https://doi.org/10.31489/2025M1/187-198 *Research article*

Approximation of a singular boundary value problem for a linear differential equation

R. Uteshova[1,*], Y. Kokotova[2]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan*
(E-mail: r.uteshova@math.kz, kokotovae@mail.ru)

This paper addresses the approximation of a bounded (on the entire real axis) solution of a linear ordinary differential equation, where the matrix approaches zero as $t \to \mp\infty$ and the right-hand side is bounded with a weight. We construct regular two-point boundary value problems to approximate the original problem, assuming the matrix and the right-hand side, both weighted, are constant in the limit. An approximation estimate is provided. The relationship between the well-posedness of the singular boundary value problem and the well-posedness of an approximating regular problem is established.

Keywords: linear differential equation, bounded solution, singular boundary value problem, approximation, well-posedness, parameterization method.

2020 Mathematics Subject Classification: 34B05, 65L10.

Introduction

In many fields of applied mathematics, systems of ordinary differential equations that involve singularities or are defined over an infinite interval frequently occur. Numerous studies (see, for example, [1–8]) have explored the existence of bounded solutions for these types of problems and the approximation of these solutions.

In the present paper, we consider the differential equation

$$\frac{dx}{dt} = A(t)x + f(t), \quad x \in \mathbb{R}^n, \quad t \in (-\infty, \infty), \tag{1}$$

where the matrix function $A(t)$ is continuous on $\mathbb{R}$ and $\|A(t)\| := \max_j \sum_{k=1}^{n} |a_{jk}(t)| \le \alpha(t)$. We assume that $\alpha(t) > 0$ is a continuous function such that

$$\int_{-\infty}^{0} \alpha(t)dt = \infty, \quad \lim_{t\to-\infty} \alpha(t) = 0, \quad \int_{0}^{\infty} \alpha(t)dt = \infty, \quad \lim_{t\to\infty} \alpha(t) = 0.$$

As is known (see, e.g. [9]), the above assumption implies that equation (1) has a bounded solution not for any function $f(t)$ continuous and bounded on the whole axis. For this reason, in [10] the existence and uniqueness of a bounded solution of equation (1) was investigated under the assumption that $f(t)$ is continuous and bounded with a weight.

We will use the following notation:

*Corresponding author. *E-mail: r.uteshova@math.kz*

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. BR20281002).

Received: 16 July 2024; *Accepted:* 28 October 2024.

$\widetilde{C}(\mathbb{R},\mathbb{R}^n)$ is the space of continuous and bounded functions $x:\mathbb{R}\to\mathbb{R}^n$ equipped with the norm $\|x\|_1=\sup\limits_{t\in\mathbb{R}}\|x(t)\|$;

$\widetilde{C}_{1/\alpha}(\mathbb{R},\mathbb{R}^n)$ is the space of functions $f:\mathbb{R}\to\mathbb{R}^n$ that are continuous and bounded with the weight $1/\alpha(t)$, i.e. $f(t)/\alpha(t)\in\widetilde{C}(\mathbb{R},\mathbb{R}^n)$, equipped with the norm $\|f\|_\alpha=\sup\limits_{t\in\mathbb{R}}\|f(t)/\alpha(t)\|$.

Problem 1 is the problem of finding a bounded on the whole axis solution of equation (1) with $f(t)\in\widetilde{C}_{1/\alpha}(\mathbb{R},\mathbb{R}^n)$.

We say that Problem 1 is well-posed with constant K if it has a unique solution $x(t)\in\widetilde{C}(\mathbb{R},\mathbb{R}^n)$ for any $f(t)\in\widetilde{C}_\alpha(\mathbb{R},\mathbb{R}^n)$, and

$$\|x\|_1\le K\|f\|_\alpha,$$

where K is a constant independent of $f(t)$.

In [10], Problem 1 was studied by the parameterization method [11] with nonuniform partition $\mathbb{R}=\bigcup\limits_{s=-\infty}^{\infty}[t_{s-1},t_s)$. For a fixed number $\theta>0$, the partition points $t_s\in\mathbb{R}$, $s\in\mathbb{Z}$, are determined as

$$t_0=0,\qquad \int\limits_{t_{s-1}}^{t_s}\alpha(t)dt=\theta.$$

Let $\widetilde{h}(\theta)$ denote a bilaterally infinite sequence of partition step sizes $h_s(\theta)=t_s-t_{s-1}$, $s\in\mathbb{Z}$, i.e. $\widetilde{h}(\theta)=(\ldots,h_s(\theta),h_{s+1}(\theta),\ldots)$. We will use the following spaces:

m_n is the space of bilaterally infinite sequences of $\lambda_s\in\mathbb{R}^n$ equipped with the norm

$$\|\lambda\|_2=\|(\ldots,\lambda_s,\lambda_{s+1},\ldots)\|_2=\sup_s\|\lambda_s\|,\quad s\in\mathbb{Z};$$

$L(m_n)$ is the space of bounded linear operators mapping m_n to itself, equipped with the induced norm;

$m_n(\widetilde{h}(\theta))$ is the space of bounded bilaterally infinite sequences of functions $x_s(t)$, each of which is continuous and bounded on its domain $[t_{s-1},t_s)$, equipped with the norm

$$\|x[t]\|_3=\|(\ldots,x_s(t),x_{s+1}(t),\ldots)\|_3=\sup_s\sup_{t\in[t_{s-1},t_s)}\|x_s(t)\|,\quad s\in\mathbb{Z}.$$

Well-posedness criteria for Problem 1 were obtained in [10] in terms of a bilaterally infinite block-diagonal matrix $Q_{\nu,\widetilde{h}(\theta)}:m_n\to m_n$ of the form

$$Q_{\nu,\widetilde{h}(\theta)}=\left\|\begin{array}{ccccccc} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & 0 & I+D_{\nu,s}(h_s(\theta)) & -I & 0 & 0 & \ldots\\ \ldots & 0 & 0 & I+D_{\nu,s+1}(h_{s+1}(\theta)) & -I & 0 & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right\|,$$

where $D_{\nu,s}(h_s(\theta))=\int\limits_{t_{s-1}}^{t_s}A(\tau_1)d\tau_1+\ldots+\int\limits_{t_{s-1}}^{t_s}A(\tau_1)\ldots\int\limits_{t_{s-1}}^{\tau_{\nu-1}}A(\tau_\nu)d\tau_\nu\ldots d\tau_1$, $s\in\mathbb{Z}$, and I is the identity matrix of order n.

1 Statement of the problem of approximation. A criterion for the well-posedness of Problem 1

In this paper we consider the issue of approximation of Problem 1 by regular two-point boundary value problems. For this purpose, we pose the following problem.

Problem 2. For a given $\varepsilon > 0$ find numbers T_1, $T_2 > 0$, real $n \times n$ matrices B, C, and vector $d \in R^n$, such that a solution $x_{T_1,T_2}(t)$ of the two-point boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in (-T_1, T_2), \tag{2}$$

$$Bx(-T_1) + Cx(T_2) = d \tag{3}$$

satisfies the inequality

$$\max_{t \in [-T_1,T_2]} ||x_{T_1,T_2}(t) - x^*(t)|| < \varepsilon,$$

where $x^*(t)$ is a solution of Problem 1.

Problem 2 is considered under the following assumptions.

Assumption 1. $\lim\limits_{t\to\mp\infty} \frac{A(t)}{\alpha(t)} = A_{(\mp)}$, and Re $\xi_j^{\mp} \neq 0$, where $\xi_j^{\mp}$ are the eigenvalues of the matrices $A_{(\mp)}$, $j = 1, 2, \ldots, n$.

Assumption 2. $\lim\limits_{t\to\mp\infty} \frac{f(t)}{\alpha(t)} = f_{(\mp)}$.

We introduce the following functions:

$$\delta_1^-(T) := \sup_{t\in(-\infty,-T]} \left\| \frac{A(t)}{\alpha(t)} - A_{(-)} \right\|, \quad \delta_1^+(T) := \sup_{t\in[T,\infty)} \left\| \frac{A(t)}{\alpha(t)} - A_{(+)} \right\|,$$

$$\delta_2^-(T) := \sup_{t\in(-\infty,-T]} \left\| \frac{f(t)}{\alpha(t)} - f_{(-)} \right\|, \quad \delta_2^+(T) := \sup_{t\in[T,\infty)} \left\| \frac{f(t)}{\alpha(t)} - f_{(+)} \right\|.$$

Obviously, $\delta_r^{\mp}(T) \to 0$ as $T \to \infty$, $r = 1, 2$.

There exist nonsingular real $n \times n$ matrices $S_{(\mp)}$ that transform the matrices $A_{(\mp)}$ into the real Jordan canonical form [12]

$$\widetilde{A}_{(\mp)} = S_{(\mp)} A_{(\mp)} S_{(\mp)}^{-1} = \left\| \begin{array}{cc} A_{11}^{\mp} & 0 \\ 0 & A_{22}^{\mp} \end{array} \right\|, \tag{4}$$

where $A_{11}^{\mp}$ and $A_{22}^{\mp}$ consist of generalized Jordan blocks associated with the eigenvalues of $A_{(\mp)}$ that have negative and positive real parts, the numbers of which we denote by $n_1^{\mp}$ and $n_2^{\mp}$, respectively. We form the $n \times n$ matrices

$$P_1 = \left\| \begin{array}{cc} I_{n_1} & 0 \\ 0 & 0 \end{array} \right\|, \qquad P_2 = \left\| \begin{array}{cc} 0 & 0 \\ 0 & I_{n_2} \end{array} \right\|,$$

where I_{n_r} are the identity matrices of orders n_r, $r = 1, 2$.

The following statement establishes the interrelation between the well-posedness of Problem 1 and that of a two-point boundary value problem.

Theorem 1. Under Assumption 1, Problem 1 is well-posed if and only if:

(i) $n_1^- = n_1^+ = n_1$ and $n_2^- = n_2^+ = n_2$;

(ii) there exist T_0^1, $T_0^2 > 0$ such that for any $T_1 > T_0^1$, $T_2 > T_0^2$ the boundary value problem (2), (3) with $B = -P_1 S_{(-)}$ and $C = P_2 S_{(+)}$, is well-posed with a constant K_1 independent of T_1, T_2.

Proof. Necessity. Let Assumption 1 be fulfilled and let Problem 1 be well-posed. Then, by Theorem 3 [10], there exist $\theta_0 > 0$ such that the matrix $Q_{1,\widetilde{h}(\theta)}$ has an inverse for all $\theta \in (0, \theta_0]$, and the estimate $\|Q_{1,\widetilde{h}(\theta)}^{-1}\|_{L(m_n)} \leq \gamma/\theta$ holds, where γ is a constant independent of $\widetilde{h}(\theta)$. For a fixed $\theta > 0$ we choose T_1 and T_2, so that $t_{-N_1} = -T_1$ and $t_{N_2} = T_2$, and construct the matrix $Q_{1,\widetilde{h}(\theta)}$. In this matrix we then replace $A(t)$ by $\alpha(t)A_{(-)}$ in the block rows numbered $-N_1, -N_1 - 1, \ldots,$ and by $\alpha(t)A_{(+)}$ in the block rows numbered $N_2, N_2+1, \ldots$, and denote the resulting matrix by Q_{θ,T_1,T_2}. Assumption 1 implies that $\|Q_{1,\widetilde{h}(\theta)} - Q_{\theta,T_1,T_2}\|_{L(m_n)} \leq \max\{\delta_1^-(T_1), \delta_1^+(T_2 - h_N(\theta))\}\theta$. Hence, by the theorem on small perturbations of boundedly invertible linear operators, if we choose T_0^1, T_0^2 satisfying $\gamma \max\{\delta_1^-(T_0^1), \delta_1^+(T_0^2 - h_N(\theta))\} \leq 1/2$, we obtain that the matrix $Q_{\theta,T_1,T_2} : m_n \to m_n$ has an inverse for all $T_1 \geq T_0^1$ and $T_2 \geq T_0^2$, and the estimate

$$\|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \leq \frac{\gamma_{T_1,T_2}}{\theta} \leq \frac{2\gamma}{\theta}$$

holds. Here $\gamma_{T_1,T_2} = \dfrac{\gamma}{1 - \gamma \max\{\delta_1^-(T_1), \delta_1^+(T_2)\}} \to \gamma$ as $T_1 \to \infty$, $T_2 \to \infty$.

We form a bilaterally infinite matrix $D = \mathrm{diag}(d_{ss})$, where $d_{ss} = S_{(-)}$ for $s = 0, -1, -2, \ldots$, and $d_{ss} = S_{(+)}$ for $s = 1, 2, \ldots$. The matrix $\widetilde{Q}_{\theta,T_1,T_2} = DQ_{\theta,T_1,T_2}D^{-1}$ has a bounded inverse and

$$\|\widetilde{Q}^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \leq \|D^{-1}\|_{L(m_n)} \|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \|D\|_{L(m_n)} \leq \zeta_1 \gamma_{T_1,T_2} \zeta_2/\theta.$$

Here $\zeta_1 = \|D^{-1}\|_{L(m_n)} = \max\left(\|S^{-1}_{(-)}\|, \|S^{-1}_{(+)}\|\right)$ and $\zeta_2 = \|D\|_{L(m_n)} = \max\left(\|S_{(-)}\|, \|S_{(+)}\|\right)$. In the matrix $\widetilde{Q}_{\theta,T_1,T_2}$ the block rows numbered $s : s \leq -N_1$, $s \geq N_2$, are of the form

$$\left\| \begin{matrix} \ldots & 0 & I + \left\| \begin{matrix} A^{\mp}_{11} & 0 \\ 0 & A^{\mp}_{22} \end{matrix} \right\| \theta & -I & 0 & \ldots \end{matrix} \right\|.$$

Rearranging the blocks in $\widetilde{Q}_{\theta,T_1,T_2}$, we obtain the matrix

$$M_{\theta,T_1,T_2} = \left\| \begin{matrix} M_{11}(\theta) & 0 & 0 & 0 & 0 \\ 0 & M_{22}(\theta) & M_{23}(\theta) & 0 & 0 \\ M_{31}(\theta) & 0 & M_{33}(\theta) & 0 & M_{35}(\theta) \\ 0 & 0 & M_{43}(\theta) & M_{44}(\theta) & 0 \\ 0 & 0 & 0 & 0 & M_{55}(\theta) \end{matrix} \right\|.$$

The one-sided infinite matrices $M_{kk}(\theta)$, $k = 1, 2, 4, 5$, are of the form

$$M_{11}(\theta) = \left\| \begin{matrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & I_{n_1^-} + A^-_{11}\theta & -I_{n_1^-} & 0 \\ \ldots & 0 & 0 & I_{n_1^-} + A^-_{11}\theta & -I_{n_1^-} \end{matrix} \right\|,$$

$$M_{22}(\theta) = \left\| \begin{matrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & I_{n_2^-} + A^-_{22}\theta & -I_{n_2^-} & 0 \\ \ldots & 0 & 0 & I_{n_2^-} + A^-_{22}\theta & -I_{n_2^-} \\ \ldots & 0 & 0 & 0 & I_{n_2^-} + A^-_{22}\theta \end{matrix} \right\|,$$

$$M_{44}(\theta) = \left\| \begin{matrix} -I_{n_1^+} & 0 & 0 & 0 & \ldots \\ I_{n_1^+} + A^+_{11}\theta & -I_{n_1^+} & 0 & 0 & \ldots \\ 0 & I_{n_1^+} + A^+_{11}\theta & -I_{n_1^+} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix} \right\|,$$

$$M_{55}(\theta) = \left\| \begin{matrix} I_{n_2^+} + A^+_{22}\theta & -I_{n_2^+} & 0 & 0 & \ldots \\ 0 & I_{n_2^+} + A^+_{22}\theta & -I_{n_2^+} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{matrix} \right\|.$$

The matrix $M_{33}(\theta)$ of dimension $[(N_1+N_2-1)n+n_1^-+n_2^+]\times(N_1+N_2)n$ is of the form

$$M_{33}(\theta)=\left\|\begin{array}{ccccccc} -P_1^{(-)} & 0 & 0 & \dots & 0 & 0 & 0 \\ I+\widetilde{A}_{-N_1+1}(\theta) & -I & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I+\widetilde{A}_{N_2-1}(\theta) & -I & \\ 0 & 0 & 0 & \dots & 0 & 0 & P_2^{(+)}(I+\widetilde{A}_{(+)}\theta) \end{array}\right\|,$$

where $P_1^{(-)}=(I_{n_1^-},0)$ is a matrix of dimension, $n_1^-\times n$, $P_2^{(+)}=(0,I_{n_2^+})$ is a matrix of dimension $n_2^+\times n$,

$$\widetilde{A}_p(\theta)=\begin{cases} S_{(-)}\int\limits_{t_{p-1}}^{t_p}A(t)dtS_{(-)}^{-1}, & p=-N_1+1,-N_1+2,\dots,1,0, \\ S_{(+)}\int\limits_{t_{p-1}}^{t_p}A(t)dtS_{(+)}^{-1}, & p=1,2,\dots,N_2-1. \end{cases}$$

In the block row of $M_{33}(\theta)$ corresponding to $p=0$, the term $-I$ is replaced by $-S_{(-)}S_{(+)}^{-1}$.

The off-diagonal nonzero blocks of the matrix M_{θ,T_1,T_2} satisfy the relations

$$\|M_{31}(\theta)\|=\|I_{n_1^-}+A_{11}^-\theta\|,\quad \|M_{23}(\theta)\|=1,\quad \|M_{43}(\theta)\|=\|I_{n_1^+}+A_{11}^+\theta\|,\quad \|M_{35}(\theta)\|=1.$$

Due to the invertibility of $\widetilde{Q}_{\theta,T_1,T_2}$, the matrix M_{θ,T_1,T_2} is also invertible, and its inverse satisfies the estimate

$$\|M_{\theta,T_1,T_2}^{-1}\|_{L(m_n)}=\|\widetilde{Q}_{\theta,T_1,T_2}^{-1}\|_{L(m_n)}\le\frac{\zeta_1\gamma_{T_1,T_2}\zeta_2}{\theta}=\frac{\widetilde{\gamma}_{T_1,T_2}}{\theta}.$$

Following the proof scheme in [13], we establish the invertibility of the matrices $M_{kk}(\theta)$, $k=\overline{1,5}$, and the estimates

$$\|[M_{kk}(\theta)]^{-1}\|\le\left[\max_{r=1,2}\left(\|S_{r,\mp}\|,\|S_{r,\mp}^{-1}\|\right)\right]^2\frac{2}{\xi\theta}=\frac{\beta}{\theta},\quad k=1,2,4,5, \tag{5}$$

$$\|[M_{33}(\theta)]^{-1}\|\le\frac{\widetilde{\gamma}_{T_1,T_2}}{\theta}. \tag{6}$$

Here $\xi=\min\left\{|\mathrm{Re}\,\xi_j^{\mp}|,j=1,2,\dots,n\right\}$ and $S_{r,\mp}$ $(r=1,2)$ are nonsingular complex matrices of order $n_r^{\mp}$ reducing $A_{rr}^{\mp}$ to Jordan form with the eigenvalues on the diagonal and $\xi/4$ or zeros on the superdiagonal.

Since the matrix $M_{33}(\theta)$ of dimension $[(N_1+N_2-1)n+n_1^-+n_2^+]\times(N_1+N_2)n$ is invertible, it follows that $n_1^-+n_2^+=n$. In view of the structure of the matrices $\widetilde{A}_{(\mp)}$, we also have $n_1^-+n_2^-=n_1^++n_2^+=n$. Hence, $n_1^-=n_1^+=n_1$, $n_2^-=n_2^+=n_2$.

By rearranging of terms in the matrix $M_{33}(\theta)$, we obtain the invertible matrix

$$N_{33}(\theta)=\left\|\begin{array}{ccccccc} -P_1 & 0 & 0 & \dots & 0 & 0 & P_2(I+\widetilde{A}_{(+)}\theta) \\ I+\widetilde{A}_{-N_1+1}(\theta) & -I & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & I+\widetilde{A}_{N_2-1}(\theta) & -I \end{array}\right\|,$$

inverse of which, by (6), satisfies the estimate

$$\|[N_{33}(\theta)]^{-1}\|=\|[M_{33}(\theta)]^{-1}\|\le\frac{\widetilde{\gamma}_{T_1,T_2}}{\theta}\le\frac{2\widetilde{\gamma}}{\theta}.$$

Let D_{N_1,N_2} denote the block diagonal matrix consisting of blocks D numbered $s = -N_1, -N_1+1, \ldots, N_2-2, N_2-1$. By premultiplying each but the first block row of $N_{33}(\theta)D_{N_1,N_2}$ with $S^{-1}_{(-)}$ or $S^{-1}_{(+)}$, respectively, we obtain the matrix $V_1(\theta)$. Its inverse satisfies the estimate

$$\|[V_1(\theta)]^{-1}\| \le \max(1,\zeta_1)\zeta_2\|[N_{33}(\theta)]^{-1}\| \le \frac{2\widetilde{\gamma}\max(1,\zeta_1)\zeta_2}{\theta} = \frac{\gamma_1}{\theta},$$

where γ_1 is independent of T_1 and T_2. Hence, by following the proof scheme of Theorem 3 in [13] and considering the specifics of our partitioning, it can be shown that for all $T_1 \ge T_0^1$ and $T_2 \ge T_0^2$, the two-point boundary value problem (2), (3) with $B = -P_1 S_{(-)}$ and $C = P_2 S_{(+)}$ is well-posed with constant K_1 independent of T_1 and T_2.

Sufficiency. Let conditions (i) and (ii) be fulfilled and let $\widetilde{Q}_1(\theta)$ denote the matrix $N_{33}(\theta)$ with the first block row scaled by $\theta > 0$. Then, adapting Theorem 3 in [13] to our partitioning, we obtain that for any $\varepsilon > 0$ there exists $\theta_1 = \theta_1(\varepsilon) > 0$ such that the matrix $\widetilde{Q}_1(\theta)$ is invertible for all $\theta \in (0,\theta_1]$, and

$$\|[\widetilde{Q}_1(\theta)]^{-1}\| \le \frac{(1+\varepsilon)\zeta_1\zeta_2 K_1}{\theta} \le \frac{(1+\varepsilon)K_1}{\theta}. \tag{7}$$

The invertibility of $\widetilde{Q}_1(\theta)$ implies that of $M_{33}(\theta)$. Taking into account the bounded invertibility of the matrices $M_{kk}(\theta)$, $k = 1,2,4,5$, and the structure of the matrix M_{θ,T_1,T_2}, we obtain that the last one has a bounded inverse. Let us show that

$$\|M^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \le \frac{\widetilde{\gamma}}{\theta}, \tag{8}$$

where $\widetilde{\gamma}$ is constant independent of θ. To this end, we consider the equation

$$M_{\theta,T_1,T_2}\mu = b, \qquad \mu, b \in m_n, \tag{9}$$

which can be rewritten as the system

$$M_{11}(\theta)\mu^{(1)} = b^{(1)},$$

$$M_{22}(\theta)\mu^{(2)} + M_{23}(\theta)\mu^{(3)} = b^{(2)}, \tag{10}$$

$$M_{31}(\theta)\mu^{(1)} + M_{33}(\theta)\mu^{(3)} + M_{35}(\theta)\mu^{(5)} = b^{(3)}, \tag{11}$$

$$M_{43}(\theta)\mu^{(3)} + M_{44}(\theta)\mu^{(4)} = b^{(4)}, \tag{12}$$

$$M_{55}(\theta)\mu^{(5)} = b^{(5)}.$$

Here $\mu = (\mu^{(1)},\mu^{(2)},\mu^{(3)},\mu^{(4)},\mu^{(5)})$ and $b = (b^{(1)},b^{(2)},b^{(3)},b^{(4)},b^{(5)})$.

The bounded invertibility of $M_{11}(\theta)$, $M_{55}(\theta)$ and estimate (5) imply the existence of $\mu^{(1)} = [M_{11}(\theta)]^{-1}b^{(1)}$ and $\mu^{(5)} = [M_{55}(\theta)]^{-1}b^{(5)}$, as well as the estimates

$$\|\mu^{(1)}\| \le \frac{\beta}{\theta}\|b^{(1)}\|, \qquad \|\mu^{(5)}\| \le \frac{\beta}{\theta}\|b^{(5)}\|. \tag{13}$$

Let us now multiply by θ the first (from the bottom) block row in equation (10), the first (of dimension n_1) and the last (of dimension n_2) block rows in (11), and the first (from the top) block row in (12). We denote the matrices transformed in this way by $M_{22,\theta}$, $M_{23,\theta}$, $M_{31,\theta}$, $M_{33,\theta}$, $M_{35,\theta}$, $M_{43,\theta}$, $M_{44,\theta}$, the vectors by $b^{(2)}_\theta, b^{(3)}_\theta, b^{(4)}_\theta$ and the equations by $(10)'$, $(11)'$ and $(12)'$. Substituting the obtained sequences $\mu^{(1)}$ and $\mu^{(5)}$ into (11)′, we determine $\mu^{(3)}$. Taking into account $\|M_{33,\theta}^{-1}\| = \|[\widetilde{Q}_1(\theta)]^{-1}\|$ and estimate (7), we obtain

$$\|\mu^{(3)}\| = \|M_{33,\theta}^{-1}\left\{b_\theta^{(3)} - M_{31,\theta}[M_{11}(\theta)]^{-1}b^{(1)} - M_{35,\theta}[M_{55}(\theta)]^{-1}b^{(5)}\right\}\| \le$$
$$\le \frac{(1+\varepsilon)\widetilde{K}_1}{\theta}\left[\|b_\theta^{(3)}\| + (1+\zeta\theta)\beta\|b^{(1)}\| + \beta\|b^{(5)}\|\right] \le \frac{(1+\varepsilon)\widetilde{K}_1}{\theta}[1 + (2+\zeta\theta)\beta]\max_{k=1,3,5}\|b^{(k)}\|, \tag{14}$$

where $\zeta = [\max(\zeta_1, \zeta_2)]^2$. The one-sided infinite matrices $M_{22,\theta}$ and $M_{44,\theta}$ have bounded inverses, and

$$\|M_{22,\theta}^{-1}\| \le \beta\frac{\xi}{2}\max\left(\frac{2}{\xi}, 1\right)\frac{1}{\theta}, \quad \|M_{44,\theta}^{-1}\| \le \beta\frac{\xi}{2}\max\left(\frac{2}{\xi}, 1\right)\frac{1}{\theta}.$$

Substituting $\mu^{(3)}$ into (10) and (12), we determine $\mu^{(2)}$ and $\mu^{(4)}$ and obtain the estimates

$$\begin{aligned}\|\mu^{(2)}\| &\le \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)(\|b_\theta^{(2)}\| + \theta\|\mu^{(3)}\|) \le \\ &\le \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)\{1 + (1+\varepsilon)\widetilde{K}_1[1 + (2+\zeta\theta)\beta]\}\max_{k=1,2,3,5}\|b^{(k)}\|,\end{aligned} \tag{15}$$

$$\begin{aligned}\|\mu^{(4)}\| &\le \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)(\|b_\theta^{(4)}\| + (1+\zeta\theta)\theta\|\mu^{(3)}\|) \le \\ &\le \beta\frac{\xi}{2\theta}\max\left(\frac{2}{\xi}, 1\right)\{1 + (1+\varepsilon)\widetilde{K}_1(1+\zeta\theta)[1 + (2+\zeta\theta)\beta]\}\max_{k=2,3,4,5}\|b^{(k)}\|.\end{aligned} \tag{16}$$

Thus, for any $b \in m_n$ equation (9) has a unique solution $\mu \in m_n$, and, by (13)–(16), the estimate

$$\|\mu\|_2 \le \frac{K}{\theta}\|b\|_2$$

holds, where

$$K = \max\{\beta,\ (1+\varepsilon)\widetilde{K}_1[1 + (2+\zeta\theta)\beta],\ (\beta\xi/2)\max(2/\xi, 1)[1 + (1+\varepsilon)\widetilde{K}_1](1 + 2\beta + \zeta\beta\theta)]\}.$$

Hence, for any $\varepsilon_1 > 0$ choosing $\theta_2 = \theta_2(\varepsilon_1) > 0$ small enough, we obtain that estimate (8) with $\widetilde{\gamma} = \widetilde{K} + \varepsilon_1 = (\beta\xi/2)\max(2/\xi, 1)[1 + \widetilde{K}_1(1+2\beta)] + \varepsilon_1$ is valid for all $\theta \in (0, \theta_2]$. Under condition (ii) the constant K_1 does not depend of T_1 and T_2, as well as the constant $\widetilde{\gamma} = \widetilde{K} + \varepsilon_1$. Thus, taking into account the estimates

$$\|\widetilde{Q}_{1,\theta} - \widetilde{Q}_{\theta,T_1,T_2}\|_{L(m_n)} \le \delta_1(T_1, T_2 - h_N(\theta_2))\theta, \quad \|\widetilde{Q}_{\theta,T_1,T_2}^{-1}\|_{L(m_n)} = \|M_{\theta,T_1,T_2}^{-1}\|_{L(m_n)} \le \frac{\widetilde{K} + \varepsilon_1}{\theta},$$

and choosing T_0^1 and T_0^2 such that $(\widetilde{K} + \varepsilon_1)\zeta\delta_1(T_0^1, T_0^2 - h_N(\theta_2)) \le 1/2$, we obtain that $\widetilde{Q}_{1,\theta}$ is invertible and $\|\widetilde{Q}_{1,\theta}^{-1}\|_{L(m_n)} \le 2\widetilde{\gamma}/\theta$. It follows then that

$$\|\widetilde{Q}_{1,\theta}\|_{L(m_n)} \le \|D^{-1}\|_{L(m_n)}\|\widetilde{Q}_{1,\theta}^{-1}\|_{L(m_n)}\|D\|_{L(m_n)} \le 2\zeta\widetilde{\gamma}/\theta.$$

Thus, by Theorem 3 in [10], Problem 1 is well-posed for $\nu = 1$. This finishes the proof.

Application of Theorem 1 allows one to obtain effective well-posedness criteria for Problem 1. But condition (ii) somewhat narrows the scope of application, since it becomes necessary to check the well-posedness of the two-point boundary value problem for all T_1 and T_2. However, if we repeat the proof of the sufficiency part of Theorem 1 setting $T_1^0 = T_0^1$, $T_2^0 = T_0^2$ and using the introduced numbers β, ξ, ζ, and then pass in the right part of the inequality to the limit, we establish the following statement.

Theorem 2. Let Assumption 1 hold and the following conditions be met:

(i) $n_1^- = n_1^+ = n_1$ and $n_2^- = n_2^+ = n_2$;

(ii) there exist $T_1^0, T_2^0 > 0$ such that the boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t), \quad t \in (-T_1^0, T_2^0), \tag{17}$$

$$-P_1 S_{(-)} x(-T_1^0) + P_2 S_{(+)} x(T_2^0) = d \tag{18}$$

is well-posed with a constant K_1 satisfying the inequality $\widetilde{K}\zeta\delta_1(T_1^0, T_2^0) < 1$ with

$$\widetilde{K} = (\beta\xi/2)\max(2/\xi, 1)[1 + (1 + 2\beta)K_1\zeta].$$

Then Problem 1 is well-posed with the constant $K = \widetilde{K}\zeta/[1 - \widetilde{K}\zeta\delta_1(T_1^0, T_2^0)]$.

2 An approximating regular boundary value problem and the estimate for the approximation

The following theorem provides an approximating two-point boundary value problem and the estimate for the approximation.

Theorem 3. Under Assumptions 1 and 2, let Problem 1 be well-posed with constant K. Then for all $T_1 \geq T_0^1$ and $T_2 \geq T_0^2$, where $T_0^1, T_0^2 > 0$ are some constants determined by $K \max(\delta_1^-(T_0^1), \delta_1^+(T_0^2)) < 1$, the boundary value problem

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in (-T_1, T_2), \tag{19}$$

$$P_1 S_{(-)} A_{(-)} x(-T_1) + P_2 S_{(+)} A_{(+)} x(T_2) = -P_1 S_{(-)} f_{(-)} - P_2 S_{(+)} f_{(+)} \tag{20}$$

has a unique solution $x_{T_1,T_2}(t)$, and

$$\begin{gathered} \max_{t \in [-T_1, T_2]} \|x_{T_1,T_2}(t) - x^*(t)\| \leq \\ \leq \frac{K}{1 - K\max(\delta_1^-(T_1), \delta_1^+(T_2))}[K\|f\|_\alpha \max(\delta_1^-(T_1), \delta_1^+(T_2)) + \max(\delta_2^-(T_1), \delta_2^+(T_2))], \end{gathered} \tag{21}$$

where $x^*(t)$ is the solution of Problem 1.

Proof. We choose $\theta > 0$ and, applying the parameterization method, obtain that the solution $(\lambda^*, u^*(t)) \in m_n \times m_n(\widetilde{h}(\theta))$ of the boundary value problem with parameter (2)–(5) in [10] satisfies the equation

$$\left[I + \int\limits_{t_{s-1}}^{t_s} A(t)dt\right]\lambda_s^* + \lambda_{s+1}^* = -\int\limits_{t_{s-1}}^{t_s} f(t)dt - \int\limits_{t_{s-1}}^{t_s} A(t)u_s^*(t)dt, \quad s \in Z. \tag{22}$$

By Theorem 3 in [10], for any $\varepsilon > 0$ there exists $\overline{\theta} = \overline{\theta}(\varepsilon)$, such that the estimate $\|Q_{1,\widetilde{h}(\theta)}^{-1}\|_{L(m_n)} \leq \frac{(1+\varepsilon)K}{\theta}$ holds for all $\theta \in (0, \overline{\theta}]$, and, in addition,

$$\left\|\int\limits_{t_{s-1}}^{t_s} A(t)u_s^*(t)dt\right\| \leq c\theta^2, \quad s \in Z,$$

where $c = [1 + (1+\varepsilon)K]e^{\overline{\theta}}\|f\|_\alpha$, then the last term in (22) can be neglected for θ small enough. Let us separate the system (22) into three parts. Replacing $A(t), f(t)$ by $\alpha(t)A_{(-)}, \alpha(t)f_{(-)}$ for $s : s \le N_1$, and by $\alpha(t)A_{(+)}, \alpha(t)f_{(+)}$ for $s : s \ge N_2$, we obtain

$$(I + A_{(-)}\theta)\lambda_{r_1} - \lambda_{r_1+1} = -f_{(-)}\theta, \qquad r_1 = -N_1, -N_1 - 1, \ldots, \tag{23}$$

$$\left[I + \int\limits_{t_{r_2-1}}^{t_{r_2}} A(t)dt\right]\lambda_{r_2} + \lambda_{r_2+1} = -\int\limits_{t_{r_2-1}}^{t_{r_2}} f(t)dt, \qquad r_2 = -N_1 + 1, \ldots, N_2 - 1, \tag{24}$$

$$(I + A_{(+)}\theta)\lambda_{r_3} - \lambda_{r_3+1} = -f_{(+)}\theta, \qquad r_3 = N_2, N_2 + 1, \ldots \tag{25}$$

We rewrite this system in the form

$$Q_{\theta,T_1,T_2}\lambda = -F_{\theta,T_1,T_2}. \tag{26}$$

If we choose $\varepsilon > 0$ to satisfy the inequality, then, by the theorem on small perturbations of boundedly invertible operators, it follows that the matrix Q_{θ,T_1,T_2} is invertible, and its inverse satisfies the estimate

$$\|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)} \le \frac{(1+\varepsilon)K}{[1 - (1+\varepsilon)K\max(\delta_1^-(T_1), \delta_1^+(T_2))]\theta}. \tag{27}$$

Hence, by Assumptions 1 and 2, we obtain the estimate for the difference between λ^* and the solution λ_{T_1,T_2} of equation (26):

$$\begin{aligned}\|\lambda_{T_1,T_2} - \lambda^*\|_2 &\le \|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)}\|F_{\theta,T_1,T_2} - F_1(\widetilde{h}(\theta)) + [F_1(\widetilde{h}(\theta)) + Q_{\theta,T_1,T_2}\lambda^*]\|_2 = \\ &= \|Q^{-1}_{\theta,T_1,T_2}\|_{L(m_n)}\|F_{\theta,T_1,T_2} - F_1(\widetilde{h}(\theta)) + [Q_{1,\widetilde{h}(\theta)}\lambda^* + G_1(u^*, \widetilde{h}(\theta)) - Q_{\theta,T_1,T_2}\lambda^*]\|_2 \le \\ &\le \frac{(1+\varepsilon)K[\max(\delta_2^-(T_1), \delta_2^+(T_2)) + K\|f\|_\alpha \max(\delta_1^-(T_1), \delta_1^+(T_2)) + c\theta]}{1 - (1+\varepsilon)K\max(\delta_1^-(T_1), \delta_1^+(T_2))}.\end{aligned} \tag{28}$$

The components of λ_{T_1,T_2} numbered with $s : s \le N_1$ and $s \ge N_2$ satisfy equations (23) and (25), respectively. Hence, the corresponding components of the vector $\mu_{T_1,T_2} = D\lambda_{T_1,T_2}$ solve the equations

$$(I + \widetilde{A}_{(-)}\theta)\mu_{r_1} - \mu_{r_1+1} = -S_{(-)}f_{(-)}\theta, \qquad r_1 = -N_1, -N_1 - 1, \ldots,$$

$$(I + \widetilde{A}_{(+)}\theta)\mu_{r_3} - \mu_{r_3+1} = -S_{(+)}f_{(+)}\theta, \qquad r_3 = N_2, N_2 + 1, \ldots$$

Then, taking into account the decomposibility of the matrices $\widetilde{A}_{(-)}$ and $\widetilde{A}_{(+)}$, we obtain that $P_1^{(-)}\mu_{r_1}$ and $P_2^{(+)}\mu_{r_3}$ satisfy the equations

$$(I_{n_1^-} + A_{11}^-\theta)P_1^{(-)}\mu_{r_1} - P_1^{(-)}\mu_{r_1+1} = -P_1^{(-)}S_{(-)}f_{(-)}\theta, \tag{29}$$

$$(I_{n_2^+} + A_{22}^+\theta)P_2^{(+)}\mu_{r_3} - P_2^{(+)}\mu_{r_3+1} = -P_2^{(+)}S_{(+)}f_{(+)}\theta. \tag{30}$$

In the proof of Theorem 1 it was shown that the matrices $M_{11}(\theta)$ and $M_{55}(\theta)$ have bounded inverses. Thus, the one-sided infinite systems (29) and (30) have the unique solutions

$$P_1^{(-)}\mu_{-N_1+1} = P_1^{(-)}\mu_{-N_1+2} = \ldots = -[A_{11}^-]^{-1}P_1^{(-)}S_{(-)}f_{(-)},$$

$$P_2^{(+)}\mu_{N_2} = P_2^{(+)}\mu_{N_2+1} = \ldots = -[A_{22}^+]^{-1}P_2^{(+)}S_{(+)}f_{(+)}.$$

Returning to the variable λ, we obtain

$$A_{11}^{-}P_1^{(-)}S_{(-)}\lambda_{-N_1+1} = -P_1^{(-)}S_{(-)}f_{(-)}, \quad A_{22}^{+}P_2^{(+)}S_{(+)}\lambda_{N_2} = -P_2^{(+)}S_{(+)}f_{(+)}.$$

Then, in view of (4), we have

$$\begin{aligned} A_{11}^{-}P_1^{(-)}S_{(-)}\lambda_{-N_1+1} =& P_1^{(-)}\widetilde{A}_{(-)}S_{(-)}\lambda_{-N_1+1} = P_1^{(-)}S_{(-)}A_{(-)}S_{(-)}^{-1}S_{(-)}\lambda_{-N_1+1} \\ =& P_1^{(-)}S_{(-)}A_{(-)}\lambda_{-N_1+1} = -P_1^{(-)}S_{(-)}f_{(-)}, \\ P_2^{(+)}S_{(+)}A_{(+)}\lambda_{N_2} =& -P_2^{(+)}S_{(+)}f_{(+)}. \end{aligned}$$

These equations together with (24) constitute a closed system in parameters λ_{-N_1+1}, λ_{-N_1+2}, ..., λ_{N_2-1}, λ_{N_2}. If estimate (27) holds, the boundary value problem (17), (18) is well-posed for all $T_1 \geq T_0^1$, $T_2 \geq T_0^2$. Taking into account that (18) multiplied by $\begin{Vmatrix} -A_{11}^{-} & 0 \\ 0 & A_{22}^{+} \end{Vmatrix}$ yields the left-hand side of the boundary condition (20), we obtain that problem (19), (20) is well-posed for all $T_1 \geq T_0^1$, $T_2 \geq T_0^2$.

Let x_{T_1,T_2} be a solution of problem (19), (20), and let $[\lambda_{T_1,T_2}]_{N_1,N_2}$ be the vector composed of those components of $\lambda_{T_1,T_2} \in m_n$ that are numbered $s = -N_1+1, -N_1+2, \ldots, N_2-1, N_2$. Since

$$\max_s \sup_{t\in[t_{s-1},t_s)} \|x_{T_1,T_2} - [\lambda_{T_1,T_2}]_{N_1,N_2}\| \leq c_1\theta,$$

where c_1 is a constant independent of θ, we obtain, in view of (28), the following estimate:

$$\max_{t\in[-T_1,T_2]} \|x_{T_1,T_2}(t) - x^*(t)\| \leq \|[\lambda_{T_1,T_2}]_{N_1,N_2} - [\lambda^*]_{N_1,N_2}\| + (c+c_1)\theta \leq$$

$$\leq \frac{(1+\varepsilon)K[K\|f\|_\alpha \max(\delta_1^-(T_1), \delta_1^+(T_2)) + \max(\delta_2^-(T_1), \delta_2^+(T_2)) + c\theta]}{1-(1+\varepsilon)K\max(\delta_1^-(T_1), \delta_1^+(T_2))} + (c+c_1)\theta.$$

Passing to the limit as $\theta \to 0$, we obtain (21). Theorem 3 is proved.

Conclusion

By approximating Problem 1 with a two-point boundary value problem and utilizing well-known results, we developed an approximate method for finding the bounded solution. The form of matrices P_1 and P_2 indicates that the approximating problem involves separated boundary conditions. Theorem 2 allows one to establish the well-posedness of the singular boundary value problem (Problem 1) using the well-posedness constant K_1 of the two-point boundary value problem, the eigenvalues $\xi_j^{\mp}$ of the limit matrices $A_{\mp}$, and the nonsingular matrices $S_{(\mp)}$. This approach provides a robust framework for addressing similar singular problems.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant no. BR20281002).

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Fazio, R. (2002). A survey on free boundary identification of the truncated boundary in numerical BVPs on infinite intervals. *Journal of Computational and Applied Mathematics, 140*(1-2), 331–344. https://doi.org/10.1016/S0377-0427(01)00526-X

2 Ronto, M., & Samoilenko, A.M. (2000). *Numerical-Analytic Methods in the Theory of Boundary-Value Problems.* World Scientific. https://doi.org/10.1142/3962

3 Mitropolsky, Yu.A., Samoilenko, A.M., & Kulik, V.L. (2002). *Dichotomies and Stability in Nonautonomous Linear Systems.* Taylor & Francis Ltd, United Kingdom. https://doi.org/10.1201/9781482264890

4 Dzhumabaev, D.S. & Uteshova, R.Ye. (2018). Weighted limit solution of a nonlinear ordinary differential equation at a singular point and its property. *Ukrainian Mathematical Journal, 69*(12), 1997–2004. https://doi.org/10.1007/s11253-018-1483-2

5 Muratbekov, M.B., & Bayandiyev, Ye.N. (2022). Existence and smoothness of solutions of a singular differential equation of hyperbolic type. *Bulletin of the Karaganda University. Mathematics Series,* 3(107), 98–104. https://doi.org/10.31489/2022M3/98-104

6 Yesbayev, A.N., & Ospanov, M.N. (2021). The solvability conditions for the second order nonlinear differential equation with unbounded coefficients in $L_2(R)$. *Bulletin of the Karaganda University. Mathematics Series,* 1(101), 104–110. https://doi.org/10.31489/2021M1/104-110

7 Uteshova, R.Ye., & Kokotova, Ye.V. (2022). On bounded solutions of linear systems of differential equations with unbounded coefficients. *Bulletin of the Karaganda University. Mathematics Series,* 4(108), 107–116. https://doi.org/10.31489/2022M4/107-116

8 Assanova, A.T., & Uteshova, R.E. (2021). A singular boundary value problem for evolution equations of hyperbolic type. *Chaos, Solitons and Fractals, 143*, article number 110517. https://doi.org/10.1016/j.chaos.2020.110517

9 Daleckii, Ju.L., & Krein, M.G. (1974). *Stability of solutions of differential equations in Banach space.* American Mathematical Society.

10 Uteshova, R.E., & Kokotova, Y.V. (2023). On the well-posedness of a singular problem for linear differential equations. *Lobachevskii Journal of Mathematics, 44*(7), 2961–2969. https://doi.org/10.1134/S1995080223070430

11 Dzhumabaev, D.S. (1989). Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. *USSR Computational Mathematics and Mathematical Physics, 29*(1), 34–46. https://doi.org/10.1016/0041-5553(89)90038-4

12 Horn, R.A., & Johnson, C.R. (2013). *Matrix Analysis.* 2nd Edition, Cambridge University Press.

13 Dzhumabaev, D.S. (1990). Approximation of the bounded solution of an ordinary linear differential equation by solutions of two-point boundary-value problems. *USSR Computational Mathematics and Mathematical Physics, 30*(2), 34–45. https://doi.org/10.1016/0041-5553(90)90074-3

Author Information[*]

Roza Uteshova (*corresponding author*) — Candidate of sciences in physics and mathematics, Associate Professor, Institute of Mathematics and Mathematical Modeling, 125 Pushkin Street, Almaty, 050010, Kazakhstan; e-mail: *r.uteshova@math.kz*; https://orcid.org/0000-0002-8809-9310

Yelena Kokotova — Candidate of sciences in physics and mathematics, K. Zhubanov Aktobe Regional University, 34 Moldagulova Avenue, Aktobe, 030000, Kazakhstan; e-mail: *kokotovae@mail.ru*; https://orcid.org/0009-0007-9625-3334

[*]The author's name is presented in the order: First, Middle and Last Names.