

D.M.Akhmanova¹, M.T.Kosmakova¹, M.I.Ramazanov¹, A.Ye.Tuimebayeva²

¹Ye.A.Buketov Karaganda State University;

²M.Kh.Dulati Taraz State University (E-mail: akner.nur@gmail.com)

On the solutions of the homogeneous mutually conjugated Volterra integral equations

The paper deals with a pair of conjugate integral equations of Volterra type of the second kind with a variable limit of integration. Feature of equations consists of incompressibility of the corresponding kernels and, consequently, of the non-applicability of the successive approximation method. By using the Carleman-Vekua method of regularization the investigated equation is reduced to the Abel equation of the second kind. The existence of the non-trivial solutions is established for homogeneous Volterra integral equation of the second kind with a special kernel and its adjoint equation in weight Lebesgue classes.

Key words: Volterra integral equation, a special kernel, non-trivial solution, the weight Lebesgue class, index.

The solution of boundary value problems of the heat equation in the degenerate domains leads to the need to study Volterra integral equations, a feature of which is that the norm of the integral operator is equal to 1 [1]. Boundary value problems for parabolic equations in the domain with the moving boundary are fundamentally different from the classical ones. Due to the dependence on the time the size of the domain the classical methods of solutions not are applied to this type of problems. From the mathematical point of view, the problematic of the problem consists in the presence of a moving boundary and the degeneracy of the solution domain at the initial time.

1 Statement of the problem. Find a non-trivial solutions of the homogeneous integral equation

$$L\varphi = (I - K)\varphi \equiv \varphi(t) - \int_0^t K(t, \tau) \varphi(\tau) d\tau = 0, \quad t > 0, \quad (1)$$

and its adjoint homogeneous equation

$$L^*\psi = (I - K^*)\psi \equiv \psi(t) - \int_t^\infty K(\tau, t) \psi(\tau) d\tau = 0, \quad t > 0, \quad (2)$$

where

$$K(t, \tau) = \frac{1}{2\sqrt{\pi}} \left(\frac{t + \tau}{(t - \tau)^{3/2}} \exp \left\{ -\frac{(t + \tau)^2}{4(t - \tau)} \right\} + \frac{1}{\sqrt{t - \tau}} \exp \left\{ -\frac{t - \tau}{4} \right\} \right). \quad (3)$$

The kernels (3) integral equations (1) and (2) have the following properties

1) $K(t, \tau) \geq 0$, $K(\tau, t) \geq 0$ and are continuous;

2) $\lim_{t \rightarrow t_0} \int_{t_0}^t K(t, \tau) d\tau = 0$, $t_0 \geq \varepsilon > 0$;

3) $\lim_{t \rightarrow +0} \int_0^t K(t, \tau) d\tau = 1$; $\lim_{t \rightarrow \infty} \int_t^\infty K(\tau, t) d\tau = 1$.

The kernel feature of equations (1) and (2) is the presence of 3) property.

In the domain $G = \{(x, t): 0 < x < t, t > 0\}$ the following homogeneous boundary value problem is reduced to the integral equation (1)

$$u_t(x, t) = u_{xx}(x, t), \quad u(x, t)|_{x=0} = 0, \quad u(x, t)|_{x=t} = 0; \quad (4)$$

if we look for its solution in the form of the sum of heat double layer potentials [2]:

$$u(x, t) = \int_0^t \frac{x}{(t - \tau)^{3/2}} \exp \left\{ -\frac{x^2}{4(t - \tau)} \right\} v(\tau) d\tau + \int_0^t \frac{x - \tau}{(t - \tau)^{3/2}} \exp \left\{ -\frac{(x - \tau)^2}{4(t - \tau)} \right\} \varphi(\tau) d\tau.$$

Where the functions $v(t)$, $\varphi(t)$ are unknown. For the unknown function $\varphi(t)$ we have the integral equation (1), and the function $v(t)$ is explicitly expressed through $\varphi(t)$. The integral equation (2) corresponds to the problem adjoint to the boundary value problem (4)

$$-v_t = v_{xx}, \quad v(x, t)|_{x=t} = 0, \quad v(x, t)|_{x=\infty} = 0.$$

2 The transformed integral equations. If we put

$$k(t, \tau) = K(t, \tau) \exp\left\{\frac{t-\tau}{4}\right\}, \quad \tilde{\varphi}(t) = \varphi(t) \exp\left\{\frac{t}{4}\right\}, \quad \tilde{\psi}(t) = \psi(t) \exp\left\{-\frac{t}{4}\right\}; \quad (5)$$

then equations (1) and (2) are equivalent to the integral equations

$$\tilde{\varphi}(t) - \int_0^t k(t, \tau) \tilde{\varphi}(\tau) d\tau = 0, \quad t > 0; \quad (6)$$

$$\tilde{\psi}(t) - \int_t^\infty k(\tau, t) \tilde{\psi}(\tau) d\tau = 0, \quad t > 0. \quad (7)$$

Singling out the the characteristic part $k_h(t, \tau)$ of the kernel of the integral operator in (6)

$$k(t, \tau) = k_h(t, \tau) + k_w(t, \tau);$$

we obtain

$$\tilde{\varphi}(t) - \int_0^t k_h(t, \tau) \tilde{\varphi}(\tau) d\tau = f(t), \quad t > 0; \quad (8)$$

where

$$k_h(t, \tau) = \frac{t}{\sqrt{\pi}(t-\tau)^{3/2}} \exp\left\{-\frac{t\tau}{t-\tau}\right\}; \quad (9)$$

$$k_w(t, \tau) = \frac{1}{\sqrt{\pi}(t-\tau)^{1/2}} \left(1 - \exp\left\{-\frac{t\tau}{t-\tau}\right\}\right); \quad (10)$$

$$f(t) = \int_0^t k_w(t, \tau) \tilde{\varphi}(\tau) d\tau. \quad (11)$$

The following lemmas are true.

Lemma 1. The inhomogeneous integral equation (8) is characteristic one for the equation (6) (temporarily we consider that the function $f(t)$ is the well-known).

The proof of Lemma 1 follows from the relations for (9) and (10)

$$\lim_{t \rightarrow +0} \int_0^t k_h(t, \tau) d\tau = 1, \quad \lim_{t \rightarrow +0} \int_0^t k_w(t, \tau) d\tau = 0.$$

It's in fact

$$\begin{aligned} \lim_{t \rightarrow 0} \int_0^t k_h(t, \tau) d\tau &= \frac{1}{\sqrt{\pi}} \lim_{t \rightarrow 0} \int_0^t \frac{t}{(t-\tau)^{3/2}} \exp\left\{-\frac{t\tau}{t-\tau}\right\} d\tau = \left\| z = \frac{t}{\sqrt{t-\tau}} \right\| = \\ &= \frac{2}{\sqrt{\pi}} \lim_{t \rightarrow 0} \int_{\sqrt{t}}^\infty \exp\left\{-(z^2-t)\right\} dz = \lim_{t \rightarrow 0} \left\{ e^t \cdot \operatorname{erfs}(\sqrt{t}) \right\} = 1. \end{aligned}$$

The validity of equality

$$\lim_{t \rightarrow 0} \int_0^t k_w(t, \tau) d\tau = 0,$$

follows from the estimate

$$k_w(t, \tau) = \frac{1}{2\sqrt{\pi}(t-\tau)^{1/2}} \left(1 - \exp\left\{-\frac{t\tau}{t-\tau}\right\}\right) \leq \frac{1}{2\sqrt{\pi}(t-\tau)^{1/2}} \cdot \frac{t\tau}{t-\tau} = \frac{t\tau}{2\sqrt{\pi}(t-\tau)^{3/2}};$$

that is the function $k_w(t, \tau)$ has a weak singularity.

Lemma 2. Equation (8) is equivalent to the integral equation

$$\tilde{\varphi}(t) = f(t) + \int_0^t r(t, \tau) f(\tau) d\tau + \frac{C_1}{\sqrt{t}}, \quad t > 0, \quad (12)$$

where

$$r(t, \tau) = \frac{1}{\sqrt{\pi}(t-\tau)^{3/2}} \sum_{n=1}^\infty n \cdot \exp\left\{-\frac{n^2 t \tau}{t-\tau}\right\}, \quad t > \tau, \quad C_1 = \operatorname{const} > 0. \quad (13)$$

To prove Lemma 2, as in [3], we reduce the integral equation (12) to an equation with a difference kernel. To do this, we will perform the replacement in it:

$$t = y^{-1}, \quad \tau = x^{-1}, \quad \tilde{\varphi}_1(y) = y^{-1/2}\tilde{\varphi}(y^{-1}), \quad f_1(y) = y^{-1/2}f(y^{-1}); \tag{14}$$

and we will reduce the equation (8) to an integral equation with a difference kernel

$$\tilde{\varphi}_1(y) - \int_y^\infty R(x-y)\tilde{\varphi}_1(x)dx = f_1(y), \tag{15}$$

where

$$R(\theta) = \frac{1}{\sqrt{\pi}\theta^{3/2}} \exp\left\{-\frac{1}{\theta}\right\}.$$

From the results of paper it follows, firstly, that the equation (15) with $f_1(y) \equiv 0$ has only a unique solution $\tilde{\varphi}_1(y) = C_1$ and, secondly, that the solution of the inhomogeneous equation (15) has a representation

$$\tilde{\varphi}_1(y) = f_1(y) + \int_y^\infty r_-(y-x)f_1(x)dx + C_1, \tag{16}$$

where

$$r_-(\theta) = \frac{1}{\sqrt{\pi}\theta^{3/2}} \sum_{n=1}^\infty n \cdot \exp\left\{-\frac{n^2}{\theta}\right\}, \quad \theta > 0.$$

Note that the representation (16) is also obtained in the paper [4].

Further performing the reverse replacements to (14) we obtain assertion of the lemma 2 (12)–(13).

3 Reduction of the equation (6) to the Abel equation and the solution of the integral equation (1).

Following [4] we substitute in (12) the function defined by relation (11). Taking (13) and changing the order of integration in the double integral, we obtain

$$\tilde{\varphi}(t) = \int_0^t [k_w(t, \tau) + J(t, \tau)] \tilde{\varphi}(\tau) d\tau + \frac{C_1}{\sqrt{t}}, \quad t > 0, \tag{17}$$

where

$$J(t, \tau) = \int_\tau^t r(t, \tau_1) k_w(\tau_1, \tau) d\tau_1;$$

and performing the necessary calculations in the latter expression we have

$$J(t, \tau) = \frac{1}{2a\sqrt{\pi}(t-\tau)} \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}. \tag{18}$$

Substituting (18) into (17), we finally have (for the sake of simplicity $C_1 = 1$ is accepted)

$$\tilde{\varphi}(t) - \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\tilde{\varphi}(\tau)}{\sqrt{t-\tau}} d\tau = \frac{1}{\sqrt{t}}, \quad t > 0. \tag{19}$$

Equation (19) is an inhomogeneous Abel integral equation of the second kind. By applying the theory of fractional differentiation and integration [5; 38–39, 41–43, 84–86], or [6, 7], we find the solution of equation (19)

$$\tilde{\varphi}(t) = \frac{1}{\sqrt{t}} + \frac{\sqrt{\pi}}{2} \exp\left\{\frac{t}{4}\right\} \left[1 + \operatorname{erf}\left(\frac{\sqrt{t}}{2}\right)\right] \tag{20}$$

which is also a solution of the equation (6).

Finally, considering the formula (5) from (20) we obtain the validity of a theorem.

Theorem 1. Homogeneous integral equation (1) has a unique non-trivial solution

$$\varphi(t) = \frac{1}{\sqrt{t}} \exp\left\{-\frac{t}{4}\right\} + \frac{\sqrt{\pi}}{2} \left[1 + \operatorname{erf}\left(\frac{\sqrt{t}}{2}\right)\right], \quad t > 0. \tag{21}$$

Moreover, the solution (21) belongs to the Lebesgue weight class

$$\sqrt{t} \exp\left\{-\frac{\varepsilon t}{4}\right\} |\varphi(t) - \sqrt{\pi}| \in L_\infty(0, \infty), \quad \varepsilon > 0, \tag{22}$$

has a feature in the point $t = 0$, and $\operatorname{Ker}\{L\} = 1$.

4 The solution of the integral equation (2). For this purpose we will solve the equation (7). With the help of substitutions

$$t = y^{-1}, \quad \tau = x^{-1}, \quad \tilde{\Psi}_1(y) = y^{-3/2} \tilde{\psi}(y^{-1}); \tag{23}$$

equation (7) we reduce to the form

$$y \cdot \tilde{\Psi}_1(y) - \int_0^y S_1(y-x) \tilde{\Psi}_1(x) dx - y \cdot \int_0^y S_2(y-x) \tilde{\Psi}_1(x) dx = 0, \tag{24}$$

where

$$S_1(z) = \frac{1}{2\sqrt{\pi \cdot z}} \left(1 - \exp\left\{-\frac{1}{z}\right\} \right), \quad S_2(z) = \frac{1}{2\sqrt{\pi \cdot z^{3/2}}} \exp\left\{-\frac{1}{z}\right\}. \tag{25}$$

Applying to the equation (24) the Laplace transformation with consideration of representations (25) [8; 216 (formula 22.1), 248 (formula 23.96)], [9] (formula 9.191), we obtain a differential equation in the Laplace images

$$-\frac{d\tilde{\Psi}_1(p)}{dp} - \frac{1}{2\sqrt{p}} \left(1 - \exp\{-2\sqrt{p}\} \right) \tilde{\Psi}_1(p) + \frac{d}{dp} \left(\exp\{-2\sqrt{p}\} \tilde{\Psi}_1(p) \right) = 0.$$

The last equality is given to the differential equation

$$\frac{d\tilde{\Psi}_1(p)}{dp} + \frac{1}{2\sqrt{p}} \cdot \coth\sqrt{p} \tilde{\Psi}_1(p) = 0. \tag{26}$$

The solution of the equation (26) is a function

$$\tilde{\Psi}_1(p) = \frac{C}{sh\sqrt{p}} = C \cdot \operatorname{cosech}\sqrt{p}, \quad C = \text{const.}$$

According to [10] (formula 1.232.3), this function can be written as a series

$$\tilde{\Psi}_1(p) = 2C \cdot \sum_{n=0}^{\infty} \exp\left\{-(2n+1)\sqrt{p}\right\}. \tag{27}$$

To the function (27) we apply the inverse transformation [8; 253 (formula 23.121)]. As a result, we have

$$\tilde{\psi}_1(y) = 2C \cdot \sum_{n=0}^{\infty} \exp\left\{-\frac{(2n+1)^2}{4y}\right\}. \tag{28}$$

Using into (28) the reverse replacements to (23) we obtain the solution of the equation (7)

$$\tilde{\psi}(t) = \frac{C}{\sqrt{\pi}} \cdot \sum_{n=0}^{\infty} (2n+1) \exp\left\{-\frac{(2n+1)^2}{4} t\right\}, \quad t > 0. \tag{29}$$

From (29) and (5) the required solution of the integral equation (2) directly follows

$$\psi(t) = \frac{C}{\sqrt{\pi}} \cdot \sum_{n=0}^{\infty} (2n+1) \exp\left\{-(n^2+n)t\right\}, \quad t > 0. \tag{30}$$

Thus, the following theorems are established.

Theorem 2. The homogeneous adjoint integral equation (2) has a unique non-trivial solution $\psi(t)$ determined by relation (30). Moreover, this solution belongs to the Lebesgue weight class

$$\frac{1}{t} \exp\left\{\frac{\varepsilon t}{4}\right\} \left| \psi(t) - \frac{1}{\sqrt{\pi}} \right| \in L_1(0, \infty), \quad \varepsilon > 0. \tag{31}$$

Theorem 3. The index of the integral operator (1) is equal to zero

$$\operatorname{index}\{L\} = \operatorname{Ker}\{L\} - \operatorname{Coker}\{L\} = 0.$$

So, we have established the weight Lebesgue classes (22) and (31) respectively in which the integral equations (1) and (2) have a unique non-trivial solution. In the next article, by substitution, we will show that the functions (21) and (30) are solutions of integral equations (1) and (2) respectively.

Note that the restriction of the obtained weight Lebesgue classes allows us to find the family of the uniqueness classes for the equations (1) and (2).

The work was financially supported by Science Committee of the MSE. (Grant № 0112 RK 00619/GF on priority «Intellectual Potential of the Country»).

References

- 1 Kim E.I. *Solution of a certain class of singular integral equations with line integrals* // Dokl. Akad. Nauk SSSR. 1957, vol. 113, p. 24–27.
- 2 Tikhonov A.N., Samarski A.A. *Equations of mathematical physics*, Publ. 5-th., Moscow: Nauka, 1977, p. 480.
- 3 Dzhemaliev M.T., Ramazanov M.I. *Loaded equations are as a perturbation of differential equations*, Almaty: Gylym, 2010, p. 174.
- 4 Akhmanova D.M., Kosmakova M.T., Ramazanov M.I. *Solution of a singular Volterra integral equation of second kind* // Bulletin of University of Karaganda. Mathematics ser. // Bull. Karaganda university. Ser. Mathematics, 2012, № 2 (66), p. 27–33.
- 5 Samko S.G., Kilbas A.A., Marichev O.I. *Integrals and derivatives of fractional order and their applications*, Minsk: Nauka and Technology, 1987, p. 688.
- 6 Polyanin A.D., Manzhirov A.V. *Reference book of Integral Equations*, Moscow: Fizmatlit, 2003, p. 608.
- 7 Brakhage K, Nickel K., Rieder P. *Auflosung der Abelschen Integralgleichung – 2. Art* // ZAMP, 1965, vol. 16, Fasc. 2, p. 295–298.
- 8 Ditkin V.A., Prudnikov A.P. *Reference book of operational calculus*, Moscow: Higher School, 1965, p. 466.
- 9 Ditkin V.A., Prudnikov A.P. *Integral transformations. Operational calculus*, Moscow: Fizmatlit, 1961, p. 385.
- 10 Gradshteyn I.S., Ryzhik I.M. *Tables of integrals, sums, rows and products*, Moscow: Fizmatgiz, 1963, p. 37.

Д.М.Ахманова, М.Т.Космакова, М.И.Рамазанов, А.Е.Түймебаева

Біртектес өзара-түйіндес Вольтерра интегралды теңдеуінің шешімі туралы

Мақалада интегралдаудың айнымалы шегімен екінші текті түйіндес интегралды Вольтерра теңдеуі қарастырылды. Теңдеудің ерекшелігі — ядроның сығылмауында және, сәйкесінше, біртіндеп жуықтау әдісін қолдана алмауда. Карлеман-Векуа регуляризация әдісін қолдана отырып, зерттелетін теңдеу екінші текті Абель теңдеуіне әкелінді. Айрықша ядромен біртектес Вольтерра екінші текті интегралды теңдеуі және оған түйіндес салмақты лебег кластарында тривиалды емес шешімнің бар екендігі анықталды.

Д.М.Ахманова, С.Т.Космакова, М.И.Рамазанов, А.Е.Түймебаева

О решениях однородных взаимно-сопряженных интегральных уравнений Вольтерра

В статье рассмотрена пара сопряженных интегральных уравнений типа Вольтерра второго рода с переменным пределом интегрирования. Особенность уравнений заключается в несжимаемости соответствующих ядер и, соответственно, в неприменимости метода последовательных приближений. Применением метода регуляризации Карлемана-Векуа исследуемое уравнение сведено к уравнению Абеля второго рода. Для однородного с особым ядром интегрального уравнения Вольтерра второго рода и его сопряженного установлено существование нетривиального решения в весовых лебеговых классах.

Список литературы

- 1 *Ким Е.И.* Решение одного класса сингулярных интегральных уравнений с линейными интегралами // Докл. АН СССР. — 1957. — Т. 113. — С. 24–27.
- 2 *Тихонов А.Н., Самарский А.А.* Уравнения математической физики. — 5-е изд. — М.: Наука, 1977. — С. 480.
- 3 *Джемалиев М.Т., Рамазанов М.И.* Нагруженные уравнения как возмущения дифференциальных уравнений. — Алматы: Ғылым, 2010. — С. 174.
- 4 *Ахманова Д.М., Космакова М.Т., Рамазанов М.И.* Решение одного сингулярного интегрального уравнения Вольтерра второго рода // Вестн. Караганд. ун-та. Сер. Математика. — 2012. — № 2 (66). — С. 27–33.
- 5 *Самко С.Г., Килбас А.А., Маричев О.И.* Интегралы и производные дробного порядка и некоторые их приложения. — Минск: Наука и техника, 1987. — 688 с.

- 6 Полянин А.Д., Манжиров А.В. Справочник по интегральным уравнениям. — М.: Физматлит, 2003. — С. 117.
- 7 Brakhage K., Nickel K., Rieder P. Auflosung der Abelschen Integralgleichung – 2. Art. // ZAMP. — 1965. — Vol. 16. — Fasc. 2. — P. 295–298.
- 8 Диткин В.А., Прудников А.П. Справочник по операционному исчислению. — М.: Высш. шк., 1965. — 466 с.
- 9 Диткин В.А., Прудников А.П. Интегральные преобразования. Операционное исчисление. — М.: Физматлит, 1961. — С. 385.
- 10 Градштейн И.С., Рыжик И.М. Таблицы интегралов, сумм, рядов и произведений // Физматгиз. — 1963. — С. 37.

Репозиторий КарГУ