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The first boundary value problem for the fractional diffusion equation in a degenerate angular domain

M.I. Ramazanov[1], A.V. Pskhu[2], M.T. Omarov[1,*]

[1] *Institute of Applied Mathematics, Karaganda Buketov University, Karaganda, Kazakhstan;*
[2] *Institute of Applied Mathematics and Automation of Kabardin-Balkar Scientific Center of RAS, Nalchik, Russia*
(E-mail: ramamur@mail.ru, pskhu@list.ru, madiomarovt@gmail.com)

This article addresses the problems observed in branching fractal structures, where super-slow transport processes can occur, a phenomenon described by diffusion equations with a fractional time derivative. The characteristic feature of these processes is their extremely slow relaxation rate, where a physical quantity changes more gradually than its first derivative. Such phenomena are sometimes categorized as processes with "residual memory". The study presents a solution to the first boundary problem in an angular domain degenerating into a point at the initial moment of time for a fractional diffusion equation with the Riemann-Liouville fractional differentiation operator with respect to time. It establishes the existence theorem of the problem under investigation and constructs a solution representation. The need for understanding these super-slow processes and their impact on fractal structures is identified and justified. The paper demonstrates how these processes contribute to the broader understanding of fractional diffusion equations, proving the theorem's existence and formulating a solution representation.

Keywords: partial differential equation, fractional calculus, angular domain, kernel, weak singularity, parabolic cylinder, Carleman-Vekua equation, general solution, unique solution, Riemann-Lioville fractional operator.

2020 Mathematics Subject Classification: 35R11.

Introduction

The paper discusses an equation of the form:

$$\left(\frac{\partial^\alpha}{\partial t^\alpha} - \frac{\partial^2}{\partial x^2}\right) u(x,t) = f(x,t), \quad (0 < \alpha < 1), \tag{1}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is a fractional derivative of an order α with respect to the variable t, starting from the point $t = 0$. This type of fractional differentiation is defined by the Riemann-Liouville operator:

$$ {}_a g^{(\nu)}(x) \equiv {}_a\mathrm{D}_x^\nu g(x) = \begin{cases} \frac{1}{\Gamma(-\nu)} \int_a^x (x-\xi)^{-\nu-1} g(\xi)d\xi, & \nu < 0, \\ \frac{1}{\Gamma(1-\nu)} \mathrm{D}_x \int_a^x (x-\xi)^{-\nu} g(\xi)d\xi, & 0 \le \nu < 1, \\ \frac{1}{\Gamma(2-\nu)} \mathrm{D}_x^2 \int_a^x (x-\xi)^{-\nu+1} g(\xi)d\xi, & 1 \le \nu < 2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots \end{cases}$$

Fractional diffusion equations (where $0 < \alpha \le 1$) have been extensively studied in recent years. This surge in interest is due to their widespread applications in physics and modeling, as referenced in sources [1–5]. The primary methodologies for exploring diffusion-wave equations are detailed in publications [6–24], while monographs [25] and [26] provide a comprehensive bibliography on the subject.

*Corresponding author. *E-mail: madiomarovt@gmail.com*

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Nearly all studies related to equation (1) have focused on initial and boundary problems in both limited and unlimited cylindrical areas. Specifically, the first boundary problem for the fractional diffusion equation in a rectangular area was examined in [17,18]. In publication [27], the first boundary problem for the fractional diffusion-wave equation in a non-cylindrical area was solved. However, the area where the solution is sought does not degenerate into a point at the initial moment in time.

The aim of this study is to solve the first boundary problem for equation (1) in a domain that is not cylindrical, but rather angular, and degenerates into a point at the initial moment in time.

In relation to the boundary value problems for the heat conduction equation with a diffusion coefficient α set to 1:

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)u(x,t)=f(x,t),$$

these problems have been investigated in non-cylindrical domains by several authors [28–32]. It is important to underline that boundary value problems for the Laplace equation in domains with evolving boundaries are distinct from the classical ones defined in fixed cylindrical domains. The reason is that the dimensions of the domain where the solution is sought are time-dependent, which makes these problems unsuitable for classical variable separation and integral transformation methods.

The potential theory approach allows reformulating the boundary value problem into a Volterra system of second kind integral equations. In such cases, if the domain's boundary does not exist at the initial time, then the corresponding system of integral equations can be solved by the method of successive approximations due to the weak singularity of their kernels. In contrast, if the boundary exists at the initial time, the integral equations of the boundary value problem might admit additional solutions, and the implementation of the Picard method encounters certain mathematical complexities. Similar issues occur for boundary value problems of the Dirichlet problem for the Laplace equation in non-cylindrical domains that originate at the initial moment in time.

1 Problem Statement

To determine a regular solution for the fractional time-derivative heat equation:

$$\left(\frac{\partial^\alpha}{\partial t^\alpha}-\frac{\partial^2}{\partial x^2}\right)u(x,t)=f(x,t),\quad (0<\alpha<1),$$

within the domain

$$D=\{(x,t):0<x<t,0<t<\infty\},$$

that adheres to the boundary conditions:

$$u(0,t)=0,\quad u(t,t)=0,\quad 0<t<\infty. \tag{2}$$

We denote $u(x,t)$ as a regular solution of equation (1) in domain D such that:

$$t^{1-\gamma}u(x,t)\in C(\overline{D})$$

for some $\gamma>0$. Additionally, the solution $u(x,t)$ must be continuous within D and possess a continuous partial derivative with respect to x, and its second-order derivative with respect to x, ${}_0D_t^v u(x,y)$, must be continuous in the variable t at fixed x inside the domain D and up to the boundary set $\{0<x<t\}$, with $u(x,t)$ fulfilling equation (1) at every point in D.

2 Main Result

Definitions are introduced as follows:

$$\beta = \frac{\alpha}{2}, \quad \omega_{\beta,\mu}(x,t) = t^{\mu-1}W\left(-\beta, \mu; -\frac{|x|}{t^{\beta}}\right),$$

$$\omega(x,t) = \omega_{\beta,0}(x,t),$$

in which

$$W(-\beta, \mu; z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\mu - \beta k)}$$

represents the Wright function, as discussed in [33].

The following statement holds true:

Theorem 1. Let the conditions be satisfied: $t^{1-\gamma}g_i(t) \in C[0,T]$, $i = 1,2$, for some $\gamma > 0$, and $t^{1-\mu}f(x,t) \in C(\overline{D}), \mu \geq 0$, if $f(x,t)$ satisfies the Holder condition with respect to the variable x.

Then the solution to problem (1)-(2) exists and can be expressed as

$$u(x,t) = \int_0^t \psi_1(\tau)\omega(x, t-\tau)d\tau + \int_0^t \psi_2(\tau)\omega(T-x, t-\tau)d\tau + F(x,t), \tag{3}$$

here

$$F(x,t) = \frac{1}{2}\int_0^t \int_0^{\tau} f(s,\tau)\omega_{\beta,\mu}(x-s, t-\tau)dsd\tau,$$

and $\psi_1(t), \psi_2(t)$ from (3) are the solutions to the system of integral equations

$$\begin{cases} \psi_1(t) + \int_0^t \psi_2(\tau)\omega(\tau, t-\tau)d\tau = -F(0,t), \\ \psi_2(t) + \int_0^t \psi_1(\tau)\omega(t, t-\tau)d\tau + \int_0^t \psi_2(\tau)\omega(\tau - \mathrm{t}, t-\tau)d\tau = -F(t,t). \end{cases} \tag{4}$$

From the first equation of this system (4), we obtain

$$\psi_1(t) = -\int_0^t \psi_2(\tau)\omega(t, \tau - t)d\tau - F(0,t),$$

and substituting $\psi_1(t)$ into the second equation of the system (4), we get [33]:

$$\begin{gathered} \psi_2(t) + \int_0^t \left(-\int_0^{\tau} \psi_2(\xi)\omega(\xi, \tau-\xi)d\xi - F(0,\tau)\right)\omega(t, t-\tau)d\tau + \\ + \int_0^t \psi_2(\tau)\omega(\tau - t, t-\tau)d\tau = -F(t,t). \end{gathered} \tag{5}$$

Substituting into the repeated integral and changing the order of integration as well as the dummy variables ξ and τ, from equation (5), we arrive at a special integral equation of the second kind in the form of a Volterra equation:

$$\psi_2(t) - \int_0^t \mathcal{K}(t,\tau)\psi_2(\tau)d\tau = \mathcal{F}(t), \tag{6}$$

here

$$\mathcal{K}(t,\tau) = \int_{\tau}^t \omega_{\beta,0}(\tau, \xi-\tau)\omega_{\beta,0}(t, t-\xi)d\xi - \omega_{\beta,0}(\tau - t, t-\tau), \tag{7}$$

and

$$\mathcal{F}(t)=\int_0^t F(0,\tau)\omega(t,\tau-t)d\tau-F(t,t). \tag{8}$$

To perform the calculation of integral (6) having (7) and (8), it is necessary to employ.

$$\int_\tau^t \omega_{\beta,0}(\tau,\xi-\tau)\omega_{\beta,0}(t,t-\xi)d\xi.$$

The convolution formula is applied to the Wright function as referenced in [33], and this is expressed through the function $\omega_{\beta,\mu}(x,t)$:

$$\int_0^y \omega_{\alpha,\delta}\left(x_1,\xi\right)\omega_{\alpha,\mu}\left(x_2,(y-\xi)\right)d\xi=\omega_{\alpha,\delta+\mu}\left(x_1+x_2,y\right).$$

Then we obtain,

$$\begin{aligned}&\int_\tau^t \omega_{\beta,0}(\tau,\xi-\tau)\omega_{\beta,0}(t,t-\xi)d\xi=\|\xi-\tau=\eta\|=\\&=\int_0^{t-\tau}\omega_{\beta,0}(\tau,\eta)\omega_{\beta,0}(t,(t-\tau)-\eta)d\eta=\\&=\omega_{\beta,0}(t+\tau,t-\tau).\end{aligned}$$

Therefore, the conclusive kernel $\mathcal{K}_\beta(t,\tau)$ is determined by the following relation:

$$\mathcal{K}_\beta(t,\tau)=\omega_{\beta,0}(t+\tau,t-\tau)-\omega_{\beta,0}(\tau-t,t-\tau). \tag{9}$$

The second term of kernel (9) has a weak singularity, since the following estimate is valid for it:

$$\omega_{\beta,0}(\tau-t,t-\tau)\le\frac{C(\beta)}{(t-\tau)^\beta}. \tag{10}$$

Indeed, by applying the estimate found in [26]:

$$|\omega_\mu(x,y)|\leqslant C(\beta,\mu,\theta)|x|^{-\theta}y^{\beta\theta+\mu-1},$$
$$\theta\geqslant\begin{cases}0, & (-\mu)\notin\mathbb{N}\cup\{0\}\\-1, & (-\mu)\in\mathbb{N}\cup\{0\}\end{cases},$$

taking into account that $\mu=0$, and choosing

$$\theta=-\frac{\beta}{1-\beta}>-1.$$

This leads to the confirmation of inequality (10). Next, we aim to demonstrate the special nature of the kernel $\mathcal{K}_\beta(t,\tau)$.

Lemma. If $0<\beta\le 1/2$, the equality holds true

$$\lim_{t\to 0}\int_0^t \mathcal{K}_\beta(t,\tau)d\tau=1. \tag{11}$$

Proof. Initially, when t is small, these inequalities are applicable:

$$\omega_{\beta,0}(t, t-\tau) \geq \omega_{\beta,0}(t+\tau, t-\tau) \geq \omega_{\beta,0}(2t, t-\tau).$$

Using equation [33]

$$D^{v}_{0t}\omega_{\beta,\mu}(x, y) = \omega_{\beta,\mu-v}(x, y)$$

these results in

$$\lim_{t\to 0}\int_0^t \omega_{\beta,0}(bt, t-\tau)d\tau = \lim_{t\to 0}\omega_{\beta,1}(bt, t) = 1, \quad b = 1, 2.$$

Therefore, considering inequality (10), we establish the validity of equality (11).

The kernel's properties make it unsuitable for solving the corresponding integral equation through the method of successive approximations. This limitation of the integral equation is due to the solution domain for the problem collapsing to a single point at the start. Otherwise, if this collapse didn't occur, the kernel for the integral equation would possess a weak singularity, enabling the use of Picard's method for finding a solution [33].

3 Solution of the special integral equation (6)

To solve the integral equation mentioned in equation (6), we apply the Carleman-Vekua method. This involves using a specific integral equation, which we refer to as the characteristic equation.

$$\psi_2(t) - \int_0^t \mathcal{K}_{\frac{1}{2}}(t, \tau)\psi_2(\tau)d\tau = \mathcal{Q}(t) \tag{12}$$

here

$$\mathcal{K}_{1/2}(t, \tau) = \frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right) + \frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\}. \tag{13}$$

Relation (13) can be verified directly using the following formula [34; 5.2.10(2)] for $(n = -2)$,

$$\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!\Gamma[1+(n-k)/2]} = \frac{1}{\sqrt{\pi}}2^{(n+1)/2}e^{-x^2/8}D_{-n-1}\left(\frac{x\sqrt{2}}{2}\right),$$

here $D_{-n-1}(z)$ is the function of a parabolic cylinder.

At the same time, the kernel $\mathcal{K}_{\frac{1}{2}}(t, \tau)$ possesses a similar property as described in equation (11):

$$\lim_{t\to 0}\int_0^t \mathcal{K}_{\frac{1}{2}}(t, \tau)d\tau = 1.$$

This means that the kernel difference $\mathcal{K}_{\frac{1}{2}}(t, \tau) - \mathcal{K}_{\beta}(t, \tau) = \widetilde{K}(t, \tau)$ has a weak singularity. We will employ the regularization method to solve the characteristic equation, known as the Carleman-Vekua equation, and to do so, we will express equation (12) in a particular form:

$$\psi_2(t) - \int_0^t \mathcal{K}_{\frac{1}{2}}(t, \tau)\psi_2(\tau)d\tau = \mathcal{Q}(t) - \int_0^t \widetilde{K}(t, \tau)\psi_2(\tau)d\tau. \tag{14}$$

Assuming the right-hand side of this equality is temporarily known and denoting it by

$$\mathcal{Q}(t) = \mathcal{F}(t) - \int_0^t \widetilde{K}(t, \tau)\psi_2(\tau)d\tau.$$

Equation (14) can be represented in the following form:

$$\mathbb{K}_{1/2}\psi_2 \equiv \psi_2(t) - \int_0^t \mathcal{K}_{\frac{1}{2}}(t,\tau)\psi_2(\tau)d\tau = Q(t). \qquad (15)$$

In [35] it is shown that the general solution of equation (15) in the weight class of functions

$$\sqrt{t}\exp\left(-\frac{t}{4a^2}\right)\varphi\left(t\right) \in L_\infty\left(0,\infty\right)$$

has the form:

$$\mathbf{K}\psi_2 \equiv \psi_2(t) - \left[\mathbb{K}_{\frac{1}{2}}\right]^{-1}\mathcal{Q}(t) = c_0\psi_0(t) \qquad (16)$$

and the function

$$\psi_0(t) = \frac{1}{\sqrt{t}}\exp\left(-\frac{t}{4a^2}\right) + \frac{\sqrt{\pi}}{2a}\,\mathrm{erf}\left(\frac{\sqrt{t}}{2a}\right) + \frac{\sqrt{\pi}}{2a}$$

is the general solution of the corresponding homogeneous integral equation.

The integral equation (16) is already solvable by the method of successive approximations and the solution to the corresponding homogeneous equation will be determined by the equality:

$$\psi_{2,0}(t) = c_0[\mathbf{K}]^{\mathbf{-1}}\left[\psi_0(t)\right].$$

Similarly, as in the work [33], it is proven that function (6) is a solution to equation (1) and satisfies conditions (2), thus proving the validity of Theorem 1.

Conclusion

It is shown that in a non-cylindrical domain that degenerates at the initial moment of time into a point, the first boundary value problem for a fractional diffusion equation with the Riemann-Liouville fractional differentiation operator with respect to a time variable is singular, that is, it may not have a unique solution.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Бұрыштық жойылмалы аядағы бөлшекті диффузия теңдеуі үшін бірінші шекаралық есеп

М.И. Рамазанов[1], А.В. Псху[2], М.Т. Омаров[1]

[1] *Қолданбалы математика институты, Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды, Қазақстан;*
[2] *РҒА Кабардин-Балқар ғылыми орталығының Қолданбалы математика және автоматтандыру институты, Нальчик, Ресей*

Мақала тармақталған фракталды құрылымдарда байқалатын мәселелерді қарастырады, мұнда уақыт бойынша бөлшектік туындылары бар диффузиялық теңдеулермен сипатталатын өте баяу транспорттық процестер болуы мүмкін. Осы процестердің ерекше белгісі — олардың өте баяу релаксация жылдамдығы, мұнда физикалық шама оның бірінші туындысынан гөрі біртіндеп өзгереді. Мұндай құбылыстар кейде «қалдық жады» бар процестер ретінде жіктеледі. Зерттеуде уақыт бойынша Риман-Лиувилль бөлшектік дифференциалдау операторы бар бөлшектік диффузиялық теңдеу үшін бұрыштық облыста, бастапқы уақыт моментінде нүктеге дегенерацияланған бірінші шекаралық есептің шешімі ұсынылған. Онда зерттелетін есептің бар екендігі туралы теорема анықталған және есептің шешімі көрсетілген. Мақалада осындай өте баяу процестерді және олардың фракталды құрылымдарға әсерін түсінудің қажеттілігі атап өтілген. Жұмыс бөлшектік диффузиялық теңдеулердің кеңірек түсінілуіне осы процестердің қалай ықпал ететінін көрсетеді, теореманың бар екендігін дәлелдейді және есептің шешімін тұжырымдайды.

Кілт сөздер: дербес туынды теңдеу, бөлшек есептеу, бұрыштық облыс, ядро, әлсіз ерекшелік, параболикалық цилиндр, Карлеман-Векуа теңдеуі, жалпы шешім, жалғыз шешім, Риман-Лиувилльдің бөлшекті операторы.

Первая краевая задача для дробного диффузионного уравнения в угловой вырождающейся области

М.И. Рамазанов[1], А.В. Псху[2], М.Т. Омаров[1]

[1] *Институт прикладной математики, Карагандинский университет имени академика Е.А. Букетова, Караганда, Казахстан;*
[2] *Институт прикладной математики и автоматизации, Кабардино-Балкарский научный центр РАН, Нальчик, Россия*

Статья рассматривает проблемы, наблюдаемые в ветвящихся фрактальных структурах, где могут происходить сверхмедленные транспортные процессы; явление, описываемое диффузионными уравнениями с дробной производной по времени. Характерной особенностью этих процессов является их крайне медленная скорость релаксации, при которой физическая величина изменяется более постепенно, чем её первая производная. Такие явления иногда классифицируются как процессы с «остаточной памятью». В исследовании представлено решение первой краевой задачи в угловой области, вырождающейся в точку в начальный момент времени, для дробного диффузионного уравнения с оператором дробного дифференцирования Римана-Лиувилля по времени. В нем устанавливается теорема существования исследуемой задачи и строится представление решения. Авторами подчёркивается необходимость понимания этих сверхмедленных процессов и их влияния на фрактальные структуры. Работа демонстрирует, как эти процессы способствуют более широкому пониманию дробных диффузионных уравнений, доказывая существование теоремы и формулируя представление решения.

Ключевые слова: уравнение в частных производных, дробное исчисление, угловая область, ядро, слабая особенность, параболический цилиндр, уравнение Карлемана-Векуа, общее решение, единственное решение, дробный оператор Римана-Лиувилля.

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*Author Information**

Murat Ibraevich Ramazanov — Doctor of physical and mathematical sciences, Professor, Institute of Applied Mathematics, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *ramamur@mail.ru*; https://orcid.org/0000-0002-2297-5488

Arsen Vladimirovich Pskhu — Doctor of physical and mathematical sciences, Docent, A. D. of IAMA KBSC RAS, Head of the Department of Fractional Calculus, The Institute of Applied Mathematics and Automation of Kabardin-Balkar Scientific Center of RAS, st. Shortanova, 89 A, Nalchik, 360000, Kabardin-Balkar Republic, Russia; e-mail: *pskhu@list.ru*; https://orcid.org/0000-0002-0506-3516

Madi Tulegenovich Omarov (*corresponding author*) — Master of natural sciences, teacher, Institute of Applied Mathematics, Karaganda Buketov University, 28 Universitetskaya street, Karaganda, 100028, Kazakhstan; e-mail: *madiomarovt@gmail.com*; https://orcid.org/0000-0002-9026-5912

*The author's name is presented in the order: First, Middle and Last Names.