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Nonpotentiality of a diffusion system and the construction of a semi-bounded functional

The wide prevalence and the systematic variational principles are used in mathematics and applications due to a series of remarkable consequences among which the possibility to establish the existence of the solutions of the initial equations, and the determination of stable approximations of the solutions of the considered equations by the so-called variational methods. In this connection, it is natural for a given system of equations to investigate the problem of the existence of its variational formulations. It can be considered as the inverse problem of the calculus of variations. The main goal of this work is to study this problem for a diffusion system of partial differential equations. A key object is the criterion of potentiality. On its ground, the nonpotentiality of the operator of the given boundary value problem with respect to the classical bilinear form is proved. This system does not admit a matrix variational multiplier of the given form. Thus, the diffusion system cannot be deduced from the classical Hamilton's principle. We posed the question that whether there exists a functional semi-bounded on solutions to the boundary value problem. We have done the algorithm of the constructive determination of such a functional. The main value of constructed functional action will be in applications of direct variational methods.

Keywords: nonpotential operators, diffusion system, semi-bounded functionals, variational multiplier.

Introduction

We consider the following system of partial differential equations (PDE) [1, 2]:

$$\tilde{N}^1(u) \equiv \sum_{i,j=1}^{n} a^{ij}(x,t,u^1)\frac{\partial^2 u^1}{\partial x^i \partial x^j} + f\left(x,t,u^1,\frac{\partial u^1}{\partial x^k}\right) - \frac{\partial u^1}{\partial t} = F^1(x,t,u^1,u^2),$$

$$\tilde{N}^2(u) \equiv \frac{\partial u^2}{\partial t} = F^2(x,t,u^1,u^2), \tag{1}$$

$$(x,t) = (x^1,...,x^n,t) \in Q_T = \Omega \times (0,T),$$

where the components u^1, u^2 of the vector u are unknown functions, the domain $\Omega \subset \mathbb{R}^n$ is bounded by the smooth surface $\partial\Omega$, $F^i : Q_T \times \mathbb{R}^2 \to \mathbb{R}(i = 1,2)$ are given differentiable functions, $f : Q_T \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a given twice differentiable function.

Denoting $F = (F^1,F^2), \tilde{N} = (\tilde{N^1},\tilde{N^2}), N = \tilde{N} - F$, we set

$$D(N) = \{(u^1,u^2) : u^1 \in C^{2,1}(\overline{Q}_T); u^2 \in C^1(\overline{Q}_T), u^i|_{t=0} = u_0^i(x^1,...,x^n),$$

$$u^i|_{t=T} = u_1^i(x^1,...,x^n), u^i|_{\partial\Omega\times(0,T)} = \psi^i(x,t)(i = 1,2)\}, \tag{2}$$

where $\psi^i(x,t), u_j^i(x) \in C(\overline{\Omega})(i = 1,2; j = 0,1)$ are given continuous functions, $\overline{\Omega} = \Omega \cup \partial\Omega$, $\overline{Q}_T = \overline{\Omega} \times [0,T]$.

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In [3], it was proved a maximum principle (comparison theorem), existence and uniqueness theorems, and also convergence of the method of successive approximations for the directional derivative problem for the diffusion system.

In [4], it was illustrated that a comparison theorem is impossible for a multicomponent diffusion system unless further conditions are placed on the monotonicity of the functions involved. The method requires the construction of a counter-example.

In [5; 172], there were recently presented results of investigation of singularly perturbed reaction-advection-diffusion problems, which are based on a further development of the asymptotic comparison principle.

In [6], the separation method was used in order to obtain sufficient conditions for the solvability of the main inverse problem in the class of first-order Ito stochastic differential systems with random perturbations from the class of Wiener processes and diffusion degenerate with respect to a part of variables.

The problem of existence of Hamilton's principle for (1), (2) has not been investigated before. In modern interpretation [7], it can be considered as an inverse problem of the calculus of variations (IPCV).

The main aim of the paper is to investigate the existence of a solution of IPCV-Hamilton's principle for problem (1), (2).

Nonpotentiality of diffusion system

Let U, V be normed linear spaces over the field of real numbers $\mathbb{R}$, $U \subseteq V$; 0_U and 0_V be the zero elements in U and V respectively; N: $D(N) \subseteq U \to R(N) \subseteq V$ be an arbitrary twice Gâteaux differentiable operator with the domain $D(N)$ and the range $R(N)$.

We set N'_u as the first Gâteaux derivative of N at the point $u \in D(N)$ defined by the formula [7]

$$N'_u h = \frac{d}{d\varepsilon} N(u + \varepsilon h)|_{\varepsilon=0} = \delta N(u, h).$$

The mapping $\Phi(u; \cdot, \cdot) : V \times U \to \mathbb{R}$, being linear in each argument and depending on the parameter $u \in U$, is called a local bilinear form.

$\Phi'_u(h; v, g)$ is defined by

$$\Phi'_u(h; v, g) = \frac{d}{d\varepsilon} \Phi(u + \varepsilon h; v, g)|_{\varepsilon=0}.$$

Φ is called a nonlocal bilinear form if it does not depend on the parameter u, that is, $\Phi(u; \cdot, \cdot) \equiv \langle \cdot, \cdot \rangle$. Then $\Phi'_u(h; v, g) \equiv 0$.

It is said that $\langle \cdot, \cdot \rangle : V \times U \to \mathbb{R}$ is a nondegenerate nonlocal bilinear form if

1) the condition $\langle v, g \rangle = 0 \ \forall g \in U$ implies that $v = 0_V$;

2) the condition $\langle v, g \rangle = 0 \ \forall v \in V$ implies that $g = 0_U$.

Definition 1. [8] The operator $N : D(N) \subseteq U \to V$ is said to be potential on the set $D(N)$ with respect to the local bilinear form $\Phi(u; \cdot, \cdot) : V \times U \to \mathbb{R}$ if there exists a functional $F_N : D(F_N) = D(N) \to \mathbb{R}$ such that $\delta F_N[u, h] = \Phi(u; N(u), h) \forall u \in D(N), \forall h \in D(N'_u)$. Here F_N is called the potential of the operator N.

For the following explanation, we need the next theorem.

Theorem 1. [9] Let $N : D(N) \subseteq U \to V$ be a Gâteaux differentiable operator on the convex set $D(N)$ and the local bilinear form $\Phi(u; \cdot, \cdot) : V \times U \to \mathbb{R}$ be such that for any fixed elements $u \in D(N)$, and $g, h \in D(N'_u)$ the function $\varphi(\varepsilon) \equiv \Phi(u + \varepsilon h; N(u + \varepsilon h), g) \in C^1[0, 1]$. Then for the potentiality of the operator N on $D(N)$ with respect to Φ , it is necessary and sufficient that

$$\begin{gathered} J_{N,h,g}(u) \equiv \Phi(u; N'_u h, g) + \Phi'_u(h; N(u), g) = \\ = \Phi(u; N'_u g, h) + \Phi'_u(g; N(u), h) \quad \forall u \in D(N), \forall g, h \in D(N'_u). \end{gathered} \tag{3}$$

In this case

$$F_N[u] = \int_0^1 \Phi(u(\lambda); N(u(\lambda)), u - u_0) d\lambda + F_N[u_0],$$

where $u(\lambda) \equiv u_0 + \lambda(u - u_0)$; u_0 – an arbitrary fixed element from $D(N)$.

Condition (3) is called the criterion of the potentiality of the operator N with respect to the local bilinear form Φ.

Remark 1. If Φ is a nonlocal bilinear form, then (3) becomes

$$\langle N'_u h, g\rangle = \langle N'_u g, h\rangle \quad \forall u \in D(N), \forall g, h \in D(N'_u). \tag{4}$$

Let us introduce the classical bilinear form by

$$\Phi_1(v, g) = \langle v, g\rangle = \int_{Q_T} \sum_{i=1}^{2} v^i(x,t) g^i(x,t)\, dxdt. \tag{5}$$

Theorem 2. Operator (1) is not potential on set (2) with respect to nonlocal bilinear form (5).

Proof. From (1), we find the Gâteaux derivative

$$N'_u = \begin{pmatrix} a_1^1 & -\frac{\partial F^1}{\partial u^2} \\ -\frac{\partial F^2}{\partial u^1} & \frac{\partial}{\partial t} - \frac{\partial F^2}{\partial u^2} \end{pmatrix},$$

where

$$a_1^1 = \sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} + a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \right) + \frac{\partial f}{\partial u^1} + \sum_{k=1}^{n} \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial}{\partial x^k} - \frac{\partial}{\partial t} - \frac{\partial F^1}{\partial u^1},$$

and $u^1_{x^k} = \frac{\partial u^1}{\partial x^k}$, $a^{ij} \equiv a^{ij}(x,t,u^1)$.

In accordance with conditions (2), we have

$$D(N'_u) = \{(h^1, h^2) : h^1 \in C^{2,1}(\overline{Q}_T), h^2 \in C^1(\overline{Q}_T), h^i|_{t=0} = 0,$$
$$h^i|_{t=T} = 0, h^i|_{\partial\Omega\times(0,T)} = 0 (i = 1, 2)\}.$$

Denoting by N'^*_u the adjoint operator to N'_u, we find

$$N'^*_u = \begin{pmatrix} a_1^{1*} & -\frac{\partial F^2}{\partial u^1} \\ -\frac{\partial F^1}{\partial u^2} & -\frac{\partial}{\partial t} - \frac{\partial F^2}{\partial u^2} \end{pmatrix},$$

where

$$a_1^{1*} = \sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} + a^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \frac{\partial^2 a^{ij}}{\partial x^j \partial x^i} + \frac{\partial a^{ij}}{\partial x^i} \frac{\partial}{\partial x^j} + \frac{\partial a^{ij}}{\partial x^j} \frac{\partial}{\partial x^i} \right) +$$
$$+ \frac{\partial f}{\partial u^1} - \sum_{k=1}^{n} \left(\frac{\partial^2 f}{\partial u^1_{x^k} \partial x^k} + \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial}{\partial x^k} \right) + \frac{\partial}{\partial t} - \frac{\partial F^1}{\partial u^1},$$

$$D(N'^*_u) = \{(v^1, v^2) : v^1 \in C^{2,1}(\overline{Q}_T), v^2 \in C^1(\overline{Q}_T), v^i|_{t=0} = 0, v^i|_{t=T} = 0, v^i|_{\partial\Omega\times(0,T)} = 0 (i = 1, 2)\}.$$

Let us prove that operator (1) does not satisfy criterion (4). For that, we find

$$\Phi_1(N'_u h, g) = \int_{Q_T} \left\{ \left(a_1^1 h^1 - \frac{\partial F^1}{\partial u^2} h^2 \right) g^1 - \left[\frac{\partial F^2}{\partial u^1} h^1 + \left(\frac{\partial F^2}{\partial u^2} - \frac{\partial}{\partial t} \right) h^2 \right] g^2 \right\} dxdt$$
$$= \int_{Q_T} \left\{ \left[\sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} h^1 + a^{ij} \frac{\partial^2 h^1}{\partial x^i \partial x^j} \right) + \right.\right.$$
$$+ \frac{\partial f}{\partial u^1} h^1 + \sum_{k=1}^{n} \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial h^1}{\partial x^k} - \frac{\partial h^1}{\partial t} - \frac{\partial F^1}{\partial u^1} h^1 - \frac{\partial F^1}{\partial u^2} h^2 \Big] g^1 -$$
$$\left. - \left(\frac{\partial F^2}{\partial u^1} h^1 + \frac{\partial F^2}{\partial u^2} h^2 - \frac{\partial h^2}{\partial t} \right) g^2 \right\} dxdt.$$

Integrating by parts, we obtain

$$\Phi_1(N'_u h, g) = \int_{Q_T} \left\{ \sum_{i,j=1}^{n} \left[\frac{\partial a^{ij}}{\partial u^1} \frac{\partial^2 u^1}{\partial x^i \partial x^j} h^1 g^1 + D_{x^i}\left(a^{ij} \frac{\partial h^1}{\partial x^j} g^1 \right) - \frac{\partial a^{ij}}{\partial x^i} \frac{\partial h^1}{\partial x^j} g^1 - \right.\right.$$
$$- a^{ij} \tfrac{\partial h^1}{\partial x^j} \tfrac{\partial g^1}{\partial x^i} \Big] + \tfrac{\partial f}{\partial u^1} h^1 g^1 + \sum_{k=1}^{n} \left[D_{x^k}\left(\frac{\partial f}{\partial u^1_{x^k}} h^1 g^1 \right) - \right.$$
$$- \tfrac{\partial^2 f}{\partial u^1_{x^k} \partial x^k} h^1 g^1 - \tfrac{\partial f}{\partial u^1_{x^k}} h^1 \tfrac{\partial g^1}{\partial x^k} \Big] - D_t(h^1 g^1) + h^1 \tfrac{\partial g^1}{\partial t} -$$
$$- \tfrac{\partial F^1}{\partial u^1} h^1 g^1 - \tfrac{\partial F^1}{\partial u^2} h^2 g^1 - \tfrac{\partial F^2}{\partial u^1} h^1 g^2 - \tfrac{\partial F^2}{\partial u^2} h^2 g^2 +$$
$$\left. + D_t(h^2 g^2) - h^2 \tfrac{\partial g^2}{\partial t} \right\} dxdt \quad \forall h, g \in D(N'_u),$$

where $D_{x^i} = \dfrac{\partial}{\partial x^i}$, $D_t = \dfrac{\partial}{\partial t}$.

By virtue of the Divergence theorem and the condition $h \in D(N'_u)$, we have

$$\int_0^T \int_\Omega \left[D_{x^i}\left(a^{ij} \frac{\partial h^1}{\partial x^j} g^1 \right) \right] dx^1 ... dx^n dt =$$
$$= \int_0^T \left[\int_{\partial\Omega} a^{ij} \frac{\partial h^1}{\partial x^j} g^1 dx^1 ... dx^{i-1} dx^{i+1} ... dx^n \right] dt = 0.$$

Similarly, we have

$$\int_0^T \int_\Omega \left[D_{x^k}\left(\frac{\partial f}{\partial u^1_{x^k}} h^1 g^1 \right) \right] dx^1 ... dx^n dt = 0,$$
$$\int_{Q_T} \Big[- D_t(h^1 g^1) + D_t(h^2 g^2) \Big] dxdt = \int_\Omega \Big(- h^1 g^1 |_{t=0}^{t=T} + h^2 g^2 |_{t=0}^{t=T} \Big) dx = 0.$$

Integrating by parts and applying the above results, we get

$$\Phi_1(N'_u h, g) = \int_{Q_T} \left\{ \left[\sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} g^1 + a^{ij} \frac{\partial^2 g^1}{\partial x^i \partial x^j} + \frac{\partial^2 a^{ij}}{\partial x^i \partial x^j} g^1 + \right.\right.\right.$$
$$+ \frac{\partial a^{ij}}{\partial x^i} \frac{\partial g^1}{\partial x^j} + \frac{\partial a^{ij}}{\partial x^j} \frac{\partial g^1}{\partial x^i} \Big) + \frac{\partial f}{\partial u^1} g^1 - \sum_{k=1}^{n} \left(\frac{\partial^2 f}{\partial u^1_{x^k} \partial x^k} g^1 + \right.$$
$$+ \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial g^1}{\partial x^k} \Big) + \frac{\partial g^1}{\partial t} - \frac{\partial F^1}{\partial u^1} g^1 - \frac{\partial F^2}{\partial u^1} g^2 \Big] h^1 - \tag{6}$$
$$\left. - \left(\frac{\partial F^1}{\partial u^2} g^1 + \frac{\partial F^2}{\partial u^2} g^2 + \frac{\partial g^2}{\partial t} \right) h^2 \right\} dxdt$$
$$= \int_{Q_T} \left\{ \left(a_1^{1*} g^1 - \frac{\partial F^2}{\partial u^1} g^2 \right) h^1 - \left[\frac{\partial F^1}{\partial u^2} g^1 + \left(\frac{\partial F^2}{\partial u^2} + \frac{\partial}{\partial t} \right) g^2 \right] h^2 \right\} dxdt.$$

On the other hand, we have

$$\Phi_1(N'_u g, h) = \int_{Q_T} \left\{ \left(a_1^1 g^1 - \frac{\partial F^1}{\partial u^2} g^2 \right) h^1 - \left[\frac{\partial F^2}{\partial u^1} g^1 + \left(\frac{\partial F^2}{\partial u^2} - \frac{\partial}{\partial t} \right) g^2 \right] h^2 \right\} dxdt. \tag{7}$$

In (6) the coefficient at h^2 is $-\left[\frac{\partial F^1}{\partial u^2} g^1 + \left(\frac{\partial F^2}{\partial u^2} + \frac{\partial}{\partial t} \right) g^2 \right]$ and in (7) it is $-\left[\frac{\partial F^2}{\partial u^1} g^1 + \left(\frac{\partial F^2}{\partial u^2} - \frac{\partial}{\partial t} \right) g^2 \right]$. It follows that $\Phi_1(N'_u h, g)$ is not indentically equal to $\Phi_1(N'_u g, h)$. Thus, criterion (4) is not satisfied.

Let us investigate the existence of the matrix variational multiplier for operator (1).

Definition 2. An invertible linear operator $M : D(M) \subset R(N) \to V$ is called a variational multiplier for the operator $N : D(N) \subset U \to V$ if the operator $\hat{N} = MN$ is potential on the set $D(N)$ with respect to the given bilinear form.

Theorem 3. There is no matrix variational multiplier of the kind

$$M = \begin{pmatrix} m_{11}(x,t) & m_{12}(x,t) \\ m_{21}(x,t) & m_{22}(x,t) \end{pmatrix} \tag{8}$$

for operator $N(1)$.

Proof. Suppose that there exists a matrix variational multiplier of form (8) and $detM \neq 0$. Then the operator $\hat{N}(u) = MN(u)$ is potential with respect to the classical bilinear form (5).

Denoting $m_{pq} \equiv m_{pq}(x,t), p, q = 1, 2$ we get

$$\begin{aligned}\Phi_1(\hat{N}'_u h, g) &= \int_{Q_T} \sum_{p=1}^{2} \left[m_{p1}\left(a_1^1 h^1 - \frac{\partial F^2}{\partial u^1} h^2 \right) g^p + m_{p2}\left(\frac{\partial h^2}{\partial t} - \frac{\partial F^2}{\partial u^1} h^1 - \frac{\partial F^2}{\partial u^2} h^2 \right) g^p \right] dxdt = \\ &= \int_{Q_T} \sum_{p=1}^{2} \Bigg\{ m_{p1} \Bigg[\sum_{i,j=1}^{n} \left(\frac{\partial^2 u^1}{\partial x^i \partial x^j} \frac{\partial a^{ij}}{\partial u^1} h^1 + a^{ij} \frac{\partial^2 h^1}{\partial x^i \partial x^j} \right) + \frac{\partial f}{\partial u^1} h^1 + \\ &\quad + \sum_{k=1}^{n} \frac{\partial f}{\partial u^1_{x^k}} \frac{\partial h^1}{\partial x^k} - \frac{\partial h^1}{\partial t} - \frac{\partial F^1}{\partial u^1} h^1 - \frac{\partial F^2}{\partial u^1} h^2 \Bigg] g^p + \\ &\quad + m_{p2} \left(\frac{\partial h^2}{\partial t} - \frac{\partial F^2}{\partial u^1} h^1 - \frac{\partial F^2}{\partial u^2} h^2 \right) g^p \Bigg\} dxdt.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\Phi_1\left(\hat{N}'_u h, g\right) &= \int_{Q_T} \sum_{p=1}^{2} \Bigg\{ \sum_{i,j=1}^{n} \Bigg[m_{p1} \frac{\partial a^{ij}}{\partial u^1} \frac{\partial^2 u^1}{\partial x^i \partial x^j} h^1 g^p + D_{x^j}\left(m_{p1} a^{ij} \frac{\partial h^1}{\partial x^i} g^p \right) - \\ &\quad - \frac{\partial m_{p1}}{\partial x^j} a^{ij} \frac{\partial h^1}{\partial x^i} g^p - m_{p1} \frac{\partial a^{ij}}{\partial x^j} \frac{\partial h^1}{\partial x^i} g^p - m_{p1} a^{ij} \frac{\partial h^1}{\partial x^i} \frac{\partial g^p}{\partial x^j} \Bigg] + \\ &\quad + m_{p1} \frac{\partial f}{\partial u^1} h^1 g^p + \sum_{k=1}^{n} \Bigg[D_{x^k}\left(m_{p1} \frac{\partial f}{\partial u^1_{x^k}} h^1 g^p \right) - \\ &\quad - \frac{\partial m_{p1}}{\partial x^k} \frac{\partial f}{\partial u^1_{x^k}} h^1 g^p - m_{p1} \frac{\partial^2 f}{\partial u^1_{x^k} \partial x^k} h^1 g^p - m_{p1} \frac{\partial f}{\partial u^1_{x^k}} h^1 \frac{\partial g^p}{\partial x^k} \Bigg] - \\ &\quad - D_t(m_{p1} h^1 g^p) + \frac{\partial m_{p1}}{\partial t} h^1 g^p + m_{p1} h^1 \frac{\partial g^p}{\partial t} - \\ &\quad - m_{p1} \frac{\partial F^1}{\partial u^1} h^1 g^p - m_{p1} \frac{\partial F^2}{\partial u^1} h^2 g^p - m_{p2} \frac{\partial F^2}{\partial u^1} h^1 g^p - m_{p2} \frac{\partial F^2}{\partial u^2} h^2 g^p + \\ &\quad + D_t(m_{p2} h^2 g^p) - \frac{\partial m_{p2}}{\partial t} h^2 g^p - m_{p2} h^2 \frac{\partial g^p}{\partial t} \Bigg\} dxdt.\end{aligned}$$

Applying the above results, we get

$$\begin{aligned}\Phi_1(\hat{N}'_u h, g) &= \int_{Q_T} \sum_{p=1}^{2} \Bigg[h^1 \Bigg(A_1 g^p + B_1 \frac{\partial g^p}{\partial x^k} + C_1 \frac{\partial g^p}{\partial x^i} + D_1 \frac{\partial g^p}{\partial x^j} + \\ &\quad + E_1 \frac{\partial^2 g^p}{\partial x^j \partial x^i} + m_{p1} \frac{\partial g^p}{\partial t} \Bigg) + h_2 \left(G_1 g^p - m_{p2} \frac{\partial g^p}{\partial t} \right) \Bigg] dxdt,\end{aligned}$$

where

$$A_1=\sum_{i,j=1}^{n}\Big(m_{p1}\frac{\partial a^{ij}}{\partial u^1}\frac{\partial^2 u^1}{\partial x^i\partial x^j}+\frac{\partial^2 m_{p1}}{\partial x^j\partial x^i}a^{ij}+\frac{\partial m_{p1}}{\partial x^j}\frac{\partial a^{ij}}{\partial x^i}+$$

$$+\tfrac{\partial m_{p1}}{\partial x^i}\tfrac{\partial a^{ij}}{\partial x^j}+m_{p1}\tfrac{\partial^2 a^{ij}}{\partial x^j\partial x^i}\Big)+m_{p1}\tfrac{\partial f}{\partial u^1}-\sum_{k=1}^{n}\Big(\frac{\partial m_{p1}}{\partial x^k}\frac{\partial f}{\partial u^1_{x^k}}+$$

$$+m_{p1}\tfrac{\partial^2 f}{\partial u^1_{x^k}\partial x^k}\Big)+\tfrac{\partial m_{p1}}{\partial t}-m_{p1}\tfrac{\partial F^1}{\partial u^1}-m_{p2}\tfrac{\partial F^2}{\partial u^1},$$

$$B_1=-m_{p1}\sum_{k=1}^{n}\frac{\partial f}{\partial u^1_{x^k}},C_1=\sum_{i,j=1}^{n}\Big(\frac{\partial m_{p1}}{\partial x^j}a^{ij}+m_{p1}\frac{\partial a^{ij}}{\partial x^j}\Big),$$

$$D_1=\sum_{i,j=1}^{n}\Big(\frac{\partial m_{p1}}{\partial x^i}a^{ij}+m_{p1}\frac{\partial a^{ij}}{\partial x^i}\Big),E_1=m_{p1}\sum_{i,j=1}^{n}a^{ij},$$

$$G_1=-\tfrac{\partial m_{p2}}{\partial t}-m_{p1}\tfrac{\partial F^2}{\partial u^1}-m_{p2}\tfrac{\partial F^2}{\partial u^2}.$$

On the other hand, we have

$$\Phi_1\left(\hat{N'}_u g,h\right)=\int_{Q_T}\sum_{p=1}^{2}\Big[m_{p1}\Big(a^1_1 g^1-\frac{\partial F^2}{\partial u^1}g^2\Big)h^p+$$

$$+m_{p2}\Big(\tfrac{\partial g^2}{\partial t}-\tfrac{\partial F^2}{\partial u^1}g^1-\tfrac{\partial F^2}{\partial u^2}g^2\Big)h^p\Big]dxdt$$

$$=\int_{Q_T}\sum_{p=1}^{2}h^p\Big[A_2g^1+B_2\frac{\partial g^1}{\partial x^k}+E_1\frac{\partial^2 g^1}{\partial x^i\partial x^j}-m_{p1}\frac{\partial g^1}{\partial t}-$$

$$-\Big(m_{p1}\tfrac{\partial F^2}{\partial u^1}+m_{p2}\tfrac{\partial F^2}{\partial u^2}\Big)g^2+m_{p2}\tfrac{\partial g^2}{\partial t}\Big]dxdt,$$

where

$$A_2=m_{p1}\Big(\sum_{i,j=1}^{n}\frac{\partial a^{ij}}{\partial u^1}\frac{\partial^2 u^1}{\partial x^i\partial x^j}+\frac{\partial f}{\partial u^1}-\frac{\partial F^1}{\partial u^1}\Big)-m_{p2}\frac{\partial F^2}{\partial u^1},B_2=m_{p1}\frac{\partial f}{\partial u^1_{x^k}}.$$

Hence,

$$\Phi_1\left(\hat{N'}_u h,g\right)-\Phi_1\left(\hat{N'}_u g,h\right)=\int_{Q_T}\Big\{h^1\Big[(A_1-A_2)g^1+\Big(A_1+$$
$$+m_{11}\frac{\partial F^2}{\partial u^1}+m_{12}\frac{\partial F^2}{\partial u^2}\Big)g^2+\Big(B_1-B_2\Big)\frac{\partial g^1}{\partial x^k}+B_1\frac{\partial g^2}{\partial x^k}+$$
$$+C_1\Big(\frac{\partial g^1}{\partial x^i}+\frac{\partial g^2}{\partial x^i}\Big)+D_1\Big(\frac{\partial g^1}{\partial x^j}+\frac{\partial g^2}{\partial x^j}\Big)+E_1\frac{\partial^2 g^2}{\partial x^j\partial x^i}+2m_{11}\frac{\partial g^1}{\partial t}\qquad(9)$$
$$+\Big(m_{21}-m_{12}\Big)\frac{\partial g^2}{\partial t}\Big]+h_2\Big[(G_1-A_2)g^1-\Big(m_{12}-m_{21}\Big)\frac{\partial g^1}{\partial t}+$$
$$+\Big(G_1+m_{21}\frac{\partial F^2}{\partial u^1}+m_{22}\frac{\partial F^2}{\partial u^2}\Big)g^2-2m_{22}\frac{\partial g^2}{\partial t}-B_2\frac{\partial g^1}{\partial x^k}-E_1\frac{\partial^2 g^1}{\partial x^i\partial x^j}\Big]\Big\}dxdt.$$

According to criterion (4), it must be

$$\Phi_1\left(\hat{N}'_u h,g\right)-\Phi_1\left(\hat{N}'_u g,h\right)=0\ \forall u\in D(N),\ \forall h,g\in D(N'_u).$$

By virtue of the arbitrariness of the functions h^k $(k=1,2)$ from (9) we obtain

$$(A_1 - A_2)g^1 + \left(A_1 + m_{11}\frac{\partial F^2}{\partial u^1} + m_{12}\frac{\partial F^2}{\partial u^2}\right)g^2 + \left(B_1 - B_2\right)\frac{\partial g^1}{\partial x^k} +$$
$$+B_1\frac{\partial g^2}{\partial x^k} + C_1\left(\frac{\partial g^1}{\partial x^i} + \frac{\partial g^2}{\partial x^i}\right) + D_1\left(\frac{\partial g^1}{\partial x^j} + \frac{\partial g^2}{\partial x^j}\right) + E_1\frac{\partial^2 g^2}{\partial x^j \partial x^i} +$$
$$+2m_{11}\frac{\partial g^1}{\partial t} + \left(m_{21} - m_{12}\right)\frac{\partial g^2}{\partial t} = 0,$$
$$(G_1 - A_2)g^1 - \left(m_{12} - m_{21}\right)\frac{\partial g^1}{\partial t} + \left(G_1 + m_{21}\frac{\partial F^2}{\partial u^1} +\right.$$
$$\left.+m_{22}\frac{\partial F^2}{\partial u^2}\right)g^2 - 2m_{22}\frac{\partial g^2}{\partial t} - B_2\frac{\partial g^1}{\partial x^k} - E_1\frac{\partial^2 g^1}{\partial x^i \partial x^j} = 0.$$

From here, by virtue of the arbitrariness of the functions g^k $(k = 1, 2)$, we get

$$m_{p1} = 0,$$
$$m_{21} - m_{12} = 0,$$
$$m_{22} = 0.$$

In total, $m_{pq}(x,t) = 0$ $(p, q = 1, 2)$. Therefore, $M = 0$ (zero matrix). It contradicts to what we have supposed above.

The construction of a semibounded functional

We have already proved that operator (1) is not potential with respect to nonlocal bilinear form (5) and there is no matrix variational multiplier of the given type. For the following exposition, we need the next theorem.

Consider an arbitrary equation

$$N(u) = 0_V, u \in D(N) \subseteq U \subseteq V, \tag{10}$$

where the operator N, in general case, is nonpotential with respect to the fixed nonlocal bilinear form $\Phi_1(\cdot,\cdot) \equiv \langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$.

Theorem 4. [8] Let: 1) $N : D(N) \subseteq U \to V$ be a twice Gâteaux differentiable operator on the convex set $D(N)$; 2) $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ be a given nonlocal bilinear form; 3) $C : D(C) \supseteq R(N) \to V$ be an arbitrary invertible linear symmetric operator, such that for any fixed elements $u \in D(N)$ and $g, h \in D(N'_u)$ the function $\varphi(\varepsilon) \equiv \left\langle N(u + \varepsilon h), CN'_{u+\varepsilon h}g\right\rangle \in C^1[0,1]$. Then the operator N is potential on $D(N)$ with respect to the following local bilinear form

$$\Phi(u; v, g) = \langle v, CN'_u g\rangle .$$

Herewith

$$F_N[u] = \frac{1}{2}\langle N(u), CN(u)\rangle . \tag{11}$$

The proof is given in [8].

Note that $\delta F_N[u,h] = \Phi(u; N(u), h) = \langle N(u), CN'_u h\rangle$.

Denoting the adjoint operator of N'_u by N'^*_u and assuming that $R(C) \subseteq D(N'^*_u)$, it follows from the last equality that $\delta F_N[u,h] = \langle N'^*_u CN(u), h\rangle$ $\forall u \in D(N)$, $\forall h \in D(N'_u)$.

Assuming that $D(N'_u) = U$ and $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ is a nonsingular continuous (in every argument) nonlocal bilinear form, we get $\delta F_N[u,h] = 0, u \in D(N), \forall h \in D(N'_u)$ if and only if

$$N_1(u) \equiv N'^*_u CN(u) = 0_V, u \in D(N). \tag{12}$$

Thus, the operator N_1 is potential on $D(N)$ with respect to the nonlocal bilinear form Φ_1.

If N'^*_u is an invertible operator, then problems (12) and (10) are equivalent in the following sense: if $\tilde{u}$ is a solution to one of them, then $\tilde{u}$ is a solution to the other ,i.e., $N(\tilde{u}) = 0_V \Leftrightarrow N_1(\tilde{u}) = 0_V$. In this case the functional (11) provides an indirect variational statement of problems (10).

If the operator C is positive definite with respect to nonlocal bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, i.e., $\langle v, Cv \rangle \geq k \|v\| \, \forall v \in D(C)$, where $k > 0$, then $F_N[u] \geq 0 \, \forall u \in D(N)$ and $F_N[\tilde{u}] = 0 \Leftrightarrow \tilde{u}$ is a solution of (10). Thus, in this case, formula (11) specifies a semi-bounded functional whose mininum is attained on the solutions to problem (10).

Note that functional (11) was obtained in another way in [10] when solving one of the statements of the inverse problem of the calculus of variations.

Let us define the operator C on $R(N)$ by the formula

$$(Cv)^j(x,t) = \int_{Q_T} K(x,t,y,\tau)\phi^j(x,t)\phi^j(y,\tau)v^j(y,\tau)dyd\tau \ (j = 1,2), \tag{13}$$

where

$$K(x,t,y,\tau) \equiv K = exp\left(\sum_{i=1}^{n} x^i y^i + t\tau\right). \tag{14}$$

$\phi^i (i = 1,2)$ are arbitrary functions of the class $C^{2,1}(\overline{Q}_T)$ such that $\phi^i(x,t) \neq 0((x,t) \in Q_T)$ and $\phi^i|_{t=0} = 0$, $\phi^i|_{t=T} = 0$, $\phi^i|_{\partial\Omega\times(0,T)} = 0 \ (i = 1,2)$. With this choice of functions ϕ^1, ϕ^2 we have $Cv \in D(N_u'^*)$.

It is also easy to see that operator (13) is symmetric on R(N). Let us show that it is positive definite. For this, we find the expansion of function (14) in the Maclaurin series

$K = \sum_{|\alpha|=0}^{\infty} \frac{1}{(\alpha!)^2} \left(x^1\right)^{\alpha_1} ... \left(x^n\right)^{\alpha_n} t^{\alpha_{n+1}} \left(y^1\right)^{\alpha_1} ... \left(y^n\right)^{(\alpha_n)} \tau^{\alpha_{n+1}}$.

Here $\alpha = (\alpha_1, ..., \alpha_{n+1})$; $\alpha_i \ (i = \overline{1, n+1})$ are nonnegative integers; $|\alpha| = \sum\limits_{i=1}^{n+1} \alpha_i$, $\alpha! = \alpha_1!...\alpha_{n+1}!$.

Using the obtained series we find

$$\begin{aligned}
\Phi_1(v,Cv) &= \int_{Q_T} \sum_{j=1}^{2} v^j(x,t) \int_{Q_T} K(x,t,y,\tau)\phi^j(x,t)\phi^j(y,\tau)v^j(y,\tau)dyd\tau dxdt \\
&= \sum_{j=1}^{2} \sum_{|\alpha|=0}^{\infty} \frac{1}{(\alpha!)^2} \int_{Q_T} (x^1)^{\alpha_1}..(x^n)^{\alpha_n} t^{\alpha_{n+1}} \phi^j(x,t)v^j(x,t)dxdt \\
&\quad \times \int_{Q_T} (y^1)^{\alpha_1}...(y^n)^{\alpha_n} \tau^{\alpha_{n+1}} \phi^j(y,\tau)v^j(y,\tau)dyd\tau \\
&\equiv \sum_{j=1}^{2} \sum_{|\alpha|=0}^{\infty} \frac{1}{(\alpha!)^2} (M^{\alpha_1 ... \alpha_{n+1} j})^2 \ \geq 0.
\end{aligned}$$

We note that all the moments $M^{\alpha^1 ... \alpha^{n+1} j}$ vanish simultaneously if and only if $v^j = 0 (j = 1,2)$ in Q_T [11].
Therefore, if $v \neq 0_V$ then $\Phi_1(v, Cv) > 0$.
Thus, the operator C of form (13) is a positive definite and invertible.
Denoting by $K \equiv K(x,t,y,\tau)$, from (1) and (13) we get

$$\begin{aligned}
(CN(u))^1(x,t) = \int_{Q_T} \Bigg\{ &- \sum_{i,j=1}^{n} \frac{\partial u^1(y,\tau)}{\partial y^j} \phi^1(x,t) D_{y^i}\left[K\phi^1(y,\tau)a^{ij}(y,\tau,u^1)\right] + \\
&+ K\phi^1(x,t)\phi^1(y,\tau) f\left(y,\tau,u^1,\frac{\partial u^1}{\partial y^k}\right) + u^1(y,\tau)\phi^1(x,t) \\
&D_\tau(K\phi^1(y,\tau)) - K\phi^1(x,t)\phi^1(y,\tau)F^1(y,\tau,u^1,u^2) \Bigg\} dyd\tau, \\
(CN(u))^2(x,t) = \int_{Q_T} &[-u^2(y,\tau)\phi^2(x,t)D_\tau(K\phi^2(y,\tau)) - \\
&- K\phi^2(x,t)\phi^2(y,\tau)F^2(y,\tau,u^1,u^2)]dyd\tau.
\end{aligned} \tag{15}$$

Using formulas (1),(5),(11),(15) we find the required functional in the form

$$F_N[u] = \frac{1}{2} \int_{Q_T} \int_{Q_T} \Bigg\{ L_1 + L_2 \Bigg\} dyd\tau dxdt,$$

where

$$L_1 = \left[\sum_{i,j=1}^{n} a^{ij}(x,t,u^1) \frac{\partial^2 u^1}{\partial x^i \partial x^j} + f\left(x,t,u^1,\frac{\partial u^1}{\partial x^k}\right) - \frac{\partial u^1(x,t)}{\partial t} - \right.$$
$$\left. -F^1(x,t,u^1,u^2)\right]\left[-\sum_{i,j=1}^{n} \frac{\partial u^1(y,\tau)}{\partial y^j}\phi^1(x,t)D_{y^i}\left[K\phi^1(y,\tau)a^{ij}(y,\tau,u^1)\right] + \right.$$
$$+K\phi^1(x,t)\phi^1(y,\tau)f\left(y,\tau,u^1,\tfrac{\partial u^1}{\partial y^k}\right) + u^1(y,\tau)\phi^1(x,t)D_\tau(K\phi^1(y,\tau)) -$$
$$\left. -K\phi^1(x,t)\phi^1(y,\tau)F^1(y,\tau,u^1,u^2)\right],$$

$$L_2 = \left[\frac{\partial u^2(x,t)}{\partial t} - F^2(x,t,u^1,u^2)\right]\left[- u^2(y,\tau)\phi^2(x,t)D_\tau(K\phi^2(y,\tau)) - \right.$$
$$\left. -K\phi^2(x,t)\phi^2(y,\tau)F^2(y,\tau,u^1,u^2)\right].$$

Integrating by parts, we get

$$F_N[u] \quad = \tfrac{1}{2}\int_{Q_T}\int_{Q_T}\left\{H_1 + H_2\right\}dyd\tau dxdt, \tag{16}$$

where

$$H_1 = \sum_{i,j=1}^{n} \frac{\partial u^1(x,t)}{\partial x^j} D_{x^i}\left\{ a^{ij}(x,t,u^1)\left[\sum_{i,j=1}^{n} \frac{\partial u^1(y,\tau)}{\partial y^j}\phi^1(x,t)D_{y^i}\right.\right.$$
$$\left[K\phi^1(y,\tau)a^{ij}(y,\tau,u^1)\right] + K\phi^1(x,t)\phi^1(y,\tau)f\left(y,\tau,u^1,\tfrac{\partial u^1}{\partial y^k}\right) +$$
$$\left.\left.+u^1(y,\tau)\phi^1(x,t)D_\tau(K\phi^1(y,\tau)) - K\phi^1(x,t)\phi^1(y,\tau)F^1(y,\tau,u^1,u^2)\right]\right\} +$$
$$+\left[f\left(x,t,u^1,\frac{\partial u^1}{\partial x^k}\right) + u^1(x,t)D_t - F^1(x,t,u^1,u^2)\right]$$
$$\left[-\sum_{i,j=1}^{n}\frac{\partial u^1(y,\tau)}{\partial y^j}\phi^1(x,t)D_{y^i}\left[K\phi^1(y,\tau)a^{ij}(y,\tau,u^1)\right] + K\phi^1(x,t)\right.$$
$$\phi^1(y,\tau)f\left(y,\tau,u^1,\tfrac{\partial u^1}{\partial y^k}\right) + u^1(y,\tau)\phi^1(x,t)D_\tau(K\phi^1(y,\tau)) -$$
$$\left. -K\phi^1(x,t)\phi^1(y,\tau)F^1(y,\tau,u^1,u^2)\right],$$

$$H_2 = u^2(x,t)u^2(y,\tau)D_t(\phi^2(x,t)D_\tau(K\phi^2(y,\tau))) +$$
$$+u^2(x,t)\phi^2(y,\tau)D_t(K\phi^2(x,t))F^2(y,\tau,u^1,u^2) + u^2(y,\tau)\phi^2(x,t)D_\tau$$
$$(K\phi^2(y,\tau))F^2(x,t,u^1,u^2) + K\phi^2(x,t)\phi^2(y,\tau)F^2(y,\tau,u^1,u^2)F^2(x,t,u^1,u^2).$$

Theorem 5. The functional of form (16) is semi-bounded on the solutions of problem (1),(2).
The theorem is proved above.

Remark 2. Functional (16) : 1) is bounded below on set (2); 2) takes a minimum value only on the solutions of problem (1), (2); 3) contains derivatives of unknown functions of lesser order, than the system of equations (1), (2); 4) the set of its stationary points contains the solution set of problem (1), (2).

Conslusions and future directions

The results of this paper can be summarized as follows.

(i) We studied the potentiality of the operator of the boundary value problem for a system of partial differential equations for diffusion. We showed that it is not potential with respect to the classical bilinear form. It means that the considered system cannot be obtained from Hamilton's variational principle.

(ii) The problem of the existence of a matrix variational multipler for (1) was investigated. We illustrated that there is no a matrix variational multiplier with elements depending on x and t.

(iii) We posed the question that whether there exists a functional semi-bounded on solutions of the given boundary value problem.

We have done the algorithm of the constructive determination of such a functional.

The main value of constructed functional (16) will be in applications of direct variational methods and its numerical performance.

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Бір диффузиялық жүйе операторының бейпотенциалдығы және жартылайшекаралы функционалды құрастыру

Математика мен қосымшаларда вариациялық принциптердің кең таралуы және жүйелі қолданылуы бірқатар керемет салдарларға байланысты, олардың арасында бастапқы теңдеулер шешімдерінің бар екендігі және вариациялық әдістермен қарастырылатын шешімдердің тұрақты жуықтауларын анықтау мүмкіндігі бар. Осыған байланысты берілген теңдеулер жүйесі үшін оның вариациялық тұжырымдарының болуы туралы мәселені зерттеу заңды. Оны вариация есептеудің кері есебі деп қарастыруға болады. Бұл жұмыстың басты мақсаты — дербес туындылы дифференциалдық теңдеулердің диффузиялық жүйесі үшін осы есепті зерттеу. Негізгі объект — бұл потенциал критерийі. Оның негізінде берілген шекаралық есеп операторының классикалық белгісіз формаға қатысты бейпотенциалдығы дәлелденді. Бұл жүйенің берілген форманың матрицалық вариациялық көбейткішін қабылдамайтындығы көрсетілген. Осылайша, берілген диффузиялық жүйені классикалық Гамильтон принципінен шығаруға болмайды. Берілген шекаралық есепті шешудің функционалды жартылай байланысы бар ма деген сұрақ қойылған. Осындай функционалды конструктивті анықтау алгоритмі жасалған. Құрылған функционалдың негізгі мәні тікелей вариациялық әдістерді қолдануда болатынында.

Кілт сөздер: бейпотенциалды операторлар, диффузиялық жүйе, жартылай шектелген функциялар, вариациялық көбейткіш.

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Непотенциальность оператора одной системы диффузии и построение полуограниченного функционала

Широкое распространение и систематическое использование вариационных принципов в математике и приложениях объясняется рядом замечательных следствий, среди которых возможность установить существование решений исходных уравнений и определение устойчивых приближений их решений так называемыми вариационными методами. В связи с этим для заданной системы уравнений естественно исследовать вопрос о существовании ее вариационных формулировок. Ее можно рассматривать как обратную задачу вариационного исчисления. Основная цель настоящей работы — исследование этой задачи для системы уравнений в частных производных диффузии. Ключевой объект — критерий потенциальности. На его основании доказана непотенциальность оператора данной краевой задачи относительно классической билинейной формы. Показано, что эта система не допускает матричный вариационный множитель данного вида. Таким образом, заданная система диффузии не может быть выведена из классического вариационного принципа Гамильтона. Поставлен вопрос о том, существует ли функционал, полуограниченный на решениях данной краевой задачи. Изложен алгоритм конструктивного определения такого функционала. Основная ценность построенного функционала заключается в применении прямых вариационных методов.

Ключевые слова: непотенциальные операторы, система диффузии, полуограниченные функционалы, вариационный множитель.

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