МАТЕМАТИКА
MATHEMATICS

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Strong approximation of Fourier series on generalized periodic Morrey spaces

In recent years, a lot of attention has been paid to study of Morrey type spaces. Many applications in partial differential equation of Morrey spaces and Lizorkin-Triebel spaces have been given in work G.Di Fazioand, M. Ragusa and the book of T. Mizuhara. The theory of generalized Triebel-Lizorkin-Morrey spaces is developed. Generalized Morrey spaces, with T. Mizuhara and E. Nakai proposed, are equipped with a parameter and a function. First we give definition of Morrey and generalized Morrey spaces. Then we recall the boundedness of periodic Hilbert transform. This will be our main tool for all wtht follows. In a more or less elementary way, we carry over the known boundedness assertions for the Hilbert transform on Morrey spaces defined on $\mathbb{R}$ to periodic Morrey spaces. Boundedness of the Hilbert transform implies uniform estimates of the operator norms of the partial sumd of the Fourier series. Then we study vector-valued Fourier-multiplier theorem for smooth multipliers. Afterwards, we study vector valued version of famous Riesz theorem. Here we concentrate on Lizorkin representations. Finally, we get an interesting characterization of the space $\mathcal{E}^s_{\varphi,p,q}(T)$ by using differences of partial sums of the Fourier series Finally, we get an interesting characterization of the space $\mathcal{E}^s_{\varphi,p,q}(T)$ by using differences of partial sums of the Fourier series and consequence for strong approximation of Fourier series on Morrey space.

Keywords: Morrey spaces, generalized periodic Morrey spaces, strong approximation, vector-valued version of the Riesz theorem.

First we recall the definition of the generalized Morrey spaces (nonperiodic and periodic). As usual, $B(x,r)$ denotes the open interval $(x-r,x+r)$. $\mathbb{T}$ denotes the one-dimensional torus, usually identified with $[-\pi,\pi]$. For a measurable set $\Omega\subset\mathbb{R}$ we use $|\Omega|$ to denote the Lebesgue measure of Ω. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0$ the natural numbers including 0. All functions are assumed to be complex-valued, i.e., we consider functions $f:\ \mathbb{R}\to\mathbb{C}$. As usual, the symbols $C, C_1, \dots, A, B, C, \dots$ denote positive constants which depend only on the fixed parameters s,p,q and λ and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line.

Definition 1. Let $0<p\le\infty$ and $0\le\lambda\le 1/p$.

(i) We say that a function $f:\ \mathbb{R}\to\mathbb{C}$ belongs to the (nonperiodic) Morrey space $M^\lambda_p(\mathbb{R})$ if $f\in L_p(B(x,r))$ for all $x\in\mathbb{R}$ and all $r>0$, and the following expression is finite

$$\|\, f\,|M^\lambda_p(\mathbb{R})\| := \sup_{x\in\mathbb{R}}\sup_{r>0}|B(x,r)|^{-\lambda}\,\|f|L_p(B(x,r))\|\,.$$

(ii) We say that a function $f:\ \mathbb{R}\to\mathbb{C}$, 2π-periodic, belongs to the periodic Morrey space $M^\lambda_p(\mathbb{T})$ if $f\in L_p(B(x,r))$ for all $x\in\mathbb{R}$ and all $r>0$ and the following expression is finite

$$\|\, f\,|M^\lambda_p(\mathbb{T})\| := \sup_{x\in\mathbb{R}}\sup_{0<r\le\pi}|B(x,r)|^{-\lambda}\,\|f|L_p(B(x,r))\|\,. \tag{1}$$

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Remark 1.

(i) Obviously we have

$$M_p^0(\mathbb{R}) = L_p(\mathbb{R}) \qquad \text{and} \qquad M_p^{1/p}(\mathbb{R}) = L_\infty(\mathbb{R})$$

in the sense of equivalent norms. In the periodic case we obtain

$$M_p^0(\mathbb{T}) = L_p(\mathbb{T}) \qquad \text{and} \qquad M_p^{1/p}(\mathbb{T}) = L_\infty(\mathbb{T})$$

in the sense of equivalent norms.

(ii) Let us mention also the trivial embeddings

$$L_\infty(\mathbb{T}) \hookrightarrow M_p^{\lambda_0}(\mathbb{T}) \hookrightarrow M_p^{\lambda_1}(\mathbb{T}) \hookrightarrow L_p(\mathbb{T})\,, \qquad \lambda_1 \le \lambda_0$$

and

$$L_u(\mathbb{T}) = M_u^0(\mathbb{T}) \hookrightarrow M_{p_1}^{\frac{1}{p_1}-\frac{1}{u}}(\mathbb{T}) \hookrightarrow M_{p_2}^{\frac{1}{p_2}-\frac{1}{u}}(\mathbb{T})\,, \qquad 0 < p_2 \le p_1 \le u < \infty\,.$$

(iii) Morrey spaces, periodic or not, are not separable if $\lambda > 0$. In some sense these Morrey spaces are relatives of L_∞, and not of L_p, $p < \infty$.

(iv) By periodicity it will be enough to restrict the supremum in (1) to $x \in [-\pi, \pi]$.

Generalized Morrey spaces have been introduced independently by Mizuhara [1] and by Nakai [2]. Here the parameter λ is replaced by a function $\varphi : (0,\infty) \to (0,\infty)$.

Definition 2. Let $0 < p < \infty$ and let $\varphi : (0,\infty) \to (0,\infty)$.

(i) Then the generalized Morrey space $M_p^\varphi(\mathbb{R})$ is the collection of all functions $f : \mathbb{R} \to \mathbb{C}$ such that $f \in L_p(B(x,r))$ for all $x \in \mathbb{R}$ and all $r > 0$ and

$$\|f|M_p^\varphi(\mathbb{R})\| := \sup_{x\in\mathbb{R}} \sup_{0<r<\infty} \varphi(r) \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy \Big)^{\frac{1}{p}} < \infty\,.$$

(ii) Then the generalized periodic Morrey space $M_p^\varphi(\mathbb{T})$ is the collection of all functions $f : \mathbb{R} \to \mathbb{C}$, 2π-periodic, such that $f \in L_p(B(x,r))$ for all $x \in \mathbb{R}$ and all $r > 0$ and

$$\|f|M_p^\varphi(\mathbb{T})\| := \sup_{x\in\mathbb{R}} \sup_{0<r\le\pi} \varphi(r) \Big(\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^p dy \Big)^{\frac{1}{p}} < \infty\,.$$

Remark 2. Clearly, if $\varphi(r) := |B(0,r)|^{-\lambda+\frac{1}{p}}$, $r > 0$, then we have coincidence $M_p^\varphi(\mathbb{T}) = M_p^\lambda(\mathbb{T})$, in particular, if $\varphi(r) := |B(0,r)|^{\frac{1}{p}}$, $r > 0$, then $M_p^\varphi(\mathbb{T}) = L_p(\mathbb{T})$.

Of course, we shall need restrictions for φ to develop a reasonable theory. We are mainly interested in smooth peturbations of $|B(0,r)|^{-\lambda+\frac{1}{p}}$. Following Nakai [3] we shall work with the following class of functions.

Definition 3. Let $0 < p < \infty$. Then $\varphi : (0,\infty) \to (0,\infty)$ belongs to the class $\mathcal{G}_p$, if φ there exist positive constants C, C' such that the inequalities

$$\varphi(t_1) \le C\,\varphi(t_2) \qquad \textit{and} \qquad t_2^{-\frac{d}{p}}\,\varphi(t_2) \le C'\,t_1^{-\frac{d}{p}}\,\varphi(t_1)$$

hold for all $0 < t_1 \le t_2 < \infty$.

In the definition of $M_p^\varphi(\mathbb{T})$, we assume that φ is in $\mathcal{G}_p$, that is, there exist some constants $C, C^{'} > 0$ such that the inequalities

$$\varphi(t_1) \le C\varphi(t_2) \quad \text{and} \quad C^{'} t_1^{-\frac{d}{p}} \varphi(t_1) \ge t_2^{-\frac{d}{p}} \varphi(t_2)$$

hold for $0 < t_1 \le t_2 < \infty$.

The nonperiodic and the periodic Hilbert transform are classical objects in harmonic analysis. We give their definitions.

Definition 4. The Hilbert transform of a given function $f \in L_1^{loc}(\mathbb{R})$ is defined as the Cauchy principal value

$$(Tf)(x) := P.V. \int_{-\infty}^{+\infty} \frac{f(t)}{x-t} dt = \lim_{\delta\downarrow 0} \int_{\{t:|t-x|>\delta\}} \frac{f(t)}{x-t} dt$$

at every point $x \in \mathbb{R}$ for which this limit exists.

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Definition 5. The periodic Hilbert transform (or conjugate function) of a periodic function $f \in L_1(\mathbb{T})$ is defined as the Cauchy principal value

$$(Hf)(x) := \widetilde{f}(x) = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} f(x-u) \cot \frac{u}{2} du =$$

$$= \frac{1}{2\pi} \lim_{\delta \downarrow 0} \int_{\delta < |u| < \pi} f(x-u) \cot \frac{u}{2} du$$

at every point $x \in \mathbb{R}$ for which this limit exists.

We recall the boundedness of the Hilbert transform on generalized periodic Morrey spaces, which have been proved in [4].

Proposition 1 [4, Theorem 2]. Let $1 < p, q, < \infty$ and $\varphi \in G_p$. We assume that there exist some $\varepsilon > 0$ and a constant $C > 0$ such that

$$\frac{t^\varepsilon}{\varphi(t)} \leq C \, \frac{r^\varepsilon}{\varphi(r)} \qquad \text{for all} \quad t \geq r > 0\,. \tag{2}$$

Then there exists a constant C such that

$$\Big\| \Big(\sum_{j=0}^{\infty} |Hf_j|^q \Big)^{1/q} \Big| M_p^\varphi(\mathbb{T}) \Big\| \leq C \Big\| \sum_{j=0}^{\infty} |f_j|^q \Big)^{1/q} \Big| M_p^\varphi(\mathbb{T}) \Big\|$$

holds for all $f \in M_p^\varphi(\mathbb{T})$.

A standard consequence of the boundedness of the periodic Hilbert transform is an estimate of the operator norm of the partial sum operator of the Fourier series.

$$S[f](x) := \sum_{k=-\infty}^{\infty} c_k e^{ikx}, \qquad x \in \mathbb{R}$$

be the Fourier series $f \in L_1(\mathbb{T})$ with $c_k(f)$. Here $c_k(f)$ is the Fourier coefficient of f given by

$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

We define

$$S_{N,M}[f](x) := \sum_{k=-N}^{M} c_k(f) e^{ikx}, \qquad N, M \in \mathbb{Z}, \quad N \leq M.$$

The next corollary is generalization of the famous Riesz theorem.

Corollary 1 [4, Theorem 5]. Let $1 < p < \infty$ and a function $\varphi \in G_p$, $\varphi \in \mathcal{G}_p$. We assume that there exist some $\varepsilon > 0$ and a constant $C > 0$ such that

$$\frac{t^\varepsilon}{\varphi(t)} \leq C \, \frac{r^\varepsilon}{\varphi(r)} \qquad \text{for all} \quad t \geq r > 0\,.$$

Then for all $N, M \in \mathbb{Z}, \quad N \leq M$, we have

$$\|S_{N,M} | M_p^\varphi(\mathbb{T}) \to M_p^\varphi(\mathbb{T})\| \leq \|H| M_p^\varphi(\mathbb{T}) \to M_p^\varphi(\mathbb{T})\|. \tag{3}$$

To reach our goal we need following vector-valued version of Corollary 1.

Theorem 1 [5, Theorem 1]. Let $1 < p < \infty$ and a function $\varphi \in G_p$, $\varphi \in \mathcal{G}_p$. We assume that there exist some $\varepsilon > 0$ and a constant $C > 0$ such that

$$\frac{t^\varepsilon}{\varphi(t)} \leq C \, \frac{r^\varepsilon}{\varphi(r)} \qquad \text{for all} \quad t \geq r > 0\,.$$

Then for all $(N_j)_j, (M_j)_j$ of complex numbers, satisfying $N_j \leq M_j$ for all j, and all sequences $(f_j)_j \subset M_p^\varphi(\mathbb{T})$ we have

$$\Big\| \Big(\sum_{j=0}^{\infty} |S_{N_j, M_j} f_j | M_p^\varphi(\mathbb{T}) \Big\| \leq C \Big(\|H| M_p^\varphi(\mathbb{T}) \to M_p^\varphi(\mathbb{T})\| \Big) \Big\| \sum_{j=0}^{\infty} |f_j|^q \Big) \frac{1}{q} | M_p^\varphi(\mathbb{T}) \Big\|. \tag{4}$$

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Remark 3. In case $\varphi(r) := |B(0,r)|^{\frac{1}{p}}$, $r > 0$, the Theorem 1 has been known before, we refer to [6] and $\varphi(r) := |B(0,r)|^{-\lambda+\frac{1}{p}}$, $r > 0$, we refer to [7].

Looking at the Fourier side Corollary 1 and Theorem 1 can be interpreted as Fourier multipliers assertion with characteristic functions of intervals as multipliers. For later use, we need a variant with smooth multipliers. Smoothness is measured in terms of the Bessel potential spaces $H_2^\kappa(\mathbb{R})$. These classes are defined as follows.

Definition 6. Let $\kappa \geq 0$. Then $H_2^\kappa(\mathbb{R})$ is the collection of all $f \in L_2(\mathbb{R})$ such that

$$\|f|H_2^\kappa(\mathbb{R})\| := \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\kappa/2}\mathcal{F}f(\xi)]|L_2(\mathbb{R})\| < \infty.$$

Theorem 2 [5, Theorem 2]. Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $\varphi \in \mathcal{G}_p$. Assume (2) is satisfied. Let $(\Lambda_j)_j$ be a given family of finite and nontrivial intervals. With d_j we denote the length of Λ_j. Let

$$\kappa > \frac{1}{2} + \frac{1}{min(p,q)}.$$

Then there exists a constant C such that the inequality

$$\Big\|\Big(\sum_{j=0}^{\infty}\Big|\sum_{k=-\infty}^{\infty} M_j(k)c_k(f_j)e^{ikx}\Big|^q\Big)^{1/q}|M_p^\varphi(\mathbb{T})\Big\| \leq$$

$$\leq C \sup_{j=0,1,\dots} \|M_j(d_j\cdot)|H_2^\kappa(\mathbb{R})\| \, \|(f_j)_j|M_p^\varphi(\mathbb{T},l_q)\|$$

holds for all sequences of functions $M_j \in H_2^\kappa(\mathbb{R})$ and all sequences $(f_j)_j$ of trigonometric polynomials such that

$$c_k(f_j) = 0 \qquad \text{if} \quad k \notin \Lambda_j, \qquad j \in \mathbb{N}_0.$$

Now let us to give the definition of generalized periodic Lizorkin-Triebel- Morrey spaces. In fact, for us the scale of the Lizorkin-Triebel-Morrey spaces will be more important one.

Let $\psi \in C_0^\infty(\mathbb{R})$ be a function such that

$$\psi(x) := \begin{cases} 1 & \text{if} \quad |x| \leq 1\,, \\ 0 & \text{if} \quad |x \geq \frac{3}{2}|. \end{cases} \tag{5}$$

Then, with $\phi_0 := \psi$,

$$\phi(x) := \phi_0(x/2) - \phi_0(x) \qquad \text{and} \qquad \phi_j(x) := \phi(2^{-j+1}x), \qquad j \in \mathbb{N}. \tag{6}$$

This implies

$$\sum_{j=0}^{\infty}\phi_j(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

We shall call $(\phi_j)_{j=0}^\infty$ a smooth dyadic decomposition of unity.

Definition 7. Let $(\phi_j)_j$ be a smooth dyadic decomposition of unity as defined (5), (6). Let $s > 0$, $0 \leq q \leq \infty$, $0 < p < \infty$ and a function $\varphi \in \mathcal{G}_p$. Assume (2). Then $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ is defined to be the set of all $f \in M_p^\varphi(\mathbb{T})$ such that

$$\|f|\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})\| := \Big\|\Big(\sum_{j=0}^{\infty}2^{jsq}\Big|\sum_{k=-\infty}^{\infty}\phi_j(k)c_k(f)e^{ikx}\Big|^q\Big)^{1/q}\Big|M_p^\varphi(\mathbb{T})\Big\| < \infty.$$

Remark 4. Taking $\varphi(r) := |B(0,r)|^{\frac{1}{p}}$, $r > 0$, we are back in the case of classical periodic Lizorkin-Triebel spaces, i.e., we have

$$\mathcal{E}^s_{\varphi,p,q}(\mathbb{T}) = F^s_{p,q}(\mathbb{T}).$$

We shall call the spaces $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ generalized periodic Lizorkin-Triebel-Morrey spaces. They represent the Lizorkin-Triebel scale built on the generalized Morrey space $M_p^\varphi(\mathbb{T})$. The nonperiodic version of this scale of spaces has been introduced by Tang and Xu in the year 2005 (for Morrey spaces). Lizorkin-Triebel-Morrey spaces, related to generalized Morrey spaces, have been considered recently by Nakamura, Noi and Sawano [7].

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Lemma 1 [8, Lemma 4]. Under the restrictions to parameters p, q, s, φ in Definition 7 the periodic Lizorkin-Triebel-Morrey spaces $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ is independent of the chosen smooth dyadic decomposition of unity, i.e., if we change the smooth dyadic decomposition of unity, then this change results in an equivalent quasi-norm.

We turn to an interesting characterization of the spaces $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ by using differences of partial sums of the Fourier series.

We define

$$S_N f(x) = S_{-N,N} f(x) = \sum_{k=-N}^{N} c_k(f) e^{ikx} \quad x \in \mathbb{R}, \quad N \in \mathbb{N}_0.$$

Here $c_k(f)$ is the Fourier coefficient of f given by

$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

Theorem 3. Let $1 < p, q < \infty, s > 0$ and $\varphi \in \mathcal{G}_p$. Assume (2) is satisfied. A function $f \in M_p^\varphi(\mathbb{T})$ belongs to $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ if and only if

$$\|f|\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})\|^* := \|S_1 f|M_p^\varphi(\mathbb{T})\| + \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |S_{2^{j+1}} f - S_{2^j} f|^q \Big)^{1/q} |M_p^\varphi(\mathbb{T}) \Big\| < \infty.$$

Furthermore the quantities $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|^*$ and $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|$ are equivalent on $M_p^\varphi(\mathbb{T})$, there exist two positive constants A,B such that for all $f \in M_p^\varphi(\mathbb{T})$,

$$A \, \|f|\mathcal{E}^s_{\varphi,p,q}\|^* \leq \|f|\mathcal{E}^s_{\varphi,p,q}\| \leq B \, \|f|\mathcal{E}^s_{\varphi,p,q}\|^*.$$

Proof of Theorem 3. We fix a dyadic decomposition of unity. Let $\psi \in C_0^\infty(\mathbb{R})$ be a function such that

$$\psi(x) := \begin{cases} 1 & \text{if} \quad |x| \leq 1, \\ 0 & \text{if} \quad |x \geq \frac{3}{2}|. \end{cases}$$

Then, $\phi_0 : \psi(x)$ Then, with $\phi_0 := \psi$,

$$\phi(x) := \phi_0(x/2) - \phi_0(x) \qquad \text{and} \qquad \phi_j(x) := \phi(2^{-j+1}x), \qquad j \in \mathbb{N},$$

we have

$$\sum_{j=0}^{\infty} \phi_j(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}.$$

It will be convenient to use abbreviation

$$f_j(x) = \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(f) e^{ikx}. \tag{7}$$

We find that

$$S_{2^{j+1}} f - S_{2^j} f = \sum_{2^j < |k| \leq 2^{j+1}} c_k(f) e^{ikx} =$$

$$= \sum_{2^j < |k| \leq 2^{j+1}} (\phi_{j-1}(k) + \phi_j(k) + \phi_{j+1}(k)) c_k(f) e^{ikx} =$$

$$= (S_{2^{j+1}} f_{j-1} - S_{2^j} f_{j-1}) + (S_{2^{j+1}} f_j - S_{2^j} f_j) + (S_{2^{j+1}} f_{j+1} - S_{2^j} f_{j+1}),$$

where we used abbreviation (7) again. Applying Theorem and generalized Minkowski inequality, we have

$$\Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |S_{2^{j+1}} f - S_{2^j} f|^q \Big)^{1/q} |M_p^\varphi(\mathbb{T}) \Big\| \leq \sum_{l=-1}^{1} \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |S_{2^{j+1}} f_{j+l} - S_{2^j} f_{j+l}|^q \Big)^{1/q} |M_p^\varphi(\mathbb{T}) \Big\| \leq$$

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$$\leq C_1 \sum_{l=-1}^{1} \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |f_{j+l}|^q \Big)^{1/q} |M_p^{\varphi}(\mathbb{T}) \Big\| \leq c_2 \|f|\mathcal{E}^s_{\varphi,p,q}\|.$$

The estimate of the term $\|S_1 f|M_p^{\varphi}(\mathbb{T})\|$ can be done in the same manner:

$$\|S_1 f|M_p^{\varphi}(\mathbb{T})\| \leq (\|H|M_p^{\varphi}(\mathbb{T}\| + 1)\|f|M_p^{\varphi}(\mathbb{T}\| = C_3 \|f|M_p^{\varphi}(\mathbb{T})\|.$$

This prove that $A\|f|\mathcal{E}^s_{\varphi,p,q}\|^* \leq \|f|\mathcal{E}^s_{\varphi,p,q}\|$ with an appropriate positive constant A. To prove

$$\|f|\mathcal{E}^s_{\varphi,p,q}\| \leq B\, \|f|\mathcal{E}^s_{\varphi,p,q}\|^*$$

we proceed similarly. Let $j \geq 1$. For brevity, we put

$$g_j := S_{2^{j+1}} f - S_{2^j} f.$$

We start with the identity

$$f_j(x) = \sum_{2^{j-1} < |k| < 32^{j-1}} \phi_j(k) c_k(f) e^{ikx}.$$

In the strength of the system's orthonormality $e^{ikx}{}_{k=-\infty}^{+\infty}$ on $[-\pi, \pi]$, we have

$$c_k(g_j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_j(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big(\sum_{2^j < |\nu| \leq 2^{j+1}} c_\nu(f) e^{i\nu x} \Big) e^{-ikx} dx = c_k(f).$$

Therefore

$$f_j(x) = \sum_{2^{j-1} < |k| < 32^{j-1}} \phi_j(k) c_k(g_{j-1} + g_j + g_{j+1}) e^{ikx} =$$

$$= \sum_{l=-1}^{1} \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(g_{j+l}) e^{ikx}.$$

Hence

$$\|f|\mathcal{E}^s_{\varphi,p,q}\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} |f_j(x)|^q \Big)^{1/q} \Big| M_p^{\varphi}(\mathbb{T}) \Big\| =$$

$$= \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big| \sum_{l=-1}^{1} \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(g_{j+l}) e^{ikx} \Big|^q \Big)^{\frac{1}{q}} \Big| M_p^{\varphi}(\mathbb{T}) \Big\| =$$

$$= \Big\| \Big(\sum_{j=0}^{\infty} \Big| \sum_{l=-1}^{1} \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(2^{js} g_{j+l}) e^{ikx} \Big|^q \Big)^{\frac{1}{q}} \Big|.$$

Then argue by using Theorem 2 instead of Theorem 1 to the last sum, considering

$$M_j(k) = \phi_j(k); \qquad f_j(k) = 2^{js} g_j(x); \qquad c_k(f_j) = c_k(2^{js} g_j); \qquad \Lambda_j = (2^j, 2^{j+1}).$$

Then we get

$$\Big\| \Big(\sum_{j=0}^{\infty} \Big| \sum_{l=-1}^{1} \sum_{k=-\infty}^{\infty} \phi_j(k) c_k(2^{js} g_{j+l}) e^{ikx} \Big|^q \Big)^{\frac{1}{q}} \Big| \leq$$

$$\leq C \cdot \sup_{j=o,1,2\ldots} \|\phi_j(d_j \cdot)|H_2^{\kappa}(\mathbb{R})\| \cdot \|2^{js}(g_j)_j|M_p^{\varphi}(\mathbb{T}, l_q)\| =$$

$$= B \cdot \Big\| \Big(\sum_{j=0}^{\infty} \Big| 2^{js}(g_j)_j \Big|^q \Big)^{\frac{1}{q}} \|M_p^{\varphi}(\mathbb{T}) \Big\| =$$

$$= B \cdot \Big\| \Big(\sum_{j=0}^{\infty} \Big| 2^{jsq} \Big| S_{2^{j+1}f - S_{2^j} f} \Big|^q \Big)^{\frac{1}{q}} \|M_p^{\varphi}(\mathbb{T}) \Big\|.$$

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Remark 5. In case $\varphi(r) := |B(0,r)|^{1/p}, r > 0$ this goes back to Lizorkin-Triebel and in case $\varphi(r) := |B(0,r)|^{-\lambda+1/p}$, $r > 0$ this theorem have been proved in [7].

To prove next theorem we add the following.

Corollary 2. [8]. Let $1 < p < \infty$, $1 \le q \le \infty$, $s > 0$ and a function $\varphi \in G_p$. Then for all $f \in \mathcal{N}^s_{\varphi,p,q}(\mathbb{T}) \cup \mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$,

$$\lim_{j\to\infty} \|S_{2^j} f - f|M^\varphi_p(\mathbb{T})\| = 0.$$

Now we are going to reach our goal in this paper. The next theorem is basic in this paper. Instead of considering dyadic subsequences of $(S_N)_N$, we switch now to the sequence $(f - S_N)_N$.

Theorem 4. Let $1 < p, q < \infty$, $s > 0$ and a function $\varphi \in \mathcal{G}_p$. A function $f \in M^\varphi_p(\mathbb{T})$ belongs to $\mathcal{E}^s_{\varphi,p,q}(\mathbb{T})$ if and only if

$$\|f|\mathcal{E}^s_{\varphi,p,q}\|^\# :=$$

$$:= \|S_1 f|M^\varphi_p(\mathbb{T})\| + \Big\|\Big(\sum_{N=1}^{\infty} N^{(s-1/q)q}|f - S_N f|^q\Big)^{1/q}|M^\varphi_p(\mathbb{T})\Big\| < \infty. \tag{8}$$

Furthermore the quantities $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|^\#$ and $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|$ are equivalent on $M^\varphi_p(\mathbb{T}^d)$, there exist two positive constants A,B such that for all $f \in M^\varphi_p(\mathbb{T})$,

$$A\|f|\mathcal{E}^s_{\varphi,p,q}\|^\# \le \|f|\mathcal{E}^s_{\varphi,p,q}\| \le B\|f|\mathcal{E}^s_{\varphi,p,q}\|^\#.$$

Proof of Theorem 4. By Theorem 3, it will be sufficient to compare $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|^\#$ and $\|\cdot|\mathcal{E}^s_{\varphi,p,q}\|^*$.

Step 1. We shall prove that

$$\|f|\mathcal{E}^s_{\varphi,p,q}\|^\# \le c_1\|f|\mathcal{E}^s_{\varphi,p,q}\|^* \tag{9}$$

with some constant C_1 independent of f. First, we split the sum $\sum_{N=1}^\infty$ into dyadic blocks. More exactly, we set

$$\sum_{N=1}^{\infty} N^{(s-1/q)q}|f - S_N f|^q = \sum_{j=0}^{\infty}\sum_{N=2^j}^{2^{j+1}-1} N^{(s-1/q)q}|f - S_N f|^q \le$$

$$\le C_2 \sum_{j=0}^{\infty} 2^{j(s-1/q)q} \sum_{N=2^j}^{2^{j+1}-1} |f - S_N f|^q,$$

where C_2 is equal to 1 if $s - 1/q \le 0$ and equal to $2^{(s-1/q)q}$ otherwise. Next, we use the identities

$$f - S_N f = f - S_{2^{j+1}} f + S_{2^{j+1}} f - S_N f$$

and

$$S_{2^{j+1}} f - S_N f = S_{N,2^{j+1}}\Big(S_{2^{j+1}} f - S_{2^j} f\Big).$$

Inserting these identities in the previous inequality and applying Theorem 1 yields

$$\Big\|\Big(\sum_{N=1}^{\infty} N^{(s-1/q)q}|f - S_N f|^q\Big)^{1/q}|M^\varphi_p(\mathbb{T}^d)\Big\| \le C_2\Big\|\Big(\sum_{j=0}^{\infty} 2^{j(s-1/q)q}2^j|f - S_{2^{j+1}} f|^q\Big)^{1/q}|M^\varphi_p(\mathbb{T}^d)\Big\| +$$

$$+C_2\Big\|\Big(\sum_{j=0}^{\infty} 2^{j(s-1/q)q} \sum_{N=2^j}^{2^{j+1}-1} |S_{N,2^{j+1}}\Big(S_{2^{j+1}} f - S_{2^j} f|^q\Big)^{1/q}|M^\varphi_p(\mathbb{T}^d)\Big\| +$$

$$+C_3\Big\|\Big(\sum_{j=0}^{\infty} 2^{j(s-1/q)q} \sum_{N=2^j}^{2^{j+1}-1} |S_{2^{j+1}} f - S_{2^j} f|^q\Big)^{1/q}|M^\varphi_p(\mathbb{T}^d)\Big\|.$$

Since

$$\Big\|\Big(\sum_{j=0}^{\infty} 2^{j(s-1/q)q} \sum_{N=2^j}^{2^{j+1}-1} |S_{2^{j+1}} f - S_{2^j} f|^q\Big)^{1/q}|M^\varphi_p(\mathbb{T}^d)\Big\| \le \Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|S_{2^{j+1}} f - S_{2^j} f|^q\Big)^{1/q}|M^\varphi_p(\mathbb{T}^d)\Big\|,$$

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it remains to estimate

$$\Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|f-S_{2^{j+1}}f|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|.$$

Obviously, as a consequence of Corollary 2, we have

$$f-S_{2^{j+1}}f=\lim_{M\to\infty}S_{2^M}f-S_{2^{j+1}}f=\lim_{M\to\infty}\sum_{l=j+1}^{M-1}S_{2^{l+1}}f-S_{2^l}f=$$

$$=\sum_{l=j+1}^{\infty}S_{2^{l+1}}f-S_{2^l}f=\sum_{l=1}^{\infty}S_{2^{l+j+1}}f-S_{2^{l+j}}f.$$

Hence it follows that

$$\Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|f-S_{2^{j+1}}f|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|\le$$

$$\le\sum_{l=1}^{\infty}\Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|S_{2^{l+j+1}}f-S_{2^{j+l}}f|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|\le$$

$$\le\sum_{l=1}^{\infty}2^{-ls}\Big\|\Big(\sum_{j=0}^{\infty} 2^{(j+l)sq}|S_{2^{l+j+1}}f-S_{2^{l+j}}f|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|\le\|f|\mathcal{E}^s_{\varphi,p,q}\|^*\Big(\sum_{l=1}^{\infty}2^{-ls}\Big).$$

Since $s>0$, the geometric series is convergent. This completes the proof of (9).

Step 2. We shall prove that

$$\|f|\mathcal{E}^s_{\varphi,p,q}\|^*\le c_4\|f|\mathcal{E}^s_{\varphi,p,q}\|^{\#} \tag{10}$$

with some constant c_4 independent of f. This time we use the identity

$$S_{2^{j+1}}f-S_{2^j}f=\sum_{2^j<\|k\|_\infty\le 2^{j+1}}c_k(f-S_Nf)e^{ikx},\quad N=2^{j-1},...,2^j-1,$$

which implies that

$$|S_{2^{j+1}}f-S_{2^j}f|^q=2^{-(j-1)}\sum_{N=2^{j-1}}^{2^j-1}\Big|\sum_{2^j<\|k\|_\infty\le 2^{j+1}}c_k(f-S_Nf)e^{ikx}\Big|^q.$$

Hence it follows that

$$\Big\|\Big(\sum_{j=0}^{\infty} 2^{jsq}|S_{2^{j+1}}f-S_{2^j}f|^q\Big)^{1/q}\Big|M_p^{\varphi}(\mathbb{T}^d)\|=$$

$$=\Big\|\Big(\sum_{j=0}^{\infty}2^{jsq}2^{-(j-1)}\sum_{N=2^j}^{2^j-1}\Big|\sum_{2^j<|k|\le 2^{j+1}}c_k(f-S_Nf)e^{ikx}\Big|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|\le$$

$$\le C_5\Big\|\Big(\sum_{j=0}^{\infty}2^{j(s-1/q)q}\sum_{N=2^j}^{2^j-1}|f-S_Nf|^q\Big)^{1/q}|M_p^{\varphi}(\mathbb{T}^d)\Big\|,$$

where we used Theorem 1 in the last step. Since

$$2^{j(s-1/q)q}\le C_6N^{(s-1/q)q},\quad 2^{j-1}\le N\le 2^j,$$

with some constant C_6 independent of j, this yields (10).

Remark 6. In case $\varphi(r):=|B(0,r)|^{1/p}, r>0$ we refer [6] and in case $\varphi(r):=|B(0,r)|^{-\lambda+1/p}, r>0$ this theorem have been proved in [9].

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Жалпыланған периодты Морри кеңiстiгiнде Фурье қатарларымен күштi қосындылау

Соңғы жылдары Морри типтi кеңiстiктердi зерттеу үлкен қызығушылық тудырды. G.Di Fazioand, M. Ragusa жұмыстарында және T. Mizuhara кiтабында Морри типтес кеңiстiктердiң дербес туындылы теңдеулердiң жиi қолданулылары көрсетiлген. T. Mizuhara және E. Nakai ұсынған жалпыланған Морри кеңiстiктерiнде параметр және функция енгiзiлген. Осы жұмыста Лизоркина-Трибель кеңiстiгi Морри кеңiстiктегi шкалада жасалған. Алдымен, Морри мен жалпыланған Морри кеңiстiктерiнiң анықтамасы берiлдi. Әрi қарай Гильберт түрлендiруiнiң шенелгендiгi көрсетiлдi. Бұл бiздiң ең басты құралымыз. Периодты емес Морри кеңiстiгiндегi Гильберт түрлендiруiнiң шенелгендiгiн қолдана отырып, периодты Морри кеңiстiгiне өткiземiз. Гильберт түрлендiруiнiң шенелгендiгi Фурье қатарының дербес қосындысының оператор нормасының бiрқалыпты бағалауын бiлдiредi. Векторлы түрдегi Фурье мультипликаторы зерттелдi. Осының негiзiнде белгiлi Рисс теоремасының векторлы түрi алынады. Мұнда бiз Лизоркин-Трибель кеңiстiктерiнiң жалпыланған Морри кеңiстiгiндегiсiн қарастырамыз. Жұмыстың соңында Фурье қатарының дербес қосындысының айырмасын қолдана отырып, жалпыланған Трибель-Лизоркин-Морри кеңiстiгiнде $\mathcal{E}^s_{\varphi,p,q}(T)$ қызықты сипаттамасы алынған және Морри кеңiстiгiнде Фурье қатарының аппроксимациялауы көрсетiлген.

Кiлт сөздер: Морри кеңiстiгi, жалпыланған Морри кеңiстiгi, қатаң қосындылау, Рисс теоремасының векторлы түрi.

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