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The Ortho-Diameters of Nikol'skii and Besov Classes in the Lorentz Spaces

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Abstract—In this paper we estimate the order of approximation of S. M. Nikol'skii and O. V. Besov classes in the norm of the anisotropic Lorentz space. We also obtain bounds for ortho-diameters of these classes.

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Let $\mathbb{R}^m$ be an m-dimensional Euclidean space of points $\overline{x} = (x_1, \dots, x_m)$ with real coordinates and let $I^m = [0, 2\pi)^m$ be the m-dimensional unit cube. Let $\overline{q} = (q_1, \dots, q_m)$, $\overline{\theta} = (\theta_1, \dots, \theta_m)$, $1 \le q_j < +\infty$, $1 \le \theta_j < +\infty$, $j = 1, \dots, m$, be given.

Denote by $L^*_{\overline{q},\overline{\theta}}(I^m)$ the anisotropic Lorentz space of Lebesgue measurable 2π-periodic functions $f(\overline{x})$, for which the value

$$\|f\|^*_{\overline{q},\overline{\theta}} = \left[\int_0^{2\pi} t_m^{\frac{\theta_m}{q_m}-1}\left[\cdots\left[\int_0^{2\pi}\left(f^{*_1,\dots,*_m}(t_1,\dots,t_m)\right)^{\theta_1} t_1^{\frac{\theta_1}{q_1}-1}dt_1\right]^{\frac{\theta_2}{\theta_1}}\cdots\right]^{\frac{\theta_m}{\theta_{m-1}}}dt_m\right]^{\frac{1}{\theta_m}}$$

is finite, where $f^{*_1,\dots,*_m}(t_1, \dots, t_m)$ is a nonincreasing permutation of the function $|f(\overline{x})|$ with respect to each variable x_j with fixed other variables (first with respect to x_1, then with respect to x_2, etc.) ([1, 2]).

For $1 \le p < +\infty$ denote by $L_p(I^m)$ the Lebesgue space with the norm

$$\|f\|_p = \left[\int_0^{2\pi}\cdots\int_0^{2\pi}|f(\overline{x})|^p dx_1\dots dx_m\right]^{\frac{1}{p}} < +\infty;$$

let $L_\infty(I^m)$ be the space of essentially bounded Lebesgue measurable functions with the norm ([3], P. 12)

$$\|f\|_\infty = \sup_{\overline{x}\in I^m} \operatorname{vrai}|f(\overline{x})|.$$

Let us expand functions $f \in L_1(I^m)$ such that

$$\int_0^{2\pi} f(\overline{x})dx_j = 0 \ \ \forall j = 1, \dots, m,$$

in the Fourier series

$$f(\overline{x}) \sim \sum_{\overline{n}\in\overset{\circ}{\mathbb{Z}}{}^m} a_{\overline{n}}(f)e^{i\langle\overline{n},\overline{x}\rangle},$$

where $a_{\overline{n}}(f)$ are the Fourier coefficients with respect to the multiple trigonometric system $\{e^{i\langle\overline{n},\overline{x}\rangle}\}$ and $\overset{\circ}{\mathbb{Z}}{}^m = \{\overline{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : k_j \ne 0, j = 1, \dots, m\}$.

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Репозиторий КарГУ

Denote by

$$\Omega_{\overline{l}}(f,\overline{t})_{\overline{p},\overline{\theta}} \equiv \Omega_{\overline{l}}(f,t_1,\dots,t_m)_{\overline{p},\overline{\theta}} = \sup_{\substack{0\le h_j\le t_j\\ j=1,\dots,m}} \left\|\Delta_{\overline{h}}^{\overline{l}} f\right\|_{\overline{p},\overline{\theta}}^{*}$$

the mixed module of smoothness of the function $f \in L^{*}_{\overline{p},\overline{\theta}}(I^m)$, where

$$\Delta_{\overline{h}}^{\overline{l}} f(\overline{x}) = \Delta_{h_1}^{l_1}(\dots\Delta_{h_m}^{l_m} f(x_1,\dots,x_m))$$

is the mixed difference of the function f.

Further we use the following denotations:

$$\delta_{\overline{s}}(f,\overline{x}) = \sum_{\overline{n}\in\rho(\overline{s})} a_{\overline{n}}(f) e^{i\langle\overline{n},\overline{x}\rangle},$$

where $\langle\overline{y},\overline{x}\rangle = \sum\limits_{j=1}^{m} y_j x_j$,

$$\rho(\overline{s}) = \{\overline{k} = (k_1,\dots,k_m)\in\mathbb{Z}^m : 2^{s_j-1} \le |k_j| < 2^{s_j},\ \ j=1,\dots,m\}.$$

Let vectors $\overline{r} = (r_1,\dots,r_m)$, $\overline{l} = (l_1,\dots,l_m)$, and natural numbers $l_j > r_j > 0$, $j = 1,\dots,m$, be given. Consider the class of S. M. Nikol'skii $H^{\overline{r}}_{\overline{p},\overline{\theta}}$ that consists of all functions $f \in L^{*}_{\overline{p},\overline{\theta}}(I^m)$ such that

$$\Omega_{\overline{l}}(f,\overline{t})_{\overline{p},\overline{\theta}} \le \prod_{j=1}^{m} t_j^{r_j}, \quad \overline{t}\in[0,1]^m.$$

The numerical sequence $\{a_{\overline{n}}\}_{\overline{n}\in Z^m} \in l_{\overline{p}}$ has the norm

$$\|\{a_{\overline{n}}\}_{\overline{n}\in Z^m}\|_{l_{\overline{p}}} = \left\{ \sum_{n_m=-\infty}^{\infty} \left[\dots \left[\sum_{n_1=-\infty}^{\infty} |a_{\overline{n}}|^{p_1} \right]^{\frac{p_2}{p_1}} \dots \right]^{\frac{p_m}{p_{m-1}}} \right\}^{\frac{1}{p_m}} < +\infty,$$

where $\overline{p} = (p_1,\dots,p_m)$, $1 \le p_j < +\infty$, $j = 1,2,\dots,m$. The class of O. V. Besov (by analogy with [4, 5]) has the form

$$B^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}} = \bigl\{ f \in L^{*}_{\overline{p},\overline{\theta}}(I^m) : \|f\|_{B^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}} = \|f\|^{*}_{\overline{p},\overline{\theta}} + \|\{2^{\langle\overline{s},\overline{r}\rangle}\|\delta_{\overline{s}}(f)\|^{*}_{\overline{p},\overline{\theta}}\}\|_{l_{\overline{\tau}}} \le 1 \bigr\}.$$

Note that in the case $\overline{p} = (p,\dots,p)$ instead of $L^{*}_{\overline{p},\overline{\theta}}(I^m)$, $H^{\overline{r}}_{\overline{p},\overline{\theta}}$, and $B^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}$ we write $L^{*}_{p,\overline{\theta}}(I^m)$, $H^{\overline{r}}_{p,\overline{\theta}}$, and $B^{\overline{r}}_{p,\overline{\theta},\overline{\tau}}$, respectively.

Let a vector $\overline{\gamma} = (\gamma_1,\dots,\gamma_m)$, $\gamma_j > 0$ be given. Put

$$Q_n^{\overline{\gamma}} = \bigcup_{\langle\overline{s},\overline{\gamma}\rangle<n} \rho(\overline{s});$$

$S_n^{\overline{\gamma}}(f,\overline{x}) = \sum\limits_{\overline{k}\in Q_n^{\overline{\gamma}}} a_{\overline{k}}(f) e^{i\langle\overline{k},\overline{x}\rangle}$ is a partial sum of the Fourier series of the function f.

Denote by $C(\alpha,\beta,\dots)$ positive values that depend on parameters written in parenthesis. In general, these values are different in different formulas.

For brevity, we use the notation $A \asymp B$ which means that positive constants c_1 and c_2 exist such that $c_1 A \le B \le c_2 A$.

In this paper, we estimate the ortho-diameters of functional classes in the Lorentz spaces with an anisotropic metric. The notion of the ortho-diameter of classes was introduced by V. N. Temlyakov [6]. He also obtained estimates for ortho-diameters of classes of S. L. Sobolev and S. M. Nikol'skii in

Репозиторий КарГУ

Lebesgue spaces [6, 7]. This research was later developed in papers of Din' Zung [8], E. M. Galeev [9], and N. N. Pustovoitov [10].

Recall the definition of the ortho-diameter. Let a set $F \subset L^*_{\overline{p},\overline{\theta}}$ be given. We understand the ortho-diameter of the set F as the value

$$d^{\perp}_M(F, L^*_{\overline{p},\overline{\theta}}) = \inf \sup_{f \in F} \Big\| f - \sum_{j=1}^{M} \langle f, u_j \rangle u_j \Big\|^*_{\overline{p},\overline{\theta}},$$

where inf is taken with respect to all orthonormalized systems $\{u_j\}_{j=1}^M$ of bounded functions. V. N. Temlyakov [6, 7] also considered the following value related to the ortho-diameter:

$$d^B_M(F, L^*_{\overline{p},\overline{\theta}}) = \inf_{G \subset \mathbb{L}_M(B)_{\overline{p},\overline{\theta}}} \sup_{f \in F} \|f - Gf\|^*_{\overline{p},\overline{\theta}}.$$

Here $B \geq 1$, $\mathbb{L}_M(B)_{\overline{p},\overline{\theta}}$ is the set of linear operators G such that their domain $D(G)$ contains all trigonometric polynomials and the set of values has the dimension M and belongs to the space $L^*_{\overline{p},\overline{\theta}}(I^m)$; in addition, for all $\overline{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$ the inequality

$$\|G e^{i\langle \overline{k}, \overline{x} \rangle}\|_2 \leq B$$

is fulfilled. The operators of orthogonal projection on a subspace of dimension M belong to $\mathbb{L}_M(1)_2$. The definitions of d^B and $d^{\perp}$ imply the inequality

$$d^B_M(F, L^*_{\overline{p},\overline{\theta}}) \leq d^{\perp}_M(F, L^*_{\overline{p},\overline{\theta}}).$$

In [7] for the class

$$H^{\overline{r}}_p = \Big\{ f \in L_p(I^m) : \Omega_{\overline{l}}(f, \overline{t})_p \leq \prod_{j=1}^{m} t_j^{r_j}, \quad \overline{t} \in [0,1]^m \Big\}$$

V. N. Temlyakov proved the following theorem.

Theorem A. *Let* $1 \leq p < q < +\infty$, $r_1 = \cdots = r_\nu < r_{\nu+1} \leq \cdots \leq r_m$. *Then the following correlation takes place*:

$$d^{\perp}_M(H^{\overline{r}}_p, L_q) \asymp M^{-(r_1 + \frac{1}{q} - \frac{1}{p})} (\log M)^{(\nu-1)(r_1 + \frac{2}{q} - \frac{1}{p})}.$$

Let us prove a generalization of this theorem for the Lorentz spaces with an anisotropic metric.

Theorem 1. *Let* $1 \leq p_j < q_j < +\infty$, $1 \leq \theta_j < +\infty$, $j = 1, \dots, m$; $q_1 = \cdots = q_\nu < q_{\nu+1} \leq \cdots \leq q_m$; $r_1 = \cdots = r_\nu < r_{\nu+1} \leq \cdots \leq r_m$; $r_1(\frac{1}{p_j} - \frac{1}{q_j}) < r_j(\frac{1}{p_1} - \frac{1}{q_1})$, $j = \nu+1, \dots, m$. *Then the following correlation takes place*:

$$d^{\perp}_M(H^{\overline{r}}_{\overline{p},\overline{\theta}}, L^*_{\overline{q},\overline{\theta}}) \asymp M^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} (\log M)^{(\nu-1)(r_1 + \frac{1}{q_1} - \frac{1}{p_1}) + \sum_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

We need the following assertions in order to prove the main results of this paper.

Theorem B ([11]). *Let* $1 \leq p_j < q_j < +\infty$, $1 \leq \theta_j < +\infty$, $j = 1, \dots, m$; $\frac{1}{p_1} - \frac{1}{q_1} < r_1 = \cdots = r_\nu < r_{\nu+1} \leq \ldots \leq r_m$, $\gamma_j = \frac{r_j}{r_1}$ $\forall j = 1, \dots, m$; $\frac{1}{p_j} - \frac{1}{q_j} = \frac{1}{p_1} - \frac{1}{q_1}$ $\forall j = 1, \dots, \nu$; $r_1(\frac{1}{p_j} - \frac{1}{q_j}) < r_j(\frac{1}{p_1} - \frac{1}{q_1})$, $j = \nu+1, \dots, m$. *Then the following correlation takes place*:

$$\sup_{f \in H^{\overline{r}}_{\overline{p},\overline{\theta}}} \|f - S^{\overline{\gamma}}_n(f)\|^*_{\overline{q},\overline{\theta}} \leq C(p,q,m,\theta) 2^{-n(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

Репозиторий КарГУ

Theorem C ([12]). *Let* $\overline{\theta} = (\theta_1, \dots, \theta_m)$, $\overline{\tau} = (\tau_1, \dots, \tau_m)$, $\overline{p} = (p_1, \dots, p_m)$, $\overline{q} = (q_1, \dots, q_m)$, $\overline{r} = (r_1, \dots, r_m)$, $\gamma_j = \frac{r_j}{r_1}$, $j = 1, \dots, m$, *and* $1 \le \theta_j, \tau_j < +\infty$, $1 \le p_j < q_j < +\infty$, $\frac{1}{p_j} - \frac{1}{q_j} < r_j$, $j = 1, \dots, m$, $r_1 = \dots = r_\nu < r_{\nu+1} \le \dots \le r_m$, $\frac{1}{p_1} - \frac{1}{q_1} = \dots = \frac{1}{p_\nu} - \frac{1}{q_\nu}$, $r_1(\frac{1}{p_j} - \frac{1}{q_j}) < r_j(\frac{1}{p_1} - \frac{1}{q_1})$, $j = \nu + 1, \dots, m$. *Then the following estimate is true*:

$$\sup_{f \in B^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}}} \|f - S_n^{\overline{\gamma}}(f)\|^*_{\overline{q},\overline{\theta}} \le C\,(p, q, \theta, r)$$
$$\times \begin{cases} 2^{-n\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} n^{\sum_{j=2}^{\nu}\left(\frac{1}{\theta_j} - \frac{1}{\tau_j}\right)}, & \theta_j < \tau_j,\ j = 1, \dots, m; \\ 2^{-n\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)}, & \tau_j \le \theta_j,\ j = 1, \dots, m; \\ 2^{-n\left(r_1 + \frac{1}{q_1} - \frac{1}{p_1}\right)} n^{\sum_{j=l+1}^{\nu}\left(\frac{1}{\theta_j} - \frac{1}{\tau_j}\right)}, & \tau_j \le \theta_j < +\infty,\ j = 1, \dots, l < \nu, \\ & \theta_j < \tau_j < +\infty,\ j = l+1, \dots, m. \end{cases}$$

Lemma 1 ([13]). *Let* $p_1 = \dots = p_\nu > p_{\nu+1} \ge \dots \ge p_m \ge 1$, $1 \le \theta_j < \infty$, $\theta_j' = \frac{\theta_j}{\theta_j - 1}$, $j = 1, \dots, m$. *Then the Dirichlet kernel* $D_{Q_n}(\overline{x}) = \sum_{\overline{k} \in Q_n} e^{i\langle \overline{k}, \overline{x}\rangle}$ *satisfies the inequality*

$$\|D_{Q_n}\|_{\overline{p},\overline{\theta}} \le C(p, \theta, m) 2^{\frac{n}{p_1}} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}.$$

Lemma 2 (ibid). *Let* $1 \le p_j < \infty$, $1 < \theta_j \le \infty$, $\theta_j' = \frac{\theta_j}{\theta_j - 1}$, $j = 1, \dots, m$ *and* $p_1 = \dots = p_\nu < p_{\nu+1} \le \dots \le p_m$. *Then any trigonometric polynomial in the form*

$$t_n(\overline{x}) = \sum_{\overline{k} \in Q_n} c_{\overline{k}} e^{i\langle \overline{k}, \overline{x}\rangle}$$

satisfies the inequality

$$\|t_n\|_\infty \le C(p, \theta, m) \|t_n\|_{\overline{p},\overline{\theta}} 2^{\frac{n}{p_1}} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j'}}.$$

Lemma 3 ([12]). *Let* $\overline{\tau} = (\tau_1, \dots, \tau_m)$, $1 \le \tau_j < +\infty$, $j = 1, \dots, m$, χ_{σ_n} *be the characteristic function of the set* $\sigma_n = \{\overline{s} : \langle \overline{s}, \overline{1}\rangle = n\}$. *Then the following correlation takes place*:

$$\left\|\{\chi_{\sigma_n}(\overline{s})\}_{\overline{s} \in \sigma_n}\right\|_{l_{\overline{\tau}}} \asymp C(\tau) n^{\sum_{j=2}^{m} \frac{1}{\tau_j}}.$$

This lemma is proved in [12] by the method of mathematical induction with respect to the dimension m.

Proof of Theorem 1. Choose a natural number $n \in \mathbb{N}$ so that $M \asymp 2^n n^{\nu-1}$.

In accordance with the definition of the ortho-diameter and Theorem B we have

$$d_M^{\perp}(H^{\overline{r}}_{\overline{p},\overline{\theta}}, L^*_{\overline{q},\overline{\theta}}) \le \sup_{f \in H^{\overline{r}}_{\overline{p},\overline{\theta}}} \|f - S_n^{\gamma}(f)\|_{\overline{q},\overline{\theta}} \le C(p, q, \theta, m) 2^{-n(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}}. \tag{1}$$

Since $M \asymp 2^n n^{\nu-1}$, we obtain

$$2^{-n(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}} \asymp M^{-(r_1 + \frac{1}{q_1} - \frac{1}{p_1})} (\log M)^{(\nu-1)(r_1 + \frac{1}{q_1} - \frac{1}{p_1}) + \sum_{j=2}^{\nu} \frac{1}{\theta_j}}. \tag{2}$$

Therefore from (1) we get the upper bound for the value $d_M^{\perp}(H^{\overline{r}}_{\overline{p},\overline{\theta}}, L^*_{\overline{q},\overline{\theta}})$ in Theorem 1.

Репозиторий КарГУ

Let us prove the lower bound. Since $r_1 = \cdots = r_\nu < r_{\nu+1} \le \cdots \le r_m$, we have $H^{r_m}_{\overline{p},\overline{\theta}} \subset H^{\overline{r}}_{\overline{p},\overline{\theta}}$. Hence $d^{\perp}_M(H^{\overline{r}}_{\overline{p},\overline{\theta}}, L^*_{\overline{q},\overline{\theta}}) \ge d^{\perp}_M(H^{r_m}_{\overline{p},\overline{\theta}}, L^*_{\overline{q},\overline{\theta}})$. Therefore, it suffices to prove the lower bound for $r_1 = \cdots = r_m = r$. To this end, let us estimate from below the value $d^B_M(H^r_{\overline{p},\overline{\theta}}, L^*_{\overline{q},\overline{\theta}})$.

Let an operator $G \in \mathbb{L}_M(B)_{\overline{q},\overline{\theta}}$ be given. Consider the operator $A = (S_{n-1} - S_{n-2})$. Then $A \in \mathbb{L}(B)_{\overline{q},\overline{\theta}}$. Choose a natural number n so as to meet the correlations

$$C(m)|Q_n| > 2B(M|Q_n|)^{\frac{1}{2}}, \quad |Q_n| \asymp M.$$

Due to the boundedness of the partial sum operator in the space $L^*_{\overline{q},\overline{\theta}}(I^m)$ ([2]) $\forall f \in T(Q_n)$ we have

$$\|f - Af\|^*_{\overline{q},\overline{\theta}} = \|(S_{n-1} - S_{n-2})(f - Gf)\|^*_{\overline{q},\overline{\theta}} \le \|f - Gf\|^*_{\overline{q},\overline{\theta}}. \tag{3}$$

Set ([7], P. 82)

$$\overline{\Phi}_{\overline{s}}(\overline{x}) = e^{i\langle \overline{k}^{\overline{s}}, \overline{x}\rangle} 2^m \prod_{j=1}^m \Phi_{2^{s_j-2}}(x_j),$$

$$k_j^{\overline{s}} = \begin{cases} 2^{s_j-1} + 2^{s_j-2}, & s_j \ge 2; \\ 1, & s_j = 1, \end{cases} \quad j = 1, \dots, m,$$

where $\Phi_l(t)$ is the Fejér kernel of the order $l-1$, $\Phi_{\frac{1}{2}}(t) = \frac{1}{2}$. Then $\|\overline{\Phi}_{\overline{s}}\|_1 = 1$.

Consider the function

$$\varphi(\overline{x}) = \sum_{\overline{s} \in \sigma_n} \prod_{j=1}^m 2^{-(1-\frac{1}{p_j})s_j} \overline{\Phi}_{\overline{s}}(\overline{x}),$$

where $\sigma_n = \{\overline{s} : \langle \overline{s}, \overline{1}\rangle = n\}$ and the set $Q_n = \bigcup_{\overline{s} \in \sigma_n} \rho(\overline{s})$. Then

$$\varphi(\overline{0}) = \sum_{\overline{s} \in \sigma_n} \prod_{j=1}^m 2^{-s_j(1-\frac{1}{p_j})} \overline{\Phi}_{\overline{s}}(\overline{0}) = 2^{-3m} \sum_{\overline{s} \in \sigma_n} \prod_{j=1}^m 2^{\frac{s_j}{p_j}}.$$

Put $\delta_j = \frac{p_1}{p_j}$, $j = 1, \dots, m$. If $\delta_j = 1$, $j = 1, \dots, \nu$, then, taking into account the inequality

$$\sum_{s=1}^n 2^{s\alpha} \ge 2^\alpha, \quad \alpha \in (-\infty, +\infty),$$

we obtain

$$\sum_{\overline{s} \in \sigma_n} \prod_{j=1}^m 2^{\frac{1}{p_j}s_j} = \sum_{\overline{s} \in \sigma_n} 2^{\frac{1}{p_1}\langle \overline{s}, \overline{\delta}\rangle} = \sum_{s_m < n} \sum_{s_{m-1} < n - s_m} \cdots \sum_{s_2 < n - \sum_{j=3}^m s_j} \sum_{s_1 = n - \sum_{j=2}^m s_j} \prod_{j=1}^m 2^{s_j \delta_j \frac{1}{p_1}}$$

$$= \sum_{s_m < n} \sum_{s_{m-1} < n - s_m} \cdots \sum_{s_2 < n - \sum_{j=3}^m s_j} \prod_{j=2}^m 2^{\frac{1}{p_1} s_j \delta_j} 2^{\frac{1}{p_1}(n - \sum_{j=2}^m s_j)}$$

$$\ge C(p,m) 2^{n \frac{1}{p_1}} \sum_{s_m < n} \sum_{s_{m-1} < n - s_m} \cdots \sum_{s_2 < n - \sum_{j=3}^m s_j} \prod_{j=2}^m 2^{\frac{1}{p_1} s_j (\delta_j - 1)} \ge C(p,m) 2^{\frac{1}{p_1} n} n^{\nu - 1}.$$

Hence,

$$\varphi(0) \ge C_1(p,m) 2^{\frac{1}{p_1} n} n^{\nu-1}, \tag{4}$$

if $p_1 = \cdots = p_\nu$.

Репозиторий КарГУ

Let $\{\psi_\nu(\overline{x})\}_{\nu=1}^{\overline{M}}$ be an orthonormal basis in A_M and

$$A(e^{i\langle \overline{k},\overline{x}\rangle}) = \sum_{\nu=1}^{\overline{M}} a_\nu^{\overline{k}} \psi_\nu(\overline{x}).$$

Then

$$\left(\sum_{\nu=1}^{\overline{M}} |a_\nu^{\overline{k}}|^2\right)^{\frac{1}{2}} \le B.$$

By assumption of theorem, $\frac{1}{p_j} - \frac{1}{q_j} < \frac{1}{p_1} - \frac{1}{q_1}$, $q_1 < q_j$, $j = \nu+1,\dots,m$. Therefore, $0 < \frac{1}{q_1} - \frac{1}{q_j} < \frac{1}{p_1} - \frac{1}{p_j}$, i.e., $p_1 < p_j$, $j = \nu+1,\dots,m$. Taking into account this inequality, in view of lemma 3.1 in [7] we have

$$\min_{\overline{y}=\overline{x}} A(\varphi(\overline{x}-\overline{y})) \le \left(M \sum_{\nu=1}^{\overline{M}} \sum_{\overline{k}\in Q_n} |a_\nu^{\overline{k}} \widehat{\varphi}(\overline{k})|^2\right)^{\frac{1}{2}}$$

$$\le \Big[B^2 M \sum_{\overline{k}\in Q_n} |\widehat{\varphi}(\overline{k})|^2\Big]^{\frac{1}{2}} \le B\Big[B^2 M \sum_{\overline{k}\in \sigma_n} 2^{s_j(\frac{2}{p_j}-1)}\Big]^{\frac{1}{2}} \le C(p,m)B\big[M 2^{n(\frac{2}{p_1}-1)} n^{\nu-1}\big]^{\frac{1}{2}}.$$

Therefore,

$$\min_{\overline{y}=\overline{x}} A(\varphi(\overline{x}-\overline{y})) \le C_2(p,m)B\big[M 2^{n(\frac{2}{p_1}-1)} n^{\nu-1}\big]^{\frac{1}{2}} \qquad (5)$$

with $p_1 = \dots = p_\nu$, $p_1 < p_j$, $j = \nu+1,\dots,m$. Formulas (4), (5) imply that

$$\varphi(0) - \min_{\overline{y}=\overline{x}} A(\varphi(\overline{x}-\overline{y})) \ge C_1(p,m) 2^{\frac{1}{p_1}n} n^{\nu-1} - C_2(p,m)B\big[M 2^{n(\frac{2}{p_1}-1)} n^{\nu-1}\big]^{\frac{1}{2}}$$

$$\times \left[C_1(p,m) - C_2(p,m)B\left(\frac{M}{2^n n^{\nu-1}}\right)^{\frac{1}{2}}\right] 2^{\frac{1}{p_1}n} n^{\nu-1}. \qquad (6)$$

Choose a natural number n so that $M \asymp 2^n n^{\nu-1}$ and

$$C_1(p,m) - C_2(p,m)B\left(\frac{M}{2^n n^{\nu-1}}\right)^{\frac{1}{2}} > C_3(p,m) > 0.$$

Then from (6) we obtain

$$\varphi(0) - \min_{\overline{y}=\overline{x}} A(\varphi(\overline{x}-\overline{y})) \ge C_3(p,m) 2^{\frac{n}{p_1}} n^{\nu-1} \qquad (7)$$

with $p_1 = \dots = p_\nu < p_j$, $j = \nu+1,\dots,m$. Since ([7], P. 83)

$$\sup_{\overline{y}} \|\varphi(\overline{x}-\overline{y}) - A(\varphi(\overline{x}-\overline{y}))\|_\infty \ge \varphi(0) - \min_{\overline{y}=\overline{x}} A(\varphi(\overline{x}-\overline{y})),$$

we have (see (7))

$$\sup_{\overline{y}} \|\varphi(\overline{x}-\overline{y}) - A(\varphi(\overline{x}-\overline{y}))\|_\infty \ge C_3(p,m) 2^{\frac{n}{p_1}} n^{\nu-1}.$$

Therefore, y^* exists such that

$$\|\varphi(\overline{x}-\overline{y^*}) - A(\varphi(\overline{x}-\overline{y^*}))\|_\infty \ge C_3(p,m) 2^{\frac{n}{p_1}} n^{\nu-1} \qquad (8)$$

with $p_1 = \dots = p_\nu < p_j$, $j = \nu+1,\dots,m$. Due to the inequality of different metrics (see Lemma 2) formula (8) gives

$$\|\varphi(\overline{x}-\overline{y^*}) - A(\varphi(\overline{x}-\overline{y^*}))\|_{\overline{q},\overline{\theta}}^* \ge C(q,p,m,\theta) 2^{\frac{n}{p_1}-\frac{n}{q_1}} n^{\sum_{j=2}^{\nu} \frac{1}{\theta_j}} \qquad (9)$$

Репозиторий КарГУ

with $p_1 = \cdots = p_\nu < p_j$, $j = \nu+1, \ldots, m$, $q_1 = \cdots = q_\nu < q_{\nu+1} \le \cdots \le q_m$.

Let us now consider the function

$$g(\overline{x}) = C2^{-nr_1}\varphi(\overline{x} - \overline{y^*}).$$

Due to the bound for the norm of the one-dimensional Fejér kernel ([14, 15]) we have

$$\|\delta_{\overline{s}}(g)\|^*_{\overline{p},\overline{\theta}} = 2^{-nr}\prod_{j=1}^{m} 2^{s_j(1-\frac{1}{p_j})}\prod_{j=1}^{m}\|\Phi_{2^{s_j-2}}(x_j)\|^*_{p_j,\theta_j} \le C(p,\theta,m)2^{-nr}. \quad (10)$$

Consequently, the function $C_0 g \in H^r_{\overline{p},\overline{\theta}}$. Now from (3) and (9) we get

$$\sup_{f\in H^r_{\overline{p},\overline{\theta}}}\|f - Gf\|^*_{\overline{q},\overline{\theta}} \ge \|g - Gg\|^*_{\overline{q},\overline{\theta}} \ge \|g - Ag\|^*_{\overline{q},\overline{\theta}}$$

$$= 2^{-nr}\|\varphi(\overline{x} - \overline{y^*}) - A(\varphi(\overline{x} - \overline{y^*}))\|^*_{\overline{q},\theta} \ge C(p,\theta,m)2^{-n(r+\frac{1}{q_1}-\frac{1}{p_1})}n^{\sum_{j=2}^{\nu}\frac{1}{\theta_j}}.$$

Since

$$d^B_M(H^r_{\overline{p},\overline{\theta}}, L^*_{\overline{q},\overline{\theta}}) \le d^{\perp}_M(H^r_{\overline{p},\overline{\theta}}, L^*_{\overline{q},\overline{\theta}}),$$

taking into account (2), from the previous inequality we obtain

$$C(p,\theta,m)M^{-(r_1+\frac{1}{q_1}-\frac{1}{p_1})}(\log M)^{(\nu-1)(r_1+\frac{1}{q_1}-\frac{1}{p_1})+\sum_{j=2}^{\nu}\frac{1}{\theta_j}} \le d^{\perp}_M(H^r_{\overline{p},\overline{\theta}}, L^*_{\overline{q},\overline{\theta}}). \qquad \square$$

Theorem 2. *Let* $1 \le p < q < +\infty$, $1 \le \theta_j < \tau_j < +\infty$, $j = 1, \ldots, m$; $r_1 = \cdots = r_\nu < r_{\nu+1} \le \cdots \le r_m$. *Then the following correlation takes place*:

$$d^{\perp}_M(B^{\overline{r}}_{p,\overline{\theta},\overline{\tau}}, L^*_{q,\overline{\theta}}) \asymp M^{-(r_1+\frac{1}{q}-\frac{1}{p})}(\log M)^{(\nu-1)(r_1+\frac{1}{q}-\frac{1}{p})+\sum_{j=2}^{\nu}(\frac{1}{\theta_j}-\frac{1}{\tau_j})}.$$

Proof. Choose a natural number $n \in \mathbb{N}$ so that $M \asymp 2^n n^{\nu-1}$. In accordance with the definition of the ortho-diameter and Theorem C with $p_1 = \cdots = p_m = p$, $q_1 = \cdots = q_m = q$ we have

$$d^{\perp}_M(B^{\overline{r}}_{p,\overline{\theta},\overline{\tau}}, L^*_{q,\overline{\theta}}) \le \sup_{f\in B^{\overline{r}}_{p,\overline{\theta},\overline{\tau}}}\|f - S_M(f)\|^*_{\overline{q},\overline{\theta}} \le C(p,q,\theta,m)2^{-n(r_1+\frac{1}{q}-\frac{1}{p})}n^{\sum_{j=2}^{\nu}(\frac{1}{\theta_j}-\frac{1}{\tau_j})}. \quad (11)$$

Since $n \asymp \log M$, we obtain

$$2^{-n(r_1+\frac{1}{q}-\frac{1}{p})}n^{\sum_{j=2}^{\nu}(\frac{1}{\theta_j}-\frac{1}{\tau_j})} \asymp M^{-(r_1+\frac{1}{q}-\frac{1}{p})}(\log M)^{(\nu-1)(r_1+\frac{1}{q}-\frac{1}{p})+\sum_{j=2}^{\nu}(\frac{1}{\theta_j}-\frac{1}{\tau_j})}.$$

Therefore formula (11) yields the upper estimate

$$d^{\perp}_M(B^{\overline{r}}_{p,\overline{\theta},\overline{\tau}}, L^*_{q,\overline{\theta}}) \le C(p,q,m,r,\tau,\theta)M^{-(r_1+\frac{1}{q}-\frac{1}{p})}(\log M)^{(\nu-1)(r_1+\frac{1}{q}-\frac{1}{p})+\sum_{j=2}^{\nu}(\frac{1}{\theta_j}-\frac{1}{\tau_j})}.$$

Let us prove the lower estimate. Since by assumption of the theorem $r_1 = \cdots = r_\nu < r_{\nu+1} \le \cdots \le r_m$, we have $B^{(r_m,\ldots,r_m)}_{p,\overline{\theta},\overline{\tau}} \subset B^{\overline{r}}_{p,\overline{\theta},\overline{\tau}}$. Hence

$$d^{\perp}_M(B^{\overline{r}}_{p,\overline{\theta},\overline{\tau}}, L^*_{q,\overline{\theta}}) \ge d^{\perp}_M(B^{(r_m,\ldots,r_m)}_{p,\overline{\theta},\overline{\tau}}, L^*_{q,\overline{\theta}}). \quad (12)$$

Therefore, it suffices to prove the lower estimate for $r_1 = \cdots = r_m = r$, i.e., $\gamma_j = 1$.

Consider the function

$$g_0(\overline{x}) = n^{-\sum_{j=2}^{m}\frac{1}{\tau_j}}2^{-n(r_1+1-\frac{1}{p})}\varphi(\overline{x} - \overline{y^*}),$$

where the function φ is defined in the proof of Theorem 1.

Then (see (10))

$$\|\delta_{\overline{s}}(g_0)\|^*_{p,\overline{\theta}} \le C(p,\theta,m)n^{-\sum_{j=2}^{m}\frac{1}{\tau_j}}\prod_{j=1}^{m}2^{s_j r}.$$

Репозиторий КарГУ

Therefore, by applying Lemma 3 we obtain

$$\left\|\left\{\prod_{j=1}^{m}2^{s_jr}\|\delta_{\overline{s}}(g_0)\|_p\right\}_{\overline{s}\in\mathbb{Z}_+^m}\right\|_{l_{\overline{\tau}}}\le C(p,q,r,m,\theta)n^{-\sum_{j=2}^{m}\frac{1}{\tau_j}}\left\|\{\chi_{\sigma_n}(\overline{s})\}_{\overline{s}\in\sigma_n}\right\|_{l_{\overline{\tau}}}\le C(p,q,r,m,\theta).$$

Hence, $C_1g_0\in B^r_{p,\overline{\theta},\overline{\tau}}$. Further, taking into account inequality (9), from (3) we obtain

$$\sup_{f\in B^r_{p,\overline{\theta},\overline{\tau}}}\|f-Gf\|^*_{q,\overline{\theta}}\ge\|g_0-Gg_0\|^*_{q,\overline{\theta}}\ge\|g_0-Ag_0\|^*_{q,\overline{\theta}}$$

$$=n^{-\sum_{j=2}^{m}\frac{1}{\tau_j}}2^{-n(r_1+1-\frac{1}{p})}\|\varphi(\overline{x}-\overline{y^*})-A(\varphi(\overline{x}-\overline{y^*}))\|^*_{q,\overline{\theta}}$$

$$\ge C(p,q,r,m,\theta)n^{m-\nu-\sum_{j=\nu+1}^{m}\frac{1}{\tau_j}}n^{\sum_{j=2}^{\nu}(\frac{1}{\theta_j}-\frac{1}{\tau_j})}2^{-n(r_1+\frac{1}{q}-\frac{1}{p})}.$$

Since $m-\nu-\sum\limits_{j=\nu+1}^{m}\frac{1}{\tau_j}\ge 0$, from the previous estimate and (12) we obtain

$$d_M^{\perp}(B^{\overline{r}}_{p,\overline{\theta},\overline{\tau}},L^*_{q,\overline{\theta}})\ge C(p,q,r,m,\theta)n^{\sum_{j=2}^{\nu}(\frac{1}{\theta_j}-\frac{1}{\tau_j})}2^{-n(r_1+\frac{1}{q}-\frac{1}{p})}.\qquad\square$$

Theorem 3. *Let* $1\le p_j<q_j<+\infty$, $1\le\theta_j<\tau_j<+\infty$, $j=1,\dots,m$; $r_1=\dots=r_\nu<r_{\nu+1}\le\dots\le r_m$; $\frac{1}{p_j}-\frac{1}{q_j}=\frac{1}{p_1}-\frac{1}{q_1}$ $\forall j=1,\dots,\nu$; $r_1(\frac{1}{p_j}-\frac{1}{q_j})<r_j(\frac{1}{p_1}-\frac{1}{q_1})$, $j=\nu+1,\dots,m$. *Then the following inequality takes place*:

$$d_M^{\perp}(B^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}},L^*_{\overline{q},\overline{\theta}})\le M^{-(r_1+\frac{1}{q_1}-\frac{1}{p_1})}(\log M)^{(\nu-1)(r_1+\frac{1}{q_1}-\frac{1}{p_1})+\sum_{j=2}^{\nu}(\frac{1}{\theta_j}-\frac{1}{\tau_j})}.$$

If $\tau_j\le\theta_j$, $j=1,\dots,m$, *then*

$$d_M^{\perp}(B^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}},L^*_{\overline{q},\overline{\theta}})\le M^{-(r_1+\frac{1}{q_1}-\frac{1}{p_1})}(\log M)^{(\nu-1)(r_1+\frac{1}{q_1}-\frac{1}{p_1})}.$$

But if $\tau_j\le\theta_j$, $j=1,\dots,l<\nu$, $\theta_j<\tau_j$, $j=l+1,\dots,m$, *then*

$$d_M^{\perp}(B^{\overline{r}}_{\overline{p},\overline{\theta},\overline{\tau}},L^*_{\overline{q},\overline{\theta}})\le C(p,q,\theta,m,l)M^{-(r_1+\frac{1}{q_1}-\frac{1}{p_1})}(\log M)^{(\nu-1)(r_1+\frac{1}{q_1}-\frac{1}{p_1})+\sum_{j=l+1}^{\nu}(\frac{1}{\theta_j}-\frac{1}{\tau_j})}.$$

The proof of this assertion follows from Theorem C.

Remark. Note that in Theorem 1 the upper estimate for the value $d_M^{\perp}(H^{\overline{r}}_{\overline{p},\overline{\theta}},L^*_{\overline{q},\overline{\theta}})$ is true without the assumption $q_1=\dots=q_\nu<q_{\nu+1}\le\dots\le q_m$. This fact follows from Theorem B.

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Репозиторий КарГУ

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