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TOTAL HARMONIC DISTORSIONS IN AN OSCILLATORY CIRCUIT WITH A FERROELECTRIC CAPACITOR WITH NEGATIVE CAPACITANCE

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An oscillating circuit including as a capacitor a two-layer ferroelectric structure exhibiting negative capacitance is considered. It is shown that the charge of such capacitor is described by the Duffing equation with a homoclinic figure of "eight". The dependence of the nonlinear distortion coefficient with respect to the voltage across the two-layer structure on the electromagnetic energy stored in the circuit is calculated. The asymptotics of the coefficient near the homoclinic figure of eight is determined. The orders of the physical quantities in the circuit are estimated.

Keywords: integrated ferroelectrics, Landau's theory of second-order phase transition, Jacobi functions, elliptic integrals, Fourier series, Parseval equation.

Introduction

Since the end of the last century, intensive researches within the framework of a new interdisciplinary scientific field have been globally carried out. This research area combines active dielectric materials with microelectronic production technologies. The field, called "integrated ferroelectrics", makes it possible to create a new generation of the elemental base of modern radio electronics, based on nonlinear effects in such compounds [1]. We should especially note that the market for devices based on such elements is not determined by records in achieving minimum topological standards due to the level of development of lithographic methods, but by the totality of our knowledge in the formation of a ferroelectric module [1].

A new drive in the development of integrated ferroelectrics is the discovery of two-layer ferroelectric systems exhibiting negative capacitance [2, 3] at room temperature. We will call such systems with negative capacitance NC capacitors. The voltage across the NC capacitor is related to q charge at its plates by the expression:

$$U = -\alpha \cdot q + \beta \cdot q^3, \quad (1)$$

which follows from the theory of ferroelectricity developed by V.L. Ginzburg in [4] within the framework of Landau's theory of second-order phase transitions.

The parameters α and β , incorporated in the formula (1), are considered positive and depend both on the properties of the materials forming the ferroelectric pair and ensuring the thermodynamic stability of the negative capacitance effect, and on the geometry of the NC capacitor [2, 3]. For the NC capacitor obtained in the experiments described in [2], $\alpha \sim 10^{10} \hat{A} \cdot \hat{E}\hat{e}^{-1}$ and $\beta \sim 0,5 \cdot 10^{29} \hat{A} \cdot \hat{E}\hat{e}^{-3}$.

This article continues started in [5] analysis of the processes of NC capacitors functioning in radio engineering generators of various types. Namely, the main element of such generators that is an oscillating circuit with a NC capacitor is described here.

1. The solution of the equation of motion for an oscillating circuit with a NC capacitor and the coefficient of nonlinear distortion for the first harmonic

The electric network of the oscillating circuit with a NC capacitor is shown in Fig. 1. Applying the second Kirchhoff law to it and using the formula (1), we obtain an ordinary differential equation for q charge of the NC-capacitor:

$$L \cdot \frac{d^2 q}{dt^2} - \alpha \cdot q + \beta \cdot q^3 = 0, \quad (2)$$

where L is the inductance value of the circuit.

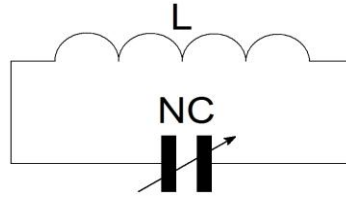


Fig.1. An oscillating circuit with a NC capacitor

Equation (2) is the Duffing equation with a homoclinic eight [6]. We make it dimensionless by introducing new variables:

$$\tau = \sqrt{\frac{\alpha}{L}} \cdot t, \quad x = \sqrt{\frac{\beta}{\alpha}} \cdot q \quad (3)$$

as well as dimensionless energy in the circuit:

$$h = \frac{H}{\alpha^2/\beta}, \quad (4)$$

where $H = \frac{L \cdot I_0^2}{2} - \frac{\alpha \cdot q_0^2}{2} + \frac{\beta \cdot q_0^4}{4}$ is the total energy in the circuit, which is preserved in consequence of equation (2), q_0 is the charge at the NC capacitor and I_0 is the current in terms of the inductance at the initial instant of time.

The typical scale of energy in the circuit is $\alpha^2/\beta \sim 1$ nJ. In variables (3), the equation (2) is written as follows:

$$\ddot{x} - x + x^3 = 0, \quad (5)$$

where the point above the dimensionless charge x means its differentiation with respect to the dimensionless time τ .

The solutions of the Duffing equation are expressed in terms of Jacobi elliptic functions [6], namely, in case that $-1/4 < h < 0$ the solution of equation (5) is [6]:

$$x(\tau) = A \cdot dn\left(\frac{2 \cdot \mathbb{K}(k_1) \cdot \tau}{T_1}, k_1\right), \quad (6)$$

where $A(h) = \sqrt{1 + \sqrt{1 + 4 \cdot h}}$, $k_1(h) = \sqrt{2 \cdot (1 - 1/A^2(h))}$, $T_1(h) = 2 \cdot \sqrt{2} \cdot \mathbb{K}(k_1(h))/A(h)$ is the dimensionless period of oscillations of a dimensionless charge at negative energy, and in the case that $h > 0$ the solution of equation (5) is [6]:

$$x(\tau) = A \cdot \operatorname{cn}\left(\frac{4 \cdot K(k_2) \cdot \tau}{T_2}, k_2\right), \tag{7}$$

where $k_2(h) = 1/k_1(h)$, $T_2(h) = 4 \cdot K(k_2(h)) / \sqrt{A^2(h) - 1}$ is the dimensionless period of oscillations of the dimensionless charge at positive energy. In both formulas (6) and (7) the starting point is chosen so that at that moment the current in the circuit is zero; and $K(k)$ means a complete elliptic integral of the first kind [7]:

$$K(k) = \int_0^1 \frac{dw}{\sqrt{(1-w^2) \cdot (1-k^2 \cdot w^2)}}. \tag{8}$$

Further, the dimensionless voltage at the NC capacitor is:

$$u(\tau) = -x(\tau) + x^3(\tau). \tag{9}$$

The graphs of dependence on the dimensionless voltage time (9) at the NC capacitor under negative and positive energy, constructed using formulas (6) and (7) are shown in Fig. 2. These graphs show that the voltage at the NC-capacitor is essentially anharmonic.

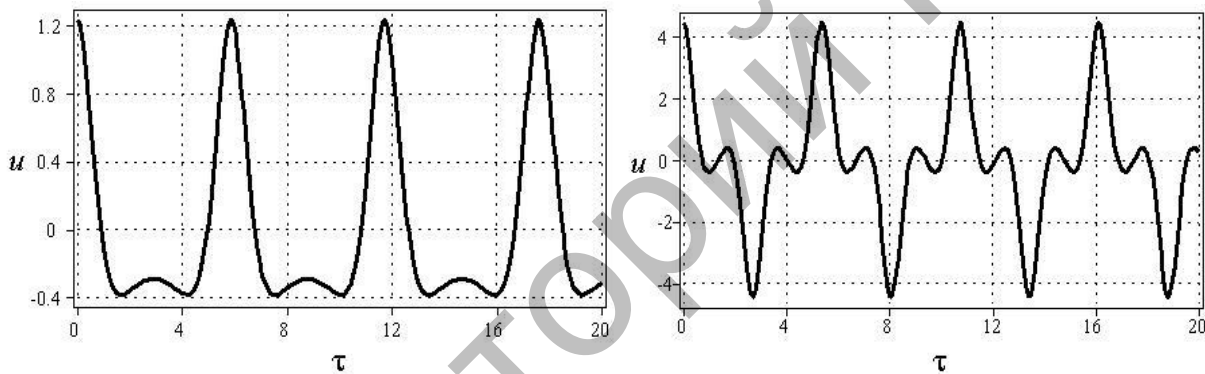


Fig. 2. Time dependence of the dimensionless voltage at the NC capacitor: on the left for $h = -0.05$, on the right for $h = 1.2$

In order that the circuit in Fig. 1 could be used in radio engineering, we should define the anharmonicity of the voltage quantitatively. The generally accepted parameter for estimating the anharmonicity of oscillations is the coefficient of nonlinear distortions (CND) with respect to the first harmonic [8]:

$$K_u(h) = \frac{1}{u_1(h)} \cdot \sqrt{\sum_{n=2}^{\infty} u_n^2(h)}, \tag{10}$$

where $u_n(h)$ ($n \in N$) are the amplitudes of Fourier harmonics of the dimensionless voltage (9).

2. Fourier series for voltage at the NC capacitor

In consequence of equation (5) $u(\tau) = -\ddot{x}(\tau)$, therefore, to determine the quantities $u_n(h)$, first we expand the solutions of (6) and (7) in Fourier series. The application of the theory of Jacobi elliptic functions [7] gives us for $-1/4 < h < 0$:

$$x(\tau, h) = x_0(h) + \sum_{n=1}^{\infty} x_n(h) \cdot \cos \frac{2 \cdot \pi \cdot n \cdot \tau}{T_1(h)}, \quad (11)$$

where

$$x_0(h) = \frac{\pi \cdot A(h)}{2 \cdot K(k_1(h))}, \quad x_n(h) = \frac{2 \cdot x_0(h)}{ch[n \cdot \gamma(k_1(h))]}, \quad (12)$$

and for $h > 0$:

$$x(\tau, h) = \sum_{n=1}^{\infty} x_n(h) \cdot \cos \frac{2 \cdot \pi \cdot (2 \cdot n - 1) \cdot \tau}{T_2(h)}, \quad (13)$$

where

$$x_n(h) = \frac{\pi \cdot A(h)}{k_2(h) \cdot K(k_2(h))} \cdot \frac{\exp[\gamma(k_2(h))/2]}{ch[n \cdot \gamma(k_2(h))]} \quad (14)$$

In the expressions (12) and (14) the γ parameter depends on the k modulus of elliptic functions as follows: $\gamma(k) = \pi \cdot K(\sqrt{1-k^2})/K(k)$

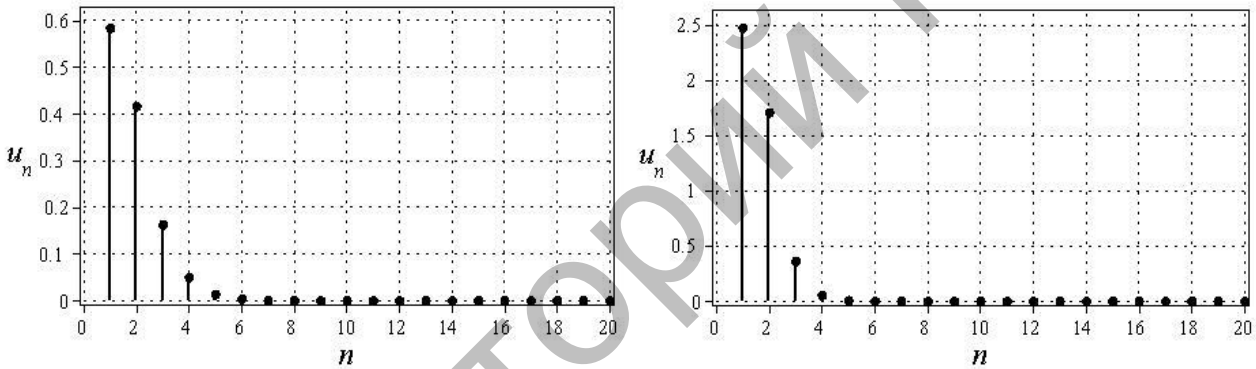


Fig. 3. Spectral components of the dimensionless voltage at the NC capacitor:
on the left for $h=-0.05$, on the right for $h=1.2$

The Fourier series for the stress (9) is obtained by twice differentiating with respect to expansion time (11) and (13), in particular for $-1/4 < h < 0$:

$$u(\tau, h) = \sum_{n=1}^{\infty} u_n(h) \cdot \cos \frac{2 \cdot \pi \cdot n \cdot \tau}{T_1(h)}, \quad (15)$$

where

$$u_n(h) = \left[\frac{2 \cdot \pi \cdot n}{T_1(h)} \right]^2 \cdot x_n(h), \quad (16)$$

and $x_n(h)$ are the Fourier coefficients (12) for the charge.

Similarly for $h > 0$:

$$u(\tau, h) = \sum_{n=1}^{\infty} u_n(h) \cdot \cos \frac{2 \cdot \pi \cdot (2 \cdot n - 1) \cdot \tau}{T_2(h)}, \quad (17)$$

where

$$u_n(h) = \left[\frac{2 \cdot \pi \cdot (2 \cdot n - 1)}{T_2(h)} \right]^2 \cdot x_n(h), \quad (18)$$

and $x_n(h)$ are Fourier coefficients (14) for the charge.

The graphs of the spectral components of the dimensionless voltage (9) at the NC capacitor under negative and positive energy, constructed using formulas (16) and (18), for the same energy values as in Fig. 2 are shown in Fig. 3.

Since for any of the expansions (15) or (17) Parseval's equality is valid:

$$\frac{2}{T} \cdot \int_0^T u^2(\tau, h) \cdot d\tau = \sum_{n=1}^{\infty} u_n^2(h), \quad (19)$$

then we can rewrite the expression (10) for CND as follows:

$$K_u(h) = \sqrt{2 \cdot \overline{\frac{u^2}{u_1^2}} - 1}, \quad (20)$$

where the bar over the letter indicates the operation of averaging with respect to the oscillation period T :

$$\overline{u^2} \equiv \frac{1}{T} \cdot \int_0^T u^2 d\tau. \quad (21)$$

The average value (21) can easily be calculated using the expression (9):

$$\overline{u^2} = \overline{x^2} - 2 \cdot \overline{x^4} + \overline{x^6}. \quad (22)$$

Thus, the calculation of the CND was reduced to the determination of the average values $\overline{x^{2l}}$ ($l = 1, 2, 3$) of the degrees of the dimensionless charge, and formulas (16) and (18) were simply necessary for determining the amplitude of the first harmonic $u_1(h)$.

3. Calculation of average values over a period for even degrees of charge

Using formula (6), for the case that $-1/4 < h < 0$ we find:

$$\overline{x^{2l}} = \frac{1}{T_1} \cdot \int_0^{T_1} A^{2l} \cdot dn^{2l} \left[\frac{2 \cdot K(k_1) \cdot \tau}{T_1}, k_1 \right] \cdot d\tau, \quad l = 1, 2, 3. \quad (23)$$

On inserting $w = sn \left[\frac{2 \cdot K(k_1) \cdot \tau}{T_1}, k_1 \right]$ the averages (23) reduce to the following integrals:

$$\overline{x^{2l}} = \frac{A^{2l}}{K(k_1)} \cdot \int_0^1 \frac{(1 - k_1^2 \cdot w^2)^l \cdot dw}{\sqrt{(1 - w^2) \cdot (1 - k_1^2 \cdot w^2)}}, \quad l = 1, 2, 3. \quad (24)$$

In their turn, the integrals in formula (24) are represented by linear combinations of elliptic integrals:

$$I_l(k) = \int_0^1 \frac{w^{2l} \cdot dw}{\sqrt{(1-w^2) \cdot (1-k^2 \cdot w^2)}}. \quad (25)$$

In case that $l \geq 2$ these integrals follow the recurrence relations [9]:

$$(2 \cdot l - 1) \cdot k^2 \cdot I_l - (2 \cdot l - 2) \cdot (k^2 + 1) \cdot I_{l-1} + (2 \cdot l - 3) \cdot I_{l-2} = 0. \quad (26)$$

Since the initial conditions for the difference equation (26) are known: $I_0(k) = \mathbf{K}(k)$,

$I_1(k) = [\mathbf{K}(k) - \mathbf{E}(k)]/k^2$, where $\mathbf{E}(k)$ denotes the complete elliptic integral of the second kind [7]:

$$\mathbf{E}(k) = \int_0^1 \sqrt{\frac{1-k^2 \cdot w^2}{1-w^2}} \cdot dw, \quad (27)$$

Then

$$I_2(k) = \frac{(2+k^2) \cdot \mathbf{K}(k) - 2 \cdot (1+k^2) \cdot \mathbf{E}(k)}{3 \cdot k^4},$$

$$I_3(k) = \frac{(8+3 \cdot k^2 + 4 \cdot k^4) \cdot \mathbf{K}(k) - (8+7 \cdot k^2 + 8 \cdot k^4) \cdot \mathbf{E}(k)}{15 \cdot k^6}, \quad (28)$$

and, using the formulas (24)-(28), we find the required average values $\overline{x^{2l}}$:

$$\overline{x^2} = A^2 \cdot \frac{\mathbf{E}(k_1)}{\mathbf{K}(k_1)}, \quad \overline{x^4} = \frac{A^4}{3} \cdot \left[k_1^2 - 1 + 2 \cdot (2 - k_1^2) \cdot \frac{\mathbf{E}(k_1)}{\mathbf{K}(k_1)} \right],$$

$$\overline{x^6} = \frac{A^6}{15} \cdot \left[-4 \cdot (2 - 3 \cdot k_1^2 + k_1^4) + (23 - 23 \cdot k_1^2 + 8 \cdot k_1^4) \cdot \frac{\mathbf{E}(k_1)}{\mathbf{K}(k_1)} \right]. \quad (29)$$

Finally, combining formulas (16), (22) and (29), we calculate the CND (20) for the dimensionless energy lying in the interval $-1/4 < h < 0$.

The situation when the dimensionless energy $h > 0$ is considered in complete analogy. Using the formula (7), for average values of even powers of the dimensionless charge, we obtain:

$$\overline{x^{2l}} = \frac{1}{T_2} \cdot \int_0^{T_2} A^{2l} \cdot cn^{2l} \left[\frac{4 \cdot \mathbf{K}(k_2) \cdot \tau}{T_2}, k_2 \right] \cdot d\tau, \quad l = 1, 2, 3. \quad (30)$$

On inserting $w = sn \left[\frac{4 \cdot \mathbf{K}(k_2) \cdot \tau}{T_2}, k_2 \right]$, the averages (30) reduce to the following integrals:

$$\overline{x^{2l}} = \frac{A^{2l}}{\mathbf{K}(k_2)} \cdot \int_0^1 \frac{(1-w^2)^l \cdot dw}{\sqrt{(1-w^2) \cdot (1-k_2^2 \cdot w^2)}}, \quad l = 1, 2, 3, \quad (31)$$

and are equals:

$$\begin{aligned} \overline{x^2} &= \frac{A^2}{k_2^2} \cdot \left[k_2^2 - 1 + \frac{E(k_2)}{K(k_2)} \right], & \overline{x^4} &= \frac{A^4}{3 \cdot k_2^4} \cdot \left[2 - 5 \cdot k_2^2 + 3 \cdot k_2^4 + 2 \cdot (2 \cdot k_2^2 - 1) \cdot \frac{E(k_2)}{K(k_2)} \right], \\ \overline{x^6} &= \frac{A^6}{15 \cdot k_2^6} \cdot \left[15 \cdot k_2^6 - 34 \cdot k_2^4 + 27 \cdot k_2^2 - 8 + (23 \cdot k_2^4 - 23 \cdot k_2^2 + 8) \cdot \frac{E(k_2)}{K(k_2)} \right]. \end{aligned} \quad (32)$$

CND (20) for dimensionless energy $h > 0$ is obtained by combining formulas (18), (22) and (32).

4. Graph of the nonlinear distortion coefficient and its asymptotics near the homoclinic eight

The final expression for CND is rather cumbersome; therefore, in Figure 4 the CND graph is presented.

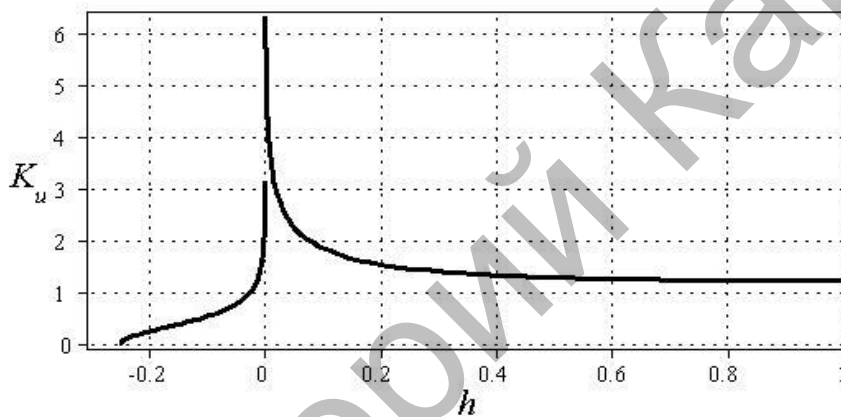


Fig. 4. The coefficient of nonlinear distortion depending on the energy h

Fig. 4 shows that $K_u(-1/4) = 0$. This is natural, since the Duffing equation (5), linearized near the equilibrium state $x = 1$, corresponding to the value $h = -1/4$, reduces to the harmonic oscillator equation, whose solution, as it is well known, does not incorporate higher harmonics.

For large values of energy $K_u(h) \approx 1$, that is, for $h \gg 1$, the squared amplitude of the first harmonic of the voltage (17) at the NC capacitor is approximately equal to the sum of the squared amplitudes of higher harmonics.

The nature of the CND inversion to infinity as the phase trajectory of Equation (5) approaches the homoclinic eight corresponding to the value $h = 0$, can be easily determined from the following considerations: in case that $h \rightarrow 0$ the oscillation period $T(h) \rightarrow +\infty$ [6]. This means that when $h \rightarrow 0$ the squared voltage (9) at the NC capacitor can be calculated with respect to the charge $x(\tau) = \sqrt{2}/ch\tau$ at the homoclinic loop. In formula (21) the integration can be extended to infinity and the figure of one in expression (20) can be neglected.

Thus, we get that near the zero energy the CND is:

$$K_u(h) \approx \begin{cases} \frac{1}{4 \cdot \pi^3} \cdot \sqrt{\frac{7}{30}} \cdot \left(\ln \frac{16}{|h|} \right)^{5/2}, & 0 < -h \ll 1 \\ \frac{1}{2 \cdot \pi^3} \cdot \sqrt{\frac{7}{15}} \cdot \left(\ln \frac{16}{h} \right)^{5/2}, & 0 < h \ll 1 \end{cases}. \quad (33)$$

Finally, since in accordance with the second of formulas (3) the dimensional voltage (1) at the NC capacitor is $U(t) = \frac{\alpha^{3/2}}{\beta^{1/2}} \cdot u(\tau)$, then the value (10) (or (20)) is the CND for the dimensional voltage (1) as well and it can be measured by existing network analyzers. The characteristic quantity of the dimensional voltage in the circuit $\alpha^{3/2} / \beta^{1/2} \sim 5 \hat{A}$.

Conclusion

In the paper, using the technique of calculating average values over oscillation period with respect to even powers of the solution of the Duffing equation (5), the authors investigated the dependence of the CND in terms of the voltage in the oscillating circuit with the NC capacitor on the energy stored in the circuit. It is necessary to know the average values (25) not only for analyzing the dependence of CND (20) on dimensionless energy h . Using the formulas (29) and (32), it is possible to determine other averages, for example, the average value of the electric energy in the NC capacitor, and the CND with respect to current, etc. By the developed above method for determining the averages over a period, the CND with respect to the voltage and current in oscillatory circuits with other nonlinear elements, such as a single-layer ferroelectric capacitor with an operating temperature above its Curie temperature, the blocked p-n junction of a semiconductor diode etc. can be calculated.

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