

## Inverse boundary value problem for a linearized equations of longitudinal waves in rods

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In this article, a question regarding the solvability of an inverse boundary value problem for the linearized equation of longitudinal waves in rods with an integral condition of the first kind was considered. For the considered inverse boundary value problem, the definition of a classical solution was introduced. Using the Fourier method, the problem was reduced to solving a system of integral equations. The method of contraction mappings is applied to prove the existence and uniqueness of a solution to the system of integral equations. The problem is to deduce the existence and uniqueness of the classical solution for the original problem.

*Keywords:* Inverse boundary value problem, longitudinal wave equations, Fourier method, classical solution.

*2020 Mathematics Subject Classification:* 49k10, 49k20.

### Introduction

At present, the theory of nonlocal problems is being intensively developed and represents an important branch of the theory of partial differential equations. Problems with nonlocal integral conditions are of great interest in this area. Conditions of this kind may appear in the mathematical modeling of phenomena related to plasma physics [1], heat propagation [2], moisture transfer in capillary-porous media [3], demography, and mathematical biology.

Inverse problems with an integral overdetermination condition for partial differential equations have been studied in many papers. Let us note the articles [4–6] and the references therein.

The study of various aspects of inverse problems of recovering the coefficients of partial differential equations, as well as the study of inverse problems by reducing them to variational formulations, is considered in the works of Kozhanov A.I. [6], Denisov A.M. [7], Ivanchov M.I. [8], and others.

Works are devoted to the study of nonlinear inverse problems for the linearized equation of longitudinal waves in rods. The questions of solvability of problems with nonlocal integral conditions for partial differential equations were studied in [5, 9–18].

#### 1 Statement of the problem and its reduction to an equivalent problem

Let  $D_T$  be a Domain,  $D_T = \{(x, t) : 0 < x < 1, 0 < t \leq T\}$ . Consider for linearized equation an inverse boundary value problem [6]

$$u_{tt}(x, t) + u_{ttxx}(x, t) - u_{xx}(x, t) = a(t)u(x, t) + f(x, t), \quad (x, t) \in D_T \quad (1)$$

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Received: 6 May 2024; Accepted: 17 December 2024.

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with the initial condition (IC)

$$u(x, 0) = \varphi(x), \quad u_t(x, T) = \psi(x) \quad (0 \leq x \leq 1). \quad (2)$$

Neumann boundary condition (BC)

$$u_x(0, t) = 0 \quad (0 \leq t \leq T), \quad (3)$$

integral condition (IgC)

$$\int_0^1 u(x, t) dx = 0 \quad (0 \leq t \leq T) \quad (4)$$

and additional conditions (AD)

$$u(0, t) = h(t) \quad (0 \leq t \leq T), \quad (5)$$

$f(x, t)$ ,  $\varphi(x)$ ,  $\psi(x)$ , and  $h(t)$  are given functions,  $u(x, t)$  and  $a(t)$  are the required functions.

*Definition.* By the classical solution of the inverse boundary value problem (1)–(5) we mean the pair  $\{u(x, t), a(t)\}$  of the functions  $u(x, t)$ ,  $a(t)$ , where  $u(x, t) \in \tilde{C}^{2,2}(\bar{D}_T)$ ,  $a(t) \in C[0, T]$ , and they satisfy the equations (1)–(5) in the ordinary sense, where

$$\tilde{C}^{2,2}(D_T) = \{u(x, t) : u(x, t) \in C^2(D_T), \quad u_{ttxx}(x, t) \in C(D_T)\}.$$

To study problem (1)–(5), we first consider the following:

$$y''(t) = a(t)y(t) \quad (0 \leq t \leq T), \quad (6)$$

$$y(0) = 0, \quad y'(T) = 0, \quad (7)$$

where  $a(t) \in C[0, T]$  are given functions, and  $y = y(t)$  is the required function. Furthermore, by solving the problem (6), (7) we mean a function  $y(t)$ , that, together with all its derivatives in equation (6), is continuous on  $[0, T]$  and satisfies conditions (6), (7) in the usual sense. The following lemma holds:

*Lemma 1.* [7] Let  $a(t) \in C[0, T]$  be such that

$$\|a(t)\|_{C[0, T]} \leq R = \text{const.}$$

Supplementarily,  $\frac{1}{2}RT^2 < 1$ . Then problem (6), (7) has only a trivial solution.

With the addition of the inverse boundary value problem (1)–(5), consider the following auxiliary inverse boundary value problem. We must define a pair  $\{u(x, t), a(t)\}$  of functions  $u(x, t) \in \tilde{C}^{2,2}(\bar{D}_T)$  and  $a(t) \in C[0, T]$ , from (2)–(3),

$$u_x(1, t) = 0 \quad (0 \leq t \leq T), \quad (8)$$

$$h''(t) + u_{ttxx}(0, t) - u_{xx}(0, t) = a(t)h(t) + f(0, t) \quad (0 \leq t \leq T). \quad (9)$$

*Theorem 1.* Let  $\varphi(x), \psi(x) \in C^1[0, 1]$ ,  $\varphi'(1) = 0$ ,  $\psi'(1) = 0$ ,  $h(t) \in C^2[0, T]$ ,  $f(x, t) \in C(\bar{D}_T)$ ,  $\int_0^1 f(x, t) dx = 0$  ( $0 \leq t \leq T$ ),  $h(t) \neq 0$  ( $0 \leq t \leq T$ ) and

$$\int_0^1 \varphi(x) dx = 0, \quad \int_0^1 \psi(x) dx = 0, \quad (10)$$

$$\varphi(0) = h(0), \quad \psi(0) = h'(T). \quad (11)$$

Then the following statements are true:

1. Each classical solution  $\{u(x, t), a(t)\}$  of problem (1)–(5) is also a solution of problem (1)–(3), (8), (9).

2. Each solution  $\{u(x, t), a(t)\}$  of problem (1)–(3), (8), (9) is a classical solution of the problem (1)–(5) if

$$\frac{1}{2} \|a(t)\|_{C[0, T]} T^2 < 1, \quad (12)$$

it is a classical solution (1)–(5).

*Proof.* Let  $\{u(x, t), a(t)\}$  be a classical solution to problem (2)–(5). Integrating equation (2), we get:

$$\begin{aligned} \frac{d^2}{dt^2} \int_0^1 u(x, t) dx + u_{ttx}(1, t) - u_{ttx}(0, t) - (u_x(1, t) - u_x(0, t)) = \\ = a(t) \int_0^1 u(x, t) dx + \int_0^1 f(x, t) dx \quad (0 \leq t \leq T). \end{aligned} \quad (13)$$

Suppose it is the case that  $\int_0^1 f(x, t) dx = 0$  ( $0 \leq t \leq T$ ), taking into account (3), (4), we identify

$$u_{ttx}(1, t) - u_x(1, t) = 0 \quad (0 \leq t \leq T). \quad (14)$$

Due to (2),  $\varphi'(1) = 0$ ,  $\psi'(1) = 0$ , therefore

$$u_x(1, 0) = \varphi'(1) = 0, \quad u_{tx}(1, T) = \psi'(1) = 0. \quad (15)$$

Obviously, problem (14), (15) has only a trivial solution,  $u_x(1, t) = 0$  ( $0 \leq t \leq T$ ), i.e. conditions (8) are satisfied.

Considering  $h(t) \in C^2[0, T]$  and differentiating (5) twice, we obtain:

$$u_{tt}(0, t) = h(t) \quad (0 \leq t \leq T). \quad (16)$$

Further, from (2) we get:

$$\frac{d^2}{dt^2} u(0, t) + u_{ttxx}(0, t) - u_{xx}(0, t) = a(t)h(t) + f(0, t) \quad (0 \leq t \leq T). \quad (17)$$

From (17), regarding to (5) and (16), we obtained (9).

Now, assume that  $\{u(x, t), a(t)\}$  is a solution to problem (1)–(3), (8), (9), and (12) is satisfied. Then from (13), taking into account (3) and (8), we deduce:

$$\frac{d^2}{dt^2} \int_0^1 u(x, t) dx - a(t) \int_0^1 u(x, t) dx = 0 \quad (0 \leq t \leq T). \quad (18)$$

As a corollary to (2) and (10), it is evident that

$$\int_0^1 u(x, 0) dx = \int_0^1 \varphi(x) dx = 0, \quad \int_0^1 u_t(x, T) dx = \int_0^1 \psi(x) dx = 0. \quad (19)$$

Since, by virtue of Lemma 1, problem (18), (19) has only a trivial solution, then  $\int_0^1 u(x, t) dx = 0$  ( $0 \leq t \leq T$ ), i.e. condition (4) is satisfied.

Further, from (9) and (17):

$$\frac{d^2}{dt^2} (u(0, t) - h(t)) = a(t)(u(0, t) - h(t)) \quad (0 \leq t \leq T). \quad (20)$$

From (2) and (11), we get:

$$u(0, 0) - h(0) = \varphi(0) - h(0) = 0. \quad (21)$$

From (20) and (21), based on Lemma 1, we conclude that condition (5) is fulfilled. The theorem is proven.

2 The Existence and uniqueness of the classical solution of the inverse boundary value problem

Component  $u(x, t)$  of solution  $\{u(x, t), a(t)\}$  of problem (1)–(3), (8), (9) is studied in the form:

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = \pi k), \tag{22}$$

$u_k(t) = m_k \int_0^1 u(x, t) \cos \lambda_k x dx$  ( $k = 0, 1, 2, \dots$ ), and

$$m_k = \begin{cases} 1, & k = 0, \\ 2, & k = 1, 2, \dots \end{cases}$$

By applying the scheme of the Fourier method from (2), we get:

$$(1 - \lambda_k^2)u_k''(t) + \lambda_k^2 u_k(t) = F_k(t; u, a) \quad (k = 0, 1, 2, \dots; 0 \leq t \leq T), \tag{23}$$

$$u_k(0) = \varphi_k, \quad u_k'(T) = \psi_k \quad (k = 0, 1, 2, \dots), \tag{24}$$

$$F_k(t; u, a) = f_k(t) + a(t)u_k(t), \quad f_k(t) = m_k \int_0^1 f(x, t) \cos \lambda_k x dx,$$

$$\varphi_k = m_k \int_0^1 \varphi(x) \cos \lambda_k x dx, \quad \psi_k = m_k \int_0^1 \psi(x) \cos \lambda_k x dx \quad (k = 0, 1, 2, \dots).$$

Solving problem (1)–(6), we obtain:

$$u_0(t) = \varphi_0 + \int_0^T m(t)u_0(t)dt + \psi_0 t + \int_0^T G_0(t, \tau)F_0(\tau; u, a)d\tau, \tag{25}$$

$$u_k(t) = \frac{ch(\beta_k(T-t))}{ch(\beta_k T)}\varphi_k + \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)}\psi_k - \frac{1}{(\lambda_k^2 - 1)\beta_k} \int_0^T G_k(t, \tau)F_k(\tau; u, a)d\tau \quad (k = 1, 2, \dots), \tag{26}$$

where

$$G_0(t, \tau) = \begin{cases} -t, & t \in [0, \tau], \\ -\tau, & t \in [\tau, T], \end{cases}$$

$$\beta_k^2 = \frac{\lambda_k^2}{\lambda_k^2 - 1} > 0,$$

$$G_k(t, \tau) = \begin{cases} -\frac{[sh(\beta_k(T+t-\tau))-sh(\beta_k(T-(t+\tau)))]}{2ch(\beta_k T)}, & t \in [0, \tau], \\ -\frac{sh(\beta_k(T-(t+\tau)))-sh(\beta_k(T-(t-\tau)))}{2ch(\beta_k T)}, & t \in [\tau, T]. \end{cases}$$

After substituting the expression from (25), (26) into (22), to define a component  $u(x, t)$  of the solution of problem (1)–(4), (8), (9), we obtain:

$$u(x, t) = \varphi_0 + \psi_0 t + \int_0^T G_0(t, \tau)F_0(\tau; u, a)d\tau + \sum_{k=1}^{\infty} \left\{ \frac{ch(\beta_k(T-t))}{ch(\beta_k T)}\varphi_k + \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)}\psi_k - \frac{1}{(\lambda_k^2 - 1)\beta_k} \int_0^T G_k(t, \tau)F_k(\tau; u, a)d\tau \right\} \cos \lambda_k x. \tag{27}$$

Now, from (8) including (22) and from (23) including (26), we get:

$$a(t) = [h(t)]^{-1} \left\{ h''(t) - f(0, t) - \sum_{k=1}^{\infty} \lambda_k^2 (u_k''(t) - u_k(t)) \right\}. \quad (28)$$

$$\begin{aligned} -\lambda_k^2 u_k''(t) + \lambda_k^2 u_k(t) &= F_k(t; u, a) - u_k''(t) = \frac{\lambda_k^2}{\lambda_k^2 - 1} F_k(t; u, a) - \frac{\lambda_k^2}{\lambda_k^2 - 1} u_k(t) = \\ &= \frac{\lambda_k^2}{\lambda_k^2 - 1} F_k(t; u, a) - \frac{\lambda_k^2}{\lambda_k^2 - 1} \left[ \frac{ch(\beta_k(T-t))}{ch(\beta_k T)} \varphi_k + \right. \\ &\left. + \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)} \psi_k - \frac{1}{(\lambda_k^2 - 1)\beta_k} \int_0^T G_k(t, \tau) F_k(\tau; u, a) d\tau \right] \quad (k = 1, 2, \dots). \end{aligned} \quad (29)$$

Aimed at defining an equation for the second component  $a(t)$  of the solution  $\{u(x, t), a(t)\}$  of problem (1)–(3), (8), (9), we substitute expression (29) into (28):

$$\begin{aligned} a(t) &= [h(t)]^{-1} \left\{ h''(t) - f(0, t) + \sum_{k=1}^{\infty} \left[ \frac{\lambda_k^2}{\lambda_k^2 - 1} F_k(t; u, a) + \right. \right. \\ &\quad \left. \left. - \frac{\lambda_k^2}{\lambda_k^2 - 1} \left[ \frac{ch(\beta_k(T-t))}{ch(\beta_k T)} \varphi_k + \frac{sh(\beta_k t)}{\beta_k ch(\beta_k T)} \psi_k - \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{1}{(\lambda_k^2 - 1)\beta_k} \int_0^T G_k(t, \tau) F_k(\tau; u, a) d\tau \right] \right] \right\} \quad (0 \leq t \leq T). \end{aligned} \quad (30)$$

Thus, the solution of problem (1)–(3), (8), (9) is reduced to the solution of system (27), (30) with respect to unknown functions  $u(x, t)$  and  $a(t)$ .

The subsequent lemma plays an essential role in the disquisition of the uniqueness question of the solution to problem (2)–(3), (8), (9).

*Lemma 2.* If  $\{u(x, t), a(t)\}$  is any solution of problem (1)–(3), (8), (9), then the functions

$$u_k(t) = m_k \int_0^1 u(x, t) \cos \lambda_k x dx \quad (k = 0, 1, 2, \dots)$$

satisfy the system consisting of equations (25), (26) on  $[0, T]$ .

*Proof.* Let  $\{u(x, t), a(t)\}$  be any solution of problem (1)–(3), (8), (9). Then multiplying both sides of equation (2) by the function  $m_k \cos \lambda_k x$  ( $k = 0, 1, 2, \dots$ ), integrating the resulting equality over  $x$  from 0 to 1 and using  $m_k \int_0^1 u_{tt}(x, t) \cos \lambda_k x dx = \frac{d^2}{dt^2} \left( m_k \int_0^1 u(x, t) \cos \lambda_k x dx \right) = u_k''(t)$  ( $k = 0, 1, 2, \dots$ ),

$$m_k \int_0^1 u_{xx}(x, t) \cos \lambda_k x dx = -\lambda_k^2 \left( m_k \int_0^1 u(x, t) \cos \lambda_k x dx \right) = -\lambda_k^2 u_k(t) \quad (k = 0, 1, 2, \dots),$$

$$\begin{aligned} m_k \int_0^1 u_{ttxx}(x, t) \cos \lambda_k x dx &= -\lambda_k^2 m_k \int_0^1 u_{tt}(x, t) \cos \lambda_k x dx = \\ &= -\lambda_k^2 \frac{d^2}{dt^2} \left( m_k \int_0^1 u(x, t) \cos \lambda_k x dx \right) = -\lambda_k^2 u_k''(t) \quad (k = 0, 1, 2, \dots), \end{aligned}$$

we obtain that Eq. (23) is satisfied.

Similarly, from (2) we obtain that condition (24) is fulfilled.

Thus,  $u_k(t)$  ( $k = 0, 1, 2, \dots$ ) is a solution to problem (23), (24). Moreover, it directly follows that the functions  $u_k(t)$  ( $k = 0, 1, 2, \dots$ ) satisfy the system (25), (26) on  $[0, T]$ . The lemma is proven.

Therefore, if  $u_k(t) = 2 \int_0^1 u(x, t) \cos \lambda_k x dx$  ( $k = 0, 1, 2, \dots$ ) is a solution to system (25), (26), then the pair  $\{u(x, t), a(t)\}$  of functions  $u(x, t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x, a(t)$  and  $a(t)$  is a solution to system (27), (30).

*Corollary 1.* Let system (27), (30) have a unique solution. Then problem (1)–(3), (8), (9) can not have more than one solution, i.e. if problem (1)–(3), (8), (9) has a solution, then it is unique.

To study the problem (1)–(3), (8), (9), we introduce two spaces.

By  $B_{2,T}^\alpha$  [8], we denote the set of all functions of the form

$$u(x, t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = k\pi),$$

considered in  $D_T$ , where each function  $u_k(t)$  ( $k = 0, 1, \dots$ ) is continuous on  $[0, T]$  and

$$J(u) = \|u_0(t)\|_{C[0,T]} + \left\{ \sum_{k=1}^{\infty} \left( \lambda_k^\alpha \|u_k(t)\|_{C[0,T]} \right)^2 \right\}^{\frac{1}{2}} < +\infty,$$

where  $\alpha \geq 0$ . We define the norm in this set as follows:

$$\|u(x, t)\|_{B_{2,T}^\alpha} = J(u).$$

$E_T^\alpha$  denotes the space  $B_{2,T}^\alpha \times C[0, T]$  with vector functions  $z(x, t) = \{u(x, t), a(t)\}$  and norm

$$\|z(x, t)\|_{E_T^\alpha} = \|u(x, t)\|_{B_{2,T}^\alpha} + \|a(t)\|_{C[0,T]}.$$

$B_{2,T}^\alpha$  and  $E_T^\alpha$  are Banach spaces.

In the space  $E_T^3$ , we define an operator:

$$\Phi(u, a, b) = \{\Phi_1(u, a), \Phi_2(u, a)\},$$

with

$$\Phi_1(u, a) = \tilde{u}(x, t) \equiv \sum_{k=0}^{\infty} \tilde{u}_k(t) \cos \lambda_k x, \quad \Phi_2(u, a, b) = \tilde{a}(t),$$

$\tilde{u}_0(t), \tilde{u}_k(t)$  ( $k = 1, 2, \dots$ ) and  $\tilde{a}(t)$  are equal to the right-hand sides of (25), (26) and (30).

It is easy to see that

$$\frac{ch(\beta_k(T-t))}{ch(\beta_k T)} < 1, \quad \frac{sh(\beta_k t)}{ch(\beta_k T)} < 1, \quad \frac{sh(\beta_k(T+t-\tau))}{ch(\beta_k T)} < 1 \quad (t \in [0, \tau]),$$

$$\frac{sh(\beta_k(T-(t+\tau)))}{ch(\beta_k T)} < 1, \quad \frac{sh(\beta_k(T-(t-\tau)))}{ch(\beta_k T)} < 1 \quad (t \in [\tau, T]),$$

$$\lambda_k^2 - 1 > \frac{1}{2} \lambda_k^2, \quad 1 < \beta_k = \frac{\lambda_k}{\sqrt{\lambda_k^2 - 1}} < \sqrt{2}, \quad \frac{1}{\sqrt{2}} < \frac{1}{\beta_k} < 1.$$

Then, we have:

$$\|\tilde{u}_0(t)\|_{C[0,T]} \leq |\varphi_0| + T \|\psi_0\| + 2T\sqrt{T} \left( \int_0^t |f_0(\tau)|^2 d\tau \right)^{\frac{1}{2}} + 2T^2 \|a(t)\|_{C[0,T]} \|u_0(t)\|_{C[0,T]}, \quad (31)$$

$$\begin{aligned} & \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|\tilde{u}_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 2 \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_k|)^2 \right)^{\frac{1}{2}} + 2 \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\psi_k|)^2 \right)^{\frac{1}{2}} + \\ & + 4\sqrt{T} \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k |f_k(\tau)|)^2 \right)^{\frac{1}{2}} + 4T \|a(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}}, \quad (32) \end{aligned}$$

$$\begin{aligned} & \|\tilde{a}(t)\|_{C[0,T]} \leq \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \|h''(t) - f(0, t)\|_{C[0,T]} + \right. \\ & + \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[ \left( \sum_{k=1}^{\infty} (\lambda_k \|f_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \|a(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \right. \\ & + \left. \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_k|)^2 \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\psi_k|)^2 \right)^{\frac{1}{2}} + 2\sqrt{T} \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k |f_k(\tau)|)^2 \right)^{\frac{1}{2}} + \right. \\ & \left. \left. + 2T \|a(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \right] \right\}. \quad (33) \end{aligned}$$

Let us assume that the data of problem (1)–(3), (8), (9), satisfy the following conditions:

- 1)  $\varphi(x) \in C^2[0, 1]$ ,  $\varphi'''(x) \in L_2(0, 1)$ ,  $\varphi'(0) = \varphi'(1) = 0$ ;
- 2)  $\psi(x) \in C^2[0, 1]$ ,  $\psi'''(x) \in L_2(0, 1)$ ,  $\psi'(0) = \psi'(1) = 0$ ;
- 3)  $f(x, t) \in C(D_T)$ ,  $f_x(x, t) \in L_2(D_T)$ ,  $f_x(0, t) = f_x(1, t) = 0$  ( $0 \leq t \leq T$ );
- 4)  $h(t) \in C^2[0, T]$ ,  $h(t) \neq 0$  ( $0 \leq t \leq T$ ).

Then from (31)–(33), we have:

$$\|\tilde{u}(x, t)\|_{B_{2,T}^3} \leq A_1(T) + B_1(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3}, \quad (34)$$

$$\begin{aligned} & \|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3} + \\ & + B_2(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3} + C_2(T) \|u(x, t)\|_{B_{2,T}^3 + D_2(T)} \|a(t)\|_{C[0,T]}, \quad (35) \end{aligned}$$

where

$$\begin{aligned} & A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T \|\psi(x)\|_{L_2(0,1)} + 2T\sqrt{T} \|f(x, t)\|_{L_2(D_T)} + \\ & + 2 \left\| \varphi'''(x) \right\|_{L_2(0,1)} + 2 \left\| \psi'''(x) \right\|_{L_2(0,1)} + 4\sqrt{T} \|f_x(x, t)\|_{L_2(D_T)}, \\ & B_1(T) = 6T, \quad A_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left\{ \|h''(t) - f(0, t)\|_{C[0,T]} + \right. \\ & + \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \left[ \left\| \|f_x(x, t)\|_{C[0,T]} \right\|_{L_2(0,1)} + \|\varphi'''(x)\|_{L_2(0,1)} + \|\psi'''(x)\|_{L_2(0,1)} + \right. \\ & \left. \left. + 2\sqrt{T} \|f_x(x, t)\|_{L_2(D_T)} \right] \right\}, \end{aligned}$$

$$B_2(T) = 2 \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left( \sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} (2T + 1).$$

From inequalities (34), (35) we conclude:

$$\|\tilde{u}(x, t)\|_{B_{2,T}^3} + \|\tilde{a}(t)\|_{C[0,T]} \leq A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3}, \tag{36}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T).$$

*Theorem 2.* Let conditions 1–3 be satisfied and

$$B(T)(A(T) + 2)^2 < 1. \tag{37}$$

Then, in  $K = K_R$  ( $\|z\|_{E_T^3} \leq R = A(T) + 2$ ), in the space  $E_T^3$ , problem (1)–(3), (8), (9) has only one solution.

*Proof.* In the space  $E_T^3$  consider the equation

$$z = \Phi z, \tag{38}$$

where  $z = \{u, a\}$  and components  $\Phi_i(u, a)$  ( $i = 1, 2$ ) of operators  $\Phi(u, a)$  are defined by the right-hand sides of equations (27) and (30).

Now, consider the operator  $\Phi(u, a)$  in the ball  $K = K_R$  of the space  $E_T^3$ . Analogously to (36), we obtain that for any  $z, z_1, z_2 \in K_R$ , the following estimates hold:

$$\|\Phi z\|_{E_T^3} \leq A(T) + B(T) \|a(t)\|_{C[0,T]} \|u(x, t)\|_{B_{2,T}^3}, \tag{39}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^3} \leq 2B(T)R (\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x, t) - u_2(x, t)\|_{B_{2,T}^3}). \tag{40}$$

As a result, the operator (38) has a unique solution for  $K = K_R$ .

Then, from estimates (39) and (40), allowing for (37), it follows that the operator  $\Phi$  acts in the ball  $K = K_R$  and is contractive.

Functions  $u(x, t)$ , as an element of space  $B_{2,T}^3$ , are continuous and have continuous derivatives  $u_x(x, t)$  and  $u_{xx}(x, t)$  in  $\bar{D}_T$ .

From (23), it is evident that

$$\left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_k''(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \sqrt{2} \left\{ \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|u_k(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \left\| \|a(t)u_x(x, t) + f_x(x, t)\|_{C[0,T]} \right\|_{L_2(0,1)} \right\}.$$

Hence, it follows that  $u_{tt}(x, t), u_{ttx}(x, t), u_{ttxx}(x, t)$  are continuous in  $\bar{D}_T$ .

It is easy to check that equation (1) and conditions (3), (4), (7), (9) are satisfied in the usual sense. Consequently,  $\{u(x, t), a(t)\}$  is a solution to problem (1)–(3), (7)–(9). By the corollary of Lemma 1, it is unique in the ball  $K = K_R$ . The theorem has been proven.

In the proof of Theorem 1 and Theorem 2, the next Theorem plays, an essential role of unique and solvability of the problem (1)–(5).

*Theorem 3.* Let all the conditions of Theorem 2 and (10), (11) be satisfied,  $\int_0^1 f(x, t)dx = 0$  ( $0 \leq t \leq T$ ), and

$$\frac{1}{2}(A(T) + 2)T^2 < 1.$$

Then, problem (1)–(5) has the only classical solution in the ball  $K = K_R$  ( $\|z\|_{E_T^3} \leq A(T) + 2$ ) from  $E_T^3$ .

### Conclusion

An inverse boundary value problem for a linearized equations of longitudinal waves in rods with integral condition of the first kind are studied.

### Author Contributions

All authors contributed equally to this work.

### Conflict of Interest

The authors declare no conflict of interest.

### References

- 1 Samarskii, A.A. (1980). O nekotorykh problemakh teorii differentsialnykh uravnenii [On some problems of the theory of differential equations]. *Differentsialnye uravneniia — Differential Equations*, 16(11), 1925–1935 [in Russian].
- 2 Ionkin, N.I. (1977). Reshenie odnoi kraevoi zadachi teorii teploprovodnosti s neklassicheskim kraevym usloviiem [Solutions of boundary value problem in heat conduction theory with nonlocal boundary conditions]. *Differentsialnye uravneniia — Differential Equations*, 13(2), 294–304 [in Russian].
- 3 Nakhushiev, A.M. (1982). Ob odnom priblizhennom metode resheniia kraevykh zadach dlia differentsialnykh uravnenii i ego prilozheniia k dinamike pochvennoi vlagi i gruntovykh vod [An approximate method for solving boundary value problems for differential equations and its application to the dynamics of ground moisture and ground water]. *Differentsialnye uravneniia — Differential Equations*, 18(1), 72–81 [in Russian].
- 4 Kozhanov, A.I., & Pul'kina, L.S. (2006). On the solvability of boundary value problems with a nonlocal boundary condition of integral form for the multidimensional hyperbolic equations. *Differential Equations*, 42(9), 1233–1246. <https://doi.org/10.1134/S0012266106090023>
- 5 Kapustin, N.Yu., & Moiseev, E.I. (1997). O spektralnykh zadachakh so spektralnym parametrov v granichnom uslovii [On spectral problems with a spectral parameter in the boundary condition]. *Differentsialnye uravneniia — Differential Equations*, 33(1), 115–119 [in Russian].
- 6 Erofeev, V.I., Kazhaev, V.V., & Semerikova, N.P. (2002). *Volny v sterzhiakh: Dispersiia. Dissipatsiia. Nelineinost* [Waves in Rods: Dispersion. Dissipation. Nonlinearity]. Moscow: Fizmatlit [in Russian].
- 7 Denisov, A.M. (1994). *Vvedenie v teoriuu obratnykh zadach* [Introduction to Theory of Inverse Problem]. Moscow: MSU [in Russian].
- 8 Ivanchov, M.I. (2003). *Inverse Problems for Equation of Parabolic Type*. Ukraine: VNTL Publishers.
- 9 Megraliev, Ya.T. (2012). Obratnaia kraevaia zadacha dlia ellipticheskogo uravneniia vtorigo poriadka s dopolnitelnym integralnym usloviiem [Inverse boundary value problem for second order elliptic equation with additional integral condition] *Vestnik Udmurtskogo universiteta. Matematika. Mekhanika. Kompjuternye nauki — Bulletin of Udmurt University. Mathematics. Mechanics. Computer sciences*, (1), 32–40 [in Russian].
- 10 Megraliev, Ya.T. (2012). On an inverse boundary value problem for a second order elliptic equation with integral condition. *Visnyk of the Lviv Univ. Series Mech. Math.*, (77), 145–156.

- 11 Juraev, D.A. (2023). The Cauchy problem for matrix factorization of the Helmholtz equation in a multidimensional unbounded domain. *Boletim da Sociedade Paranaense de Matemática*, 41, 1–18. <https://doi.org/10.5269/bspm.63779>
- 12 Mehraliyev, Y.T., Allahverdiyeva, S., & Ramazanova, A.T. (2023). On one coefficient inverse boundary value problem for a linear pseudoparabolic equation of the fourth order. *AIMS Mathematics*, 8(2), 2622–2633. <https://doi.org/10.3934/math.2023136>
- 13 Mehraliyev, Y.T., Huntul, M.J., Ramazanova, A.T., Tamsir, M., & Emadifar, H. (2022). An inverse boundary value problem for transverse vibrations of a bar. *Boundary Value Problems*, 2022, 96. <https://doi.org/10.1186/s13661-022-01679-x>
- 14 Mehraliyev, Y.T., Ramazanova, A.T., & Huntul, M.J. (2022). An inverse boundary value problem for a two-dimensional pseudo-parabolic equation of third order. *Results in Applied Mathematics*, 14, 100274. <https://doi.org/10.1016/j.rinam.2022.100274>
- 15 Mehraliyev, Y.T., Sadikhzade, R., & Ramazanova, A.T. (2023). Two-dimensional inverse boundary value problem for a third-order pseudo-hyperbolic equation with an additional integral condition. *European Journal of Pure and Applied Mathematics*, 16(2), 670–686. <https://doi.org/10.29020/nybg.ejpam.v16i2.4743>
- 16 Juraev, D.A., Shokri, A., & Marian, D. (2022). Solution of the Ill-Posed Cauchy Problem for Systems of Elliptic Type of the First Order. *Fractal and Fractional*, 6(7), 358. <https://doi.org/10.3390/fractalfract6070358>
- 17 Mehraliyev, Y.T., Ramazanova, A.T., & Sevdimaliyev, Y.M. (2020). An inverse boundary value problem for the equation of flexural vibrations of a bar with an integral conditions of the first kind. *Journal of Mathematical Analysis*, 11(5), 1–12.
- 18 Farajov, A.S. (2022). On a Solvability of the Nonlinear Inverse Boundary Value Problem for the Boussinesq equation. *Advanced Mathematical Models & Applications*, 7(2), 241–248.

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