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## Ehrenfeucht theories without powerful digraphs<sup>\*</sup>

All Ehrenfeucht theories are based on digraphs having the local pairwise intersection property. In this case, all known Ehrenfeucht theories are based on digraphs having the global pairwise intersection property. In the paper, we construct examples of Ehrenfeucht theories with powerful digraphs having the local pairwise intersection property, without the global pairwise intersection property, and thus without powerful digraphs.

*Key words:* Ehrenfeucht theory, powerful digraph, local pairwise intersection property, syntactic generic construction.

### 1 Introduction

In [1], the notion of powerful digraph is defined and it is proven that powerful digraphs are locally (i. e., with the local pairwise intersection property) presented in Ehrenfeucht theories (i. e., having finitely many but more than one pairwise non-isomorphic countable models). All known examples of Ehrenfeucht theories are constructed on a base of powerful digraphs with the global pairwise intersection property:

- dense linear orders [2–4];
- dense ordered trees [5–8];
- powerful digraphs with unbounded lengths of shortest paths and dense transitive closures [9];
- powerful digraphs with unbounded lengths of shortest paths and non-dense transitive closures [10].

In the paper, we construct examples of Ehrenfeucht theories with powerful types having the local pairwise intersection property and without the global pairwise intersection property.

We use without specifications the standard graph-theoretic and model-theoretic notions [11, 12], the system of notions in [1, 9, 10] as well as general principles for constructions of generic structures based on the syntactic approach [13]. All considered theories are first order, complete, countable, and without finite models.

### 2 Construction and results

Let  $p(x)$  be a non-principal type of a theory  $T$ ,  $M$  be a countable saturated model of  $T$ . Recall [1, 10] that the type  $p(x)$  has the *local pairwise intersection property* and denote it by  $(\text{LPIP})_p$  if for each formula  $\varphi(x) \in p(x)$ , there exists a formula  $\psi(x, y)$  of  $T$  such that for every  $a, b \in p(M)$ , there is a tuple  $c$  with  $M \models \varphi(c) \wedge \psi(c, a) \wedge \psi(c, b)$ .

If for  $\psi(x, y)$ , a stronger property holds:

for every  $a, b \in p(M)$ , there is a tuple  $c \in p(M)$  with  $M \models \psi(c, a) \wedge \psi(c, b)$ ,

it is called the *global pairwise intersection property* and it is denoted by  $(\text{GPIP})_p$  (with respect to  $\psi(x, y)$ ).

We omit the index  $\cdot_p$  in  $(\text{LPIP})_p$  and  $(\text{GPIP})_p$  if the type  $p$  is fixed.

Now we describe a modification of the construction in [9] allowing to produce a small generic theory with a non-principal powerful type  $p(x)$  having the  $(\text{LPIP})_p$  and without  $(\text{GPIP})_p$ .

Let  $\langle A, Q, W \rangle$  be a  $c$ -graph. We replace the graph language introducing a sequence of pairwise disjoint binary predicates  $Q_n$ ,  $n \in \omega$ , such that  $Q = \bigcup_{n \in \omega} Q_n$ , and substitute in  $W$ , instead of records on existence of external paths connecting elements in  $A$ , the records on existence of these paths with an information of colors

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$m_i$  for sequential arcs  $(b_i, b_{i+1}) \in Q_{m_i}$  forming these paths. Moreover, we allow, in obtained record  $W'$ , for any two elements  $a, b \in A$ , to write a formula-definable information of form:

$$\begin{aligned} \varphi_{k,m}(a,b) = & \exists x \left( \bigwedge_{i=0}^k \neg \text{Col}_i(x) \wedge \left( \bigvee_{i=0}^m Q_i(x,a) \right) \wedge \left( \bigvee_{i=0}^m Q_i(x,b) \right) \right) \wedge \\ & \wedge \neg \exists y \left( \bigwedge_{i=0}^k \neg \text{Col}_i(y) \wedge \left( \bigvee_{i=0}^{m-1} \bigwedge_{j=0}^k Q_i^j(y,a) \right) \wedge \left( \bigvee_{i=0}^{m-1} \bigwedge_{j=0}^k Q_i^j(y,b) \right) \right) \end{aligned}$$

such that the set of formulas over  $A$ , correspondent to  $W'$ , is consistent. The obtained structure  $\langle A, Q_n, W' \rangle_{n \in \omega}$  is called a *cv-graph*.

The concepts of  $c$ -embedding and  $c$ -isomorphism are obviously transformed to the class of *cv-graphs* of the language  $\langle Q_0, Q_1, \dots, Q_n, \dots \rangle$  with the preservation, via mappings, of arc colors and of satisfiability for  $\varphi_{k,m}(a,b)$  in the records.

A *cv-graph*  $\langle A, Q_{n,A}, W_A \rangle_{n \in \omega}$  is called a *cv-subgraph* of *cv-graph*  $\langle B, Q_{n,B}, W_B \rangle_{n \in \omega}$  (written  $A \subseteq_{cv} B$ ) if  $A \subseteq B$ ,  $Q_{n,A} = Q_{n,B} \cap A^2$ ,  $n \in \omega$ , and  $W_A$  consists of all records in  $W_B$ , for which pairs of elements in  $A$  are used, and also of records, induced by the *cv-graph*  $\langle B, Q_{n,B}, W_B \rangle_{n \in \omega}$ , on the existence or on the absence of elements  $a$  and  $b$  in accordance with  $\varphi_{k,m}(a,b)$ .

Let  $\langle A, Q_{n,A}, W_A \rangle_{n \in \omega}$ ,  $\langle B, Q_{n,B}, W_B \rangle_{n \in \omega}$ , and  $\langle C, Q_{n,C}, W_C \rangle_{n \in \omega}$  be *cv-graphs* such that  $A = B \cap C$ ,  $Q_{n,A} = Q_{n,B} \cap Q_{n,C}$ ,  $n \in \omega$ , and  $W_A$  consists of all records, included both in  $W_B$  and in  $W_C$ , in which pairs of elements in  $A$  are used, that induced by *cv-graphs*  $B$  and  $C$ . A *cv-graph*

$$\langle B \cup C, Q_{n,B} \cup Q_{n,C}, W_B \cup W_C \cup W \rangle_{n \in \omega}$$

is called a *cv-amalgam* of  $B$  and  $C$  over  $A$  if  $W$  consists of all possible formulas, describing, for any elements  $b \in B \setminus A$  and  $c \in C \setminus A$ , the existence and colors of elements  $d$ , for which there are  $(d,b)$ - and  $(d,c)$ -paths (if their existence is implied by relations  $Q_{n,B} \cup Q_{n,C}$  and  $W_B \cup W_C$ ), as well as the existence of elements  $e$  of all finite colors  $m$ , not exceeding  $\min\{\text{Col}(b), \text{Col}(c)\}$  and such that

$$\models \left( \bigvee_{i=0}^m Q_i(e,b) \right) \wedge \left( \bigvee_{i=0}^m Q_i(e,c) \right).$$

The record  $W$  also contains a finite set of formulas of form  $\varphi_{m-1,m}(b,c)$ ,  $m \leq \min\{\text{Col}(b), \text{Col}(c)\}$ ,  $m \in \omega$ , for other pairs  $(b,c)$ , where  $b \in B \setminus A$  and  $c \in C \setminus A$ .

We denote by  $\tilde{K}_0$  the class of all *cv-graphs* that can be obtained by all possible transformations above for  $c$ -graphs. Denote by  $\tilde{K}_0^*$  the subclass of  $\tilde{K}_0$ , generated by taking of *cv-subgraphs*, *cv-isomorphic copies*, and free *cv-amalgams*, as well as by routings and deroutings starting with the following *cv-graphs*  $\langle A, Q_{n,A}, W_A \rangle_{n \in \omega}$ , where the index  $\cdot_n$  in  $(a,b)_n$  means that  $(a,b) \in Q_n$ :

$$\Gamma_{\alpha, \beta, \gamma, n_1, n_2, n_3, W} = \left\langle \{0,1,2\}, \left\{ (0,1)_{n_1}, (1,2)_{n_2}, (0,2)_{n_3} \right\}, W \right\rangle,$$

where  $\text{Col}(0) = \alpha$ ,  $\text{Col}(1) = \beta$ ,  $\text{Col}(2) = \gamma$ ,  $\alpha \leq \beta \leq \gamma$ ,  $\gamma \in \omega \cup \{\infty\}$ ,  $W$  consists of nonempty finite set of formulas,  $\varphi_{m-1,m}(0,1)$ ,  $m \leq \min\{\text{Col}(0), \text{Col}(1)\}$ ,  $m \in \omega$ , and of nonempty finite set of formulas  $\varphi_{m-1,m}(0,2)$ ,  $m \leq \min\{\text{Col}(0), \text{Col}(2)\}$ ,  $m \in \omega$

**Lemma 2.1** ( $\tilde{\cdot}$ -amalgamation Lemma). *The class  $\tilde{K}_0^*$  has the  $\tilde{\cdot}$ -amalgamation property, that is, for any *cv-embeddings*  $f_0: A \rightarrow_{cv} B$  and  $g_0: A \rightarrow_{cv} C$ , where  $A, B, C \in \tilde{K}_0^*$ , there exist a *cv-graph*  $D \in \tilde{K}_0^*$  and *cv-embeddings*  $f_1: B \rightarrow_{cv} D$  and  $g_1: C \rightarrow_{cv} D$  such  $f_0 \circ f_1 = g_0 \circ g_1$ .*

Proof is obvious.  $\square$

Denote by  $\tilde{K}$  the class of colored acyclic digraphs in which any finite subgraph forms a cv-graph in  $\tilde{K}_0^*$ .

**Theorem 2.2.** *There exists countable, colored, saturated digraph  $\tilde{M} \in \tilde{K}$  satisfying the following:*

(1) *if  $f: A \rightarrow_{cv} \tilde{M}$  and  $g: A \rightarrow_{cv} B$  are cv-embeddings and  $B \in \tilde{K}_0^*$ , then there exists a cv-embedding  $h: B \rightarrow_{cv} \tilde{M}$  such that  $f = g \circ h$ ;*

(2) *if  $A$  and  $B$  are cv-isomorphic cv-subgraphs of  $\tilde{M}$ , then  $tp_{\tilde{M}}(A) = tp_{\tilde{M}}(B)$ ;*

(3) *the coloring of restriction  $\tilde{M}|_{\tilde{Q}}$  of  $\tilde{M}$  to the graph language  $\langle Q_0, Q_1, \dots, Q_n, \dots \rangle$  is inessential and  $Q_n$ -ordered for each  $n \in \omega$ ;*

(4) *the type  $p_\infty(x)$  has (LPIP) but does not have (GPIP).*

Proof is similar to the proof of [9, Theorem 1.3]. The construction of the saturated structure  $\tilde{M}$  repeats steps of construction for the structure  $M$  in the proof of [9, Theorem 1.3] with replacing of  $K_0^*$  by the class  $\tilde{K}_0^*$  and of  $K_0$  by  $\tilde{K}$ . The property (LPIP) for  $p_\infty(x)$  holds, since, by the generic construction, for any two elements  $a$  and  $b$ , having the same color at least  $n+1$ , there exists an element  $c$  of color  $n$  such that

$\models \left( \bigvee_{i=0}^n Q_i(c, a) \right) \wedge \left( \bigvee_{i=0}^n Q_i(c, b) \right)$ . The absence of (GPIP) is followed since, by compactness, for any  $n \in \omega$

there exist realizations  $a$  and  $b$  of  $p_\infty(x)$ , for which there are no realizations  $c$  of  $p_\infty(x)$  such that

$$(c, a), (c, b) \in \bigcup \{Q_i^j \mid 0 \leq i \leq n, j \in \omega\}. \quad \square$$

Using the construction of  $\tilde{M}$ , we have that for any two realizations  $a$  and  $b$  of  $p_\infty(x)$ , there exist numbers  $m, n \in \omega$  and a realization  $c$  of  $p_\infty(x)$  such that  $\models Q_m(c, a) \wedge Q_n(c, b)$ . Moreover, an expansion of  $\tilde{M}$  by binary predicates  $Q_n$ ,  $n \in \omega^*$ , and with (LPIP)  $\wedge$   $\neg$ (GPIP) admits the following property: for any realizations  $a_1, \dots, a_k$  of  $p_\infty(x)$ , there exist  $n_1, \dots, n_k \in \omega$  and a *special* realization  $c$  of  $p_\infty(x)$  such that  $\models Q_{n_1}(c, a_1) \wedge \dots \wedge Q_{n_k}(c, a_k)$ . Therefore, an obvious modification of construction in [9, Sections 1 and 2], applied for an expansion of the generic structure  $\tilde{M}$ , states the following theorem.

**Theorem 2.3.** *There exists a generic theory  $T$  satisfying the following conditions:*

(1)  $|\text{RK}(T)| = 2$ ;

(2) *some powerful type  $p \in S(T)$  has (LPIP) and does not have (GPIP).*

Repeating the construction in [9, Section 3] we obtain an Ehrenfeucht theory with three countable models, based on the structure  $\tilde{M}$ :

**Theorem 2.4.** *There exists a generic Ehrenfeucht theory  $T$  with  $I(T, \omega) = 3$  and such that each non-principal type  $p(x)$  has (LPIP) and does not have (GPIP).*

Repeating the proof of Theorem 4.1 in [9], based on the structure  $\tilde{M}$ , we obtain the basic theorem of [9] with (LPIP) and without (GPIP):

**Theorem 2.5.** *For any finite preordered set  $\langle X; \leq \rangle$  with the least element  $x_0$  and the greatest class  $\tilde{x}_1$  in the ordered factor set  $\langle X; \leq \rangle / \sim$  with respect to  $\sim$  (where  $x \sim y \Leftrightarrow x \leq y$  and  $y \leq x$ ), and for any function  $f: X \rightarrow \omega$  satisfying the conditions  $f(\tilde{x}_0) = 0$ ,  $f(\tilde{x}_1) > 0$  for  $|X| > 1$ , and  $f(\tilde{y}) > 0$  for  $|\tilde{y}| > 1$ , there exist a complete theory  $T$ , with (LPIP)  $\wedge$   $\neg$ (GPIP)-structures of non-principal types, and an isomorphism  $g: \langle X; \leq \rangle \xrightarrow{\sim} \text{RK}(T)$  such that  $\text{IL}(g(\tilde{y})) = f(\tilde{y})$  for any  $\tilde{y} \in X / \sim$ .*

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### Үстем орграфтарды қамтымайтын эренфойхт теориялары

Қос-қостан қиылымды локалдық қасиетке ие негізі орграфтар болып табылады. Бұл жағдайда қос-қостан қиылысымды басты қасиетке ие болу барлық белгілі эренфойхтық теоремалар үстем графтардың негізінде құрылады деп болжанады. Мақалада қос-қостан қиылысымды локалдық қасиетке ие, бірақ қос-қостан қиылысымды басты қасиетке ие емес және үстем орграфтар құрылымын қамтымайтын үстем типті эренфойхтық теориялардың мысалдары құрылған.

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### Эренфойхтовы теории, не имеющие властных орграфов

Основой построения всех эренфойхтовых теорий являются орграфы, обладающие локальным свойством попарного пересечения. При этом все известные эренфойхтовы теории строятся на базе властных орграфов, предполагающих наличие глобального свойства попарного пересечения. В статье приведены примеры эренфойхтовых теорий с властными типами, обладающими локальным свойством попарного пересечения, но не имеющими глобального свойства попарного пересечения и, тем самым, не содержащими структур властных орграфов.