-complete.

Let $\overline{\Delta} = \Delta \cup \{\varphi \mid \varphi \in \Sigma_1, \varphi \in T_{S_{act}}\}$ denote the result of the existential completion procedure applied to the theory Δ. In other words, $\overline{\Delta}$ is $\exists$-complete and is also a Robinson theory. We now show that $\overline{\Delta} \in RSp(\mathfrak{C}TS_{act})$, which implies that the theory $\overline{\Delta}$ is perfect.

Let us assume the opposite, that is, suppose $\overline{\Delta} \notin RSp(\mathfrak{C}_{T_{S_{act}}})$. This implies that $\mathfrak{C}_{T_{S_{act}}} \notin \mathrm{Mod}(\overline{\Delta})$. However, this cannot be the case because $\mathfrak{C}_{T_{S_{act}}} \models \Delta$, and for any sentence $\varphi \in \overline{\Delta} \setminus \Delta$, we have $\varphi \in T_{S_{act}}$. Therefore, $\mathfrak{C}_{T_{S_{act}}} \models \varphi$, which means that $\mathfrak{C}_{T_{S_{act}}} \in \mathrm{Mod}(\overline{\Delta})$. This leads to a contradiction, so we conclude that $\overline{\Delta} \in RSp(\mathfrak{C}_{T_{S_{act}}})$.

Since $\mathfrak{C}_{T_{S_{act}}}$ is saturated, it follows that $\overline{\Delta}$ is a perfect Robinson theory. Hence, by Lemma 1, we obtain the equivalence: $HR^*(\dddot{\Delta}_{\mathfrak{U}}, \dddot{\Delta}_{\mathfrak{G}}) \overset{S}{\bowtie} \overline{\Delta}^* \Leftrightarrow HR(\dddot{\Delta}_{\mathfrak{U}}, \dddot{\Delta}_{\mathfrak{G}}) \overset{S}{\rtimes} \overline{\Delta}$, where $\overline{\Delta} = T'_{S_{act}}$.

Conclusion

This study has explored the fundamental aspects of Jonsson theories and the associated Jonsson spectra of their model classes, with a particular focus on the Robinson spectrum and the relationship between syntactic and semantic similarity. By analyzing how these concepts interact within the framework of model-theoretic structures, we highlighted the relevance of definability, compactness, and saturation in understanding the classification and behavior of models determined by Jonsson theories.

A promising and relatively unexplored direction for future research involves extending these ideas to the setting of positive Jonsson theories. This includes formulating a precise definition of the positive Jonsson spectrum and investigating how the syntactic-semantic correspondence and model-theoretic equivalences, such as K_T-equivalence, manifest in this more restrictive yet expressive framework. Foundational concepts and definitions for developing positive model theory in the context of Jonssonness are already outlined in [4, 17], offering a solid starting point for such an investigation.

Altogether, the theoretical insights presented in this paper offer a clearer understanding of classical Jonsson structures and establish a meaningful foundation for advancing future research on their positive counterparts.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors have no conflict of interest to declare.

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