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Convergence of Double Fourier Series of Functions from Symmetric Spaces

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Abstract—We establish conditions for the convergence of double Fourier series in the trigonometric system of functions belonging to a symmetric space.

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In the present paper, we prove assertions on the convergence of double Fourier series in the trigonometric system of functions belonging to a symmetric space.

Suppose that $\overline{x} = (x_1, \dots, x_s) \in \mathbb{R}^s$, $I^s = [0, 2\pi)^s$, and $X(\psi)$ is a symmetric function space 2π-periodic in each variable x_j, $j = 1, \dots, s$, with the fundamental function ψ (see [1, p. 123]). The fundamental function $\varphi(t)$ of a symmetric space can be assumed concave, nondecreasing, and continuous on $[0, 1]$, with $\varphi(0) = 0$ (see [1, p. 137]). Such functions are called Φ-*functions*.

For the function $\psi(t)$, let

$$\alpha_\psi = \underline{\lim}_{t \to 0} \frac{\psi(2t)}{\psi(t)}, \qquad \beta_\psi = \overline{\lim}_{t \to 0} \frac{\psi(2t)}{\psi(t)}.$$

It is well known that if $X(\psi)$ is a symmetric space, then $1 \le \alpha_\psi \le \beta_\psi \le 2$.

As examples of symmetric spaces, consider the following:

- $L_q(I^s)$ is a Lebesgue space with norm

$$\|f\|_q = \left(\int_{I^s} |f(\overline{x}|^q \, d\overline{x} \right)^{1/q}, \qquad 1 \le q < \infty;$$

- $L_{\psi,\infty}(I^s)$ is a Marcinkiewicz space with norm

$$\|f\|_{\psi,\infty} = \sup_{\tau > 0} \frac{\psi(\tau)}{\tau} \int_0^\tau f^*(t) \, dt,$$

  where $f^*(t)$ is a nonincreasing rearrangement of the function $|f(\overline{x})|$, $\overline{x} \in I^s$, and $\psi(t)$ is a Φ-function (see [1, p. 137]).

Let $a_{\overline{n}}(f) = a_{n_1,\dots,n_s}(f)$ be the Fourier coefficients of a function $f \in L_1$ in the multiple system $\{\prod_{j=1}^s e^{i n_j x_j}\}$. By $C(\alpha, q, \dots)$ we denote positive constants depending on the parameters indicated above, which are, in general, different in different formulas.

In the proof of the main results, we shall use the following assertions.

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Lemma A ([2, Lemma 3]). *Let the Φ-functions $\varphi(x), \psi(x)$, $x \in [0,1]$, be given. If $1 < \beta_\psi < \alpha_\varphi < 2$, then, for the function*

$$\theta(x) = \begin{cases} \dfrac{\varphi(x)}{\psi(x)}, & x \in (0,1], \\ 0, & x = 0, \end{cases}$$

there exists a Φ-function $\theta_1(x)$ such that $\theta(x) \asymp \theta_1(x)$, $x \in (0,1]$, and $\alpha_{\theta_1} > 1$.

Lemma B ([2, Lemma 4]). *If $1 < \alpha_\psi \le \beta_\psi < 2$ for the Φ-functions $\psi(x)$, $x \in [0,1]$, then, for any $q \in (0,+\infty)$, the following relations hold:*

$$\int_0^x \frac{\psi^q(t)}{t}\,dt = O\big(\psi^q(x)\big), \qquad x \to +0,$$

$$\int_x^1 [t\psi^q(t)]^{-1}\,dt = O\big(\psi^{-q}(x)\big), \qquad x \to +0.$$

Let us present the main results.

Theorem 1. *Suppose that $f \in L_{\psi,\infty}(I^s)$, $\sqrt{2} < \alpha_\psi$, $\beta_\psi < 2$. Then the following inequality holds:*

$$\psi\left(\frac{1}{|Q|}\right)\left|\sum_{\overline{k}\in Q} a_{\overline{k}}(f)\right| \le C \cdot \|f\|_{\psi,\infty},$$

where Q is an arbitrary finite subset of $\mathbb{Z}^s$ and $|Q|$ is the number of its elements.

Proof. Suppose that $f \in L_{\psi,\infty}(I^s)$. Set

$$f_1(\overline{x}) = \begin{cases} f(\overline{x}) & \text{if } |f(\overline{x})| \le f^*(\tau), \quad \tau > 0, \\ 0 & \text{otherwise}, \end{cases} \qquad f_0(\overline{x}) = f(\overline{x}) - f_1(\overline{x}), \quad \overline{x} \in I^s.$$

It is well known that (see [3])

$$f_1^*(t) \le \begin{cases} f^*(\tau), & 0 \le t < \tau, \\ f^*(t), & \tau \le t, \end{cases} \qquad f_0^*(t) \le \begin{cases} f^*(t), & 0 \le t < \tau, \\ 0, & \tau \le t. \end{cases} \tag{1}$$

Using the property of equimeasurable functions, relations (1), and the definition of the Marcinkiewicz space, we can write

$$\|f_0\|_1 = \|f_0^*\|_1 \le \frac{\tau}{\psi(\tau)} \cdot \|f\|_{\psi,\infty}. \tag{2}$$

Using the definition of the function f_1, inequality (1), and Lemmas A and B, and invoking the condition $\sqrt{2} < \alpha_\psi \le \beta_\psi < 2$, we obtain

$$\|f_1\|_2 = \|f_1^*\|_2 \le C \cdot \frac{\sqrt{\tau}}{\psi(\tau)} \|f\|_{\psi,\infty}. \tag{3}$$

It follows from inequality (2) that

$$\left|\sum_{\overline{k}\in Q} a_{\overline{k}}(f_0)\right| \le \sum_{\overline{k}\in Q} \|f_0\|_1 \le c \cdot |Q| \cdot \frac{\tau}{\psi(\tau)} \|f\|_{\psi,\infty}. \tag{4}$$

Applying Hölder's inequality for the sum, Parseval's inequality, and estimate (3), we find

$$\left|\sum_{\overline{k}\in Q} a_{\overline{k}}(f_1)\right| \le \sqrt{|Q|}\left(\sum_{\overline{k}\in Q} |a_{\overline{k}}(f_1)|^2\right)^{1/2} = \sqrt{|Q|} \cdot \|f_1\|_2 \le C \cdot \sqrt{|Q|} \cdot \frac{\sqrt{\tau}}{\psi(\tau)} \|f\|_{\psi,\infty}. \tag{5}$$

From inequalities (4) and (5), we obtain

$$\Big|\sum_{\overline{k}\in Q} a_{\overline{k}}(f)\Big| \le c\frac{\sqrt{|Q|}}{\psi(\tau)}\big\{\sqrt{|Q|}\cdot\tau+\sqrt{\tau}\,\big\}\cdot\|f\|_{\psi,\infty}.$$

Setting $\tau=|Q|^{-1}$ in this inequality, we obtain the assertion of the theorem. □

Let us now study questions relating to the convergence of the series

$$\sum_{(n_1,n_2)\in\mathbb{N}^2} a_{n_1n_2}\cdot e^{in_1x}\cdot e^{in_2y}, \tag{6}$$

whose coefficients satisfy the conditions

$$a_{m_1m_2}\ge a_{n_1n_2} \qquad \text{for}\quad m_i\le n_i,\quad i=1,2, \tag{7}$$

$$a_{n_1n_2}\to 0 \qquad \text{for}\quad n_1+n_2\to+\infty. \tag{8}$$

Questions relating to the convergence of multiple trigonometric series with monotone coefficients were studied by Dyachenko [4] (see also the survey [5]).

The following assertions concerning the Fourier series of functions from symmetric spaces are valid.

Theorem 2. *Suppose that $X(\psi)$ is a symmetric space and $\sqrt{2}<\alpha_\psi,\ \beta_\psi<2^{3/4}$. If the Fourier coefficients of a function $f\in X(\psi)$ satisfy condition* (7)*, then its Fourier series is Pringsheim convergent almost everywhere on* $(0,2\pi)^2$.

Proof. By Lemma 3 from [4], it suffices to prove the convergence of the series

$$\sum_{(k_1,k_2)\in\mathbb{N}^2} \Delta a_{k_1k_2}(f)\cdot e^{ik_1x_1}\cdot e^{ik_2x_2}, \tag{9}$$

where

$$\Delta a_{nm}(f)=a_{nm}(f)-a_{n+1,m}(f)-a_{n,m+1}(f)+a_{n+1,m+1}(f).$$

Using condition (7), Theorem 1, and the inclusion $X(\psi)\subset L_{\psi,\infty}$ (see [1, p. 162]) we obtain

$$a_{n_1n_2}(f)\le C\cdot\|f\|_X\cdot[n_1n_2\cdot\psi(n_1^{-1}\cdot n_2^{-1})]^{-1}. \tag{10}$$

Since $\beta_\psi<2^{3/4}$ by the assumption of the theorem, there exists a number p_0 such that the inequality $\beta_\psi<2^{1/p_0}<2^{3/4}$ holds. Therefore, by Lemma B, we have

$$\Big(\frac{1}{n_1n_2}\Big)^{1/p_0}\cdot\Big[\psi\Big(\frac{1}{n_1n_2}\Big)\Big]^{-1}\le C \tag{11}$$

for any $(n_1,n_2)\in\mathbb{N}^2$.

Now we choose a number δ so that $2(1-1/p_0)(2-\delta)>1$. Such a number δ exists, because $4/3<p_0<2$.

Further, using inequalities (10), (11), and Lemma 5 from [4], we can write

$$\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}|\Delta a_{n_1n_2}(f)|^2\ln n_1\cdot\ln n_2\le C\cdot\sum_{n=1}^{\infty}a_{n,n}^{2-\delta}(f)\le C\cdot\sum_{n=1}^{\infty}\Big[\frac{1}{n^{2(1-1/p_0)}}\Big]^{2-\delta}<+\infty.$$

Therefore, by the Kaczmarz theorem (see [5, p. 105]), the series (9) is Pringsheim convergent a.e. on $(0,2\pi)^2$. Therefore, by Lemma 3, from [4] we obtain the assertion of the theorem. □

Remark 1. If $\sqrt{2}<\alpha_\psi,\ \beta_\psi<2$, then there exists a series (6) with coefficients satisfying conditions (7), (8) which is the Fourier series of some function $f\in X(\psi)$ divergent in squares on an everywhere dense subset of the square $(0,2\pi)^2$.

To justify the above assertion, it suffices to note that there exists a number p_0 satisfying the inequality $\sqrt{2} < 2^{1/p_0} < \alpha_\psi < 2$, i.e., $L_{p_0} \subset X(\psi)$, and then use Theorem 7 from [4].

Remark 2. If $2^{3/4} < \alpha_\psi,\ \beta_\psi < 2$, then there exists a series of the form (6) whose coefficients satisfy conditions (7) (8) and

$$\sup_{(n_1,n_2)\in\mathbb{N}^2}\left[n_1n_2\psi\left(\frac{1}{n_1n_2}\right)\right]a_{n_1n_2} < +\infty,$$

such that each sequence $\{S_{m_k,n_k}(x,y)\}$, where $m_k, n_k \to +\infty$ as $k \to +\infty$, is divergent almost everywhere on $(0,2\pi)^2$.

Explanation. Since $2^{3/4} < \alpha_\psi$, there exists a number p_0 such that $2^{3/4} < 2^{1/p_0} < \alpha_\psi < 2$, i.e., we have $L_{4/3} \subset L_{p_0} \subset X(\psi)$. Now, considering the series constructed in the proof of Theorem 8 from [4] and using the property of its coefficients and Lemma B, we can easily verify that

$$\sup_{(n_1,n_2)\in\mathbb{N}^2}\left[n_1n_2\psi\left(\frac{1}{n_1n_2}\right)\right]a_{n_1n_2} < C\cdot\sum_{k=4}^{\infty}2^{-k(2/p_0-3/2)} < +\infty.$$

It was proved in [4] that any sequence $\{S_{m_k,n_k}(x,y)\}$ is divergent almost everywhere on $(0,2\pi)^2$.

Remark 3. In the case $\psi(t) = t^{1/q}$, $1 < q < 2$, Theorem 1 was proved in [6, p. 117] for regular systems.

The conditions for the convergence of double Fourier series of functions from the Orlicz space follow from the results announced in [7].

The condition for the convergence of double trigonometric Fourier series from anisotropic Lebesgue spaces was established in [8].

Theorems 1 and 2 were announced in [9]. Theorem 1 in [9] contains a misprint: instead of $1/|Q|$, one should read $\psi(1/|Q|)$.

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