https://doi.org/10.31489/2025M1/143-154 *Research article*

Some methods for solving boundary value problems for polyharmonic equations

M.T. Sabirzhanov[1], B.D. Koshanov[2,*], N.M. Shynybayeva[2], P.Zh. Kozhobekova[1]

[1] *Osh State University, Osh, Kyrgyzstan;*
[2] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan*
(E-mail: smskg@bk.ru, koshanov@math.kz, shynybayeva001@mail.ru, pardaz@mail.ru)

This article consists of three sections. In the first section the concept of Vekua space is introduced, where for elliptic systems of the first order, the theorem on the representation of the solution of a homogeneous equation and the theorem on the continuity of the solution of an inhomogeneous equation are valid. In the second section the Vekua method for solving boundary value problems for a polyharmonic equation is described. In the third section the Otelbaev method describes the correct boundary value problems for a polyharmonic equation in a multidimensional sphere.

Keywords: first order elliptic system, polyharmonic equation, continuity of solution, boundary value problem, integral representations of solution.

2020 Mathematics Subject Classification: 31B30, 31B20, 31B10.

1 Vekua space for analytic functions

The question of the existence of solutions in the classical sense and of methods for finding these solutions has not lost interest. It is particularly interesting to consider equations of order higher than two using the method of potential theory.

Recently, the theory of boundary value problems for polyharmonic equations and elliptic systems has attracted the attention of mathematicians, due to their great theoretical and practical importance. For example, hydrodynamic and elasticity problems can be formulated using these equations.

The object of our research is polyharmonic equations. Vekua's method is applicable for fixed boundary conditions on the surface of the domain, and the equation itself can change, i.e. minor terms can be added to the main equation. Otelbaev's method is used for fixed equations, but the boundary conditions can vary.

The problem is that in these methods under what conditions both methods are applicable. This article is devoted to this problem of the applicability of the Vekua and Otelbaev methods.

In [1] I.N. Vekua proved that for a first-order elliptic system

$$Lu = \partial_{\overline{z}} u + a(z)u + b(z)\overline{u} = f, \quad z \in \Omega, \tag{1.1}$$

when $a, b, f \in L_p(\Omega)$, $p > 2$, any solution from $W_2^1(\Omega)$ is continuous in Ω, and any solution of the corresponding homogeneous system is representable in the form

$$u(z) = \Phi(z)e^{\omega(z)}, \tag{1.2}$$

*Corresponding author. *E-mail: koshanov@math.kz*

This research was funded by the grant AP 19678182 of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

Received: 5 June 2024; *Accepted:* 10 December 2024.

where $\Phi(z)$ is analytic in Ω and $\omega(z)$ are continuous functions in Ω. Here Ω is any open set, $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, $i^2 = -1$.

It is not possible to transfer this result to the case $1 \le p \le 2$. Therefore, the problem naturally arises: What should be the Banach space B of a function that is continuously embedded in L_1, such that a) any solution of equation (1.1) from $L_{1,loc}$, when $a, b, f \in B$, is continuous in Ω and b) any solution of the corresponding homogeneous system can be represented in the form (1.2), where $\omega(z)$ is continuous in Ω?

This problem was first posed by M.O. Otelbaev in his works [2, 3] and in the same article gave a comprehensive answer to this problem. To solve this problem, the paper introduces the concept of Vekua spaces (V-spaces), namely: a Banach function space B is a Vekua space if for any $a(z)$, $b(z)$ and $f(z)$, the theorem on the continuity of the solution of an inhomogeneous equation and the theorem on the representation of solutions of a homogeneous equation are valid.

Let $\mathbb{C}$ be the complex plane of points $z = x+iy$, and let Q be the square $\{-\pi \le x \le \pi, -\pi \le y \le \pi\}$. Let us assume that a certain norm $\|\cdot\|_B$ is defined on the set of trigonometric polynomials $\mathcal{F}$. Let $B(Q)$ denote the Banach space obtained by completing $\mathcal{F}$ with respect to the $\|\cdot\|_B$ norm.

Let us give the exact definition of Vekua space (V-space). Throughout what follows we will assume that $B(Q)$ satisfies the following three properties:

1°. A multiplication operator is defined in the space $B(Q)$. The operator of multiplication by the characteristic function of any rectangle located in Q, and the operator of multiplication by any function $\psi \in C^\infty_\pi(Q)$ are bounded, where $C^\infty_\pi(Q)$ is the space of infinitely smooth periodic functions with period 2π in each variable x, y.

2°. If $f \in B(Q)$ and $a \in \mathbb{C}$, then $f(z+a), |f| \in B(Q)$ and $\|f(z+a)\|_B \le C\,\|f\|_B$, $\|\,|f|\,\|_B \le C\,\|f\|_B$. Here and below, C will denote, generally speaking, various positive constants.

3°. $B(Q)$ is continuously embedded in $L_1(Q)$ $(B(Q) \hookrightarrow L_1(Q))$.

Let $\mathbb{P}_1(Q)$ denote the completion of $\mathcal{F}$ with respect to norm

$$\langle f\rangle_{1,Q} = \sup_{z\in Q} \int\limits_Q P(z-\zeta)\,|f(\zeta)|\,dQ_\zeta,$$

where $P(\cdot)$ is periodic function, with period 2π in each variable, such that

$$P(z) = \begin{cases} |z|^{-1}, \text{ at } & |z| \le 1, \quad z \in Q, \\ 1, \quad\ \text{ at } & |z| \ge 1, \quad z \in Q. \end{cases} \tag{1.3}$$

We will denote the integral operator with kernel $P(z-\xi)$ by P. Let's introduce one more operator

$$Tf = \int\limits_Q T(z-\zeta)\,f(\zeta)\,dQ_\zeta,$$

where $T(\cdot)$ is a continuous function for $|z| > 0$, $z \in Q$, and 2π-periodic function for each variable, such that

$$T(z) = \begin{cases} C\,|z|^{-1} + K_1(z), \text{ at } & |z| \le 1, \quad z \in Q, \\ K_2(z), \quad\quad\quad\ \text{ at } & |z| \ge 1, \quad z \in Q. \end{cases} \tag{1.4}$$

Here, $K_1(z)$ and $K_2(z)$ are continuous functions for $|z| > 0$, in addition, $|K_j(z)| \le C\,|z|^{-1+\varepsilon_0}$, $\varepsilon_0 > 0$, $j = 1, 2$.

Let us recall the definition of Lorentz spaces.

Let $1 \le p,\ q < \infty$. The completion of $\mathcal{F}$ by the norm

$$|f : \mathcal{L}(p,q)| = \left(\int_0^\infty \{[\mu(z \in Q : |f(z) \ge t|)]^{\frac{1}{p}} t\}\frac{dt}{t}\right)^{\frac{1}{q}},$$

where $\mu(\cdot)$ is the Lebesgue measure, will be called the Lorentz space $\mathcal{L}(p,q)$.

The main result of the work [2] is the following statement.

Theorem 1.1. A function space B with properties $1^\circ - 3^\circ$ is a Vekua space if and only if $B \hookrightarrow \mathbb{P}_1$.

This result implies that a symmetric space is a V space if and only if it is continuously embedded in the Lorentz space $\mathcal{L}(2;1)$.

Thus, we can say that the widest space to which Vekua's theory can be extended is $\mathbb{P}_1(\cdot)$, and among all symmetric spaces, this is $\mathcal{L}(2;1)$.

In the prove of the main result, we used information about the complete continuity of some integral operators, in particular the operators introduced in (1.3) and (1.4). Such statements play a very important role in Vekua's theory [1].

Theorem 1.2. Let $B^s_{p,\theta}$ be completion of $\mathcal{F}$ according to the Besov norm

$$\|f\|_{B^s_{p,\theta}(Q)} = \left(\|f\|^\theta_{W^{n,p}(Q)} + \int\limits_0^\infty \left(\frac{\omega^2_p(f^{(n)},t)}{t^2} \right)^\theta \right)^{1/\theta},$$

where $\omega^2_p(f^{(n)},t) = \sup\limits_{|h|\le t} \|\Delta^2_h f\|_p$ is continuity modulus, $\Delta_h f = f(x+h) - f(x)$.

If $s \ge \frac{2}{p} - 1, \theta = 1$, or $s > \frac{2}{p} - 1$ and $p \ge 1$, then $B^s_{p,\theta}(Q)$ is V-space.

Remark 1.1. It can be shown that if the relations on s, p, θ specified in the theorem are violated, then $B^s_{p,\theta}(Q)$ is not a Vekua space.

Remark 1.2. We will say that $\varphi(\cdot) \in B_{loc}$ in a neighborhood of the point z_0 if $\psi(z)\varphi(z) \in B(Q)$ for $\psi(z) \in C^\infty_0(Q), \psi(z) = 1$ in the neighborhood of z_0.

The theory constructed in [2] is also applicable in local Vekua spaces.

Corollary of Theorem 1.2. Let $\overset{\circ}{W}{}^s_p(Q)$ be completion of $C^\infty_0(Q)$ by the norm

$$\|(-\Delta)^{s/2}u\|_{L_p(R^n)} = \left(\int\limits_{R^n} |F^{-1}|\xi|^s F u|^p dx \right)^{1/p},$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$, F is Fourier transform. It can be shown that $\overset{\circ}{W}{}^s_p(Q)$ is a Vekua space if and only if $s > \frac{2}{p} - 1$. In particular, for $s = 0$, the space $L_p(Q)$ is a Vekua space if and only if $p > 2$.

2 On one Vekua method for solving boundary value problems for polyharmonic equations

Now we turn to the study of similar problems for polyharmonic equations in an arbitrary multidimensional domain.

In the monograph by I.N.Vekua [1], the calculus theory of the simplest problems for polyharmonic equations is given, namely: it is required to solve the equation

$$\Delta^m u + a_1 \Delta^{m-1} u + a_2 \Delta^{m-2} u + \cdots + a_m u = 0 \tag{2.1}$$

under boundary conditions

$$u|_s = \varphi_0,\ \Delta u|_s = \varphi_1,\ \Delta^2 u|_s = \varphi_2,\ \ldots,\ \Delta^{m-1} u|_s = \varphi_{m-1}, \tag{2.2}$$

where $\varphi_0, \ldots, \varphi_{m-1}$ are given functions on the boundary $S = \partial\Omega$ of a bounded domain $\Omega \subseteq \mathbb{R}^n$, $a_i \in \mathbb{R}$, $i = \overline{1,m}$.

The solution of this problem (2.1), (2.2) is decomposed into the solution of m Dirichlet problems for equations of the form

$$\Delta v - k_i^2 v = 0,\ i = 1, 2, \ldots, m,$$

where k_i^2 are the roots of the characteristic equation

$$p^m + a_1 p^{m-1} + a_2 p^{m-2} + \cdots + a_m = 0.$$

Similarly, the boundary value problem

$$\partial_n u|_s = \psi_0,\ \partial_n \Delta u|_s = \psi_1,\ \partial_n \Delta^2 u|_s = \psi_2,\ \ldots,\ \partial_n \Delta^{m-1} u|_s = \psi_{m-1}$$

reduces to m Neumann problems for equations (2.1). Here, $\partial_n = \frac{\partial}{\partial n}$ is outward normal to boundary $S = \partial\Omega$.

For the case of two independent variables I.N. Vekua gave a general theory of linear boundary value problems based on the methods of the theory of analytic functions and on the theory of singular integral equations with Cauchy kernels. The main works in this area are the monographs of I.N. Vekua [1], N.I. Muskhelishvili [4].

Let us briefly outline the idea of the method proposed by I.N. Vekua. Let us assume that the problem of finding a function $u(x),\ x = (x_1, ..., x_n)$, that satisfies the equation

$$\Delta^m u = f(x),\ x \in \Omega \tag{2.3}$$

and homogeneous boundary conditions

$$R_1(u) = 0,\ ...,\ R_m(u) = 0, \tag{2.4}$$

in a simply connected domain Ω, admits a solution for any function $f(x) \in L_p(\Omega)$, $p \geq 1$, and the solution of this problem (2.3), (2.4) is represented in the form

$$u(x) = L_0 f = \int_\Omega G(x, y) f(y) dy,\ dy = dy_1 ... dy_n. \tag{2.5}$$

It is important to note that for the case when Ω is a multidimensional ball and $R_k = \partial^{k-1}/\partial n^{k-1}$ (k-th outer normal to the boundary surface), function G is constructed explicitly (see, for example, [5–8]).

Considering now the problem of finding a solution to the more general equation

$$F(x, u, Du, ..., D^{2m} u) = 0,\ D^p u = \partial^p u / \partial x_1^{k_1} ... \partial x_n^{k_n},\ \sum_{j=1}^n k_j = p$$

with the same boundary conditions

$$R_1(u) = 0,\ ...,\ R_m(u) = 0,$$

we can look for its solution in the form (2.5). This will lead us for $f(x)$ to the functional equation $F(x, L_0 f, ..., L_{2m} f) = 0$ with operators $L_k f = D^k L_0 f,\ k = 0, 1, ..., 2m$.

The operators $L_k f$ are linear and completely continuous for $k \leq 2m-1$. As for the operators $L_{2m} f$, their boundedness in L_p, $p > 1$, is proved by using Zygmund-Calderon equality [9], which generalizes of the well-known Riesz inequality for the singular operator with a Cauchy type kernel. In this way, the problem with unbounded operators $D^k u$ is reduced to the equivalent problem of studying the functional equation $F(x, L_0 f, ..., L_{2m} f) = 0$ with bounded operators $L_k f$. Using the basic principles of functional analysis, it is possible to prove the solvability of this equation for a very wide range of problems for linear and quasilinear differential equations of elliptic type. It should be noted that this method allows the study of boundary value problems with minimal assumptions regarding the coefficients of the equation and the domain. In addition, by using the embedding theorems of S.L. Sobolev [10] and using formula (2.5), it is possible to prove almost extremely accurate theorems on the nature of the smoothness of the generalized solution depending on the degree of smoothness of the coefficients.

3 Description of correct boundary value problems for polyharmonic equations in a ball

Let m be a natural number and in an n-dimensional ball $\Omega = \{x : |x| < r\}$ consider *Dirichlet problem for a polyharmonic equation*

$$\Delta^m u(x) = f(x), \quad x \in \Omega, \tag{3.1}$$

$$\frac{\partial^j u(x)}{\partial n_x^j} = \varphi_j(x), \ \ 0 \le j \le m-1, \ \ x \in \partial\Omega. \tag{3.2}$$

The classical solution $u(x) \in C^{2m}(\Omega) \cap C^{m-1}(\overline{\Omega})$ of the Dirichlet problem (3.1), (3.2) exists, is unique, and it represented using the Green's function $G_{2m,n}(x,y)$ in the following form [10]

$$u(x) = \int_\Omega G_{2m,n}(x,y) f(y) dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}(x,y) \cdot \Delta_y^{m-1-j} \varphi(y) - \right.$$

$$\left. - \Delta_y^j G_{2m,n}(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} \varphi(y) \right] dS_y,$$

where $\frac{\partial}{\partial n_y}$ is outer normal to boundary $\partial\Omega$.

The Green's function of the Dirichlet problem (3.1), (3.2) is defined as follows

$$\Delta^m G_{2m,n}(x,y) = \delta(x-y), \quad x, y \in \Omega, \tag{3.3}$$

$$\frac{\partial^j G_{2m,n}(x,y)}{\partial n_x^j} = 0, \ \ x \in \partial\Omega, \ \ y \in \Omega, \ \ 0 \le j \le m-1, \tag{3.4}$$

where $\delta(x-y)$ is the Dirac delta function.

In further studies we will use the following notation

$$X^2 = X^2(x,y) = |x-y|^2, \quad Y^2 = Y^2(x,y) = \left|\frac{y}{r}\right|^2 \left| x - \frac{y}{|y|^2} r^2 \right|^2,$$

$$Z^2 = Z^2(x,y) = \left(1 - \left|\frac{y}{r}\right|^2\right)\left(1 - \left|\frac{x}{r}\right|^2\right) r^2.$$

Theorem 3.1. [5] a) In the case of odd n, as well as for even n, if $2m < n$ the Green's function of the Dirichlet problem (3.3), (3.4) can be represented as

$$G_{2m,n}(x,y) = d_{2m,n} \Big[X^{2m-n} - Y^{2m-n} -$$

$$- \sum_{k=1}^{m-1} \frac{(-1)^k}{k!} (m - \frac{n}{2}) ... \Big(m - \frac{n}{2} - k + 1 \Big) Y^{2m-n-2k} Z^{2k} \Big], \tag{3.5}$$

and

$$d_{2m,n} = \frac{1}{(m-1)!\,(2m-n)(2(m-1)-n)...(4-n)(2-n)} \cdot \frac{\Gamma(n/2)}{2^m \pi^{n/2}},$$

where $\Gamma(\cdot)$ is gamma function;

b) In the case of even n and $2m \geq n$, the Green's function of the Dirichlet problem (3.3), (3.4) can be represented as

$$G_{2m,n}(x,y) = d_{2m,n}\Big[X^{2m-n}\ln X - Y^{2m-n}\ln Y -$$

$$-\sum_{\nu=1}^{m-n/2}(-1)^{\nu}C_{\nu}^{m-n/2}\Big[\ln Y + \sum_{\mu=m-n/2+1-\nu}^{m-n/2}\frac{1}{2\mu}\Big]Z^{2\nu}Y^{2m-2\nu-n} +$$

$$+(-1)^{m-n/2}\sum_{\nu=1}^{n/2-1}\frac{2^{2m+2\nu-n}}{2\nu C_{\nu+n/2}^{m+\nu}}Z^{2(m+\nu)}Y^{-2\nu-n}\Big], \tag{3.6}$$

and

$$d_{2m,n} = \frac{(-1)^{n/2-1}}{\Gamma(m)\Gamma(m-n/2+1)\cdot 2^{2m-1}\pi^{n/2}}.$$

In this case, $\Omega = \{x \in R^n : |x| < r\}$ or Ω is simply connected domains homeomorphic to the ball. Let us choose the domain of definition of the maximal operator $\widehat{L}$

$$D(\widehat{L}) = W_2^{2m}(\Omega).$$

On the domain $D(\widehat{L})$ we define the operator $\widehat{L}$ by the formula

$$\widehat{L}u = \Delta^m u, \ \forall u \in D(\widehat{L}).$$

Recall that the domain of the maximal operator

$$R(\widehat{L}) = L_2(\Omega),$$

and $Ker\widehat{L}$ its kernel is not trivial.

The Dirichlet boundary value problem for the polyharmonic equation

$$L_0 u := \begin{cases} \Delta_x^m u(x) = f(x), \quad x \in \Omega = \{x : |x| < r\}, \\ \dfrac{\partial^j u(x)}{\partial n_x^j} = 0, \ \ 0 \leq j \leq m-1, \ \ x \in \partial\Omega, \end{cases}$$

has a unique solution $u(x)$ for any $f \in L_2(\Omega)$, which has an integral representation

$$L_0^{-1}f = u(x) = \int_{\Omega} G_{2m,n}^D(x,y)f(y)dy, \tag{3.7}$$

where $G_{2m,n}^D(x,y) \equiv G_{2m,n}(x,y)$ is Green's function of the Dirichlet problem from (3.5) or (3.6).

Note that the zero Dirichlet boundary conditions for a polyharmonic equation are equivalent to the following boundary conditions for the same equation.

Theorem 3.2. a) For any $f \in L_2(\Omega)$, the function $u(x)$, given by formula (3.7) with $m = 2p$, is a solution to the boundary value problem:

$$\Delta_x^m u(x) = f(x), \quad x \in \Omega,$$

$$u(x)\,|_{\partial\Omega} = 0, \ \ \frac{\partial}{\partial n_x}u(x)\Big|_{\partial\Omega} = 0, \ \ \Delta_x u(x)\,|_{\partial\Omega} = 0, \ \ \frac{\partial}{\partial n_x}\Delta_x u(x)\Big|_{\partial\Omega} = 0,$$

$$\ldots\ldots\ldots \Delta_x^{p-1}u(x)\,|_{\partial\Omega}=0,\ \frac{\partial}{\partial n_x}\Delta_x^{p-1}u(x)\,\bigg|_{\partial\Omega}=0.$$

b) For any $f\in L_2(\Omega)$, the function $u(x)$, given by formula (3.7) with $m=2p+1$, is a solution to the boundary value problem:

$$\Delta_x^m u(x)=f(x),\quad x\in\Omega,$$

$$u(x)\,|_{\partial\Omega}=0,\ \frac{\partial}{\partial n_x}u(x)\,\bigg|_{\partial\Omega}=0,\ \Delta_x u(x)\,|_{\partial\Omega}=0,\ \frac{\partial}{\partial n_x}\Delta_x u(x)\,\bigg|_{\partial\Omega}=0,$$

$$\ldots\ldots\ldots\ \frac{\partial}{\partial n_x}\Delta_x^{p-1}u(x)\,\bigg|_{\partial\Omega}=0,\ \Delta_x^p u(x)\,|_{\partial\Omega}=0.$$

Based on the representation of the solution (3.7) of the Dirichlet problem, we present other well-posed boundary value problems for an inhomogeneous polyharmonic equation. To do this, we apply Otelbaev's theorem [5] to describe correct restrictions of the maximal operator $\widehat{L}$.

Now we can describe the domain of the maximal operator $\widehat{L}$ in terms of the Green's function $G_{2m,n}$.

Lemma 3.1. [5] The domain of the maximal operator $\widehat{L}$ has the representation

$$D(\widehat{L})=\{w:\ w(x)=\int_\Omega G_{2m,n}(x,y)f(y)dy+$$

$$+\sum_{j=0}^{m-1}\int_{\partial\Omega}\left[\frac{\partial\Delta_y^j G_{2m,n}(x,y)}{\partial n_y}\cdot\Delta_y^{m-1-j}h(y)-\Delta_y^j G_{2m,n}(x,y)\cdot\frac{\partial\Delta_y^{m-1-j}h(y)}{\partial n_y}\right]dS_y,$$

$$\forall f\in L_2(\Omega),\ \forall h\in W_2^{2m}(\Omega).$$

In particular, if

$$\Delta_y^{m-1-j}h(y)|_{y\in\partial\Omega}=0,\ \frac{\partial}{\partial n_y}\Delta_y^{m-1-j}h(y)|_{y\in\partial\Omega}=0,\ j=0,...,m-1,$$

then we obtain that the $D(L_0)$ is domain of the operator L_0.

Now the next question arises: how to describe the domains of other possible correct restrictions of the maximal operator $\widehat{L}$?

Let K be an operator putting each function $f(x)\in L_2(\Omega)$ into correspondence to a unique function $h(x)\in W_2^{2m}(\Omega)$, such that $\|Kf\|_{L_2(\Omega)}\le C\|f\|_{L_2(\Omega)}$.

Using the chosen operator K, we construct the set

$$D_K=\{\,w(x)\in D(\widehat{L})\,:\,h=Kf\}.$$

On the set D_K we define the operator

$$\widehat{L}\Big|_{D_K}=L_K.$$

From Otelbaev's theorem [5] it follows that L_K is a correct restriction of the maximal operator $\widehat{L}$. In conclusion, we give another description of the operator L_K in terms of boundary conditions.

Theorem 3.3. [5] Let K be an arbitrary continuous operator acting from $L_2(\Omega)$ to $D(\widehat{L})$. Then the inhomogeneous operator equation $L_K w=f$ is equivalent to the following boundary value problem

a) for $m=2p$

$$\Delta_x^m w(x)=f(x),\quad x\in\Omega,$$

$$w\,|_{\partial\Omega}=K(\Delta_x^m w)|_{\partial\Omega},\ \frac{\partial}{\partial n_x}w\,\bigg|_{\partial\Omega}=\frac{\partial}{\partial n_x}K(\Delta_x^m w)\bigg|_{\partial\Omega},\ldots\ldots\ldots$$

$$\Delta_x^{p-1}w\,\big|_{\partial\Omega} = \Delta_x^{p-1}K(\Delta_x^m w)\big|_{\partial\Omega},\ \ \frac{\partial}{\partial n_x}\Delta_x^{p-1}w\,\bigg|_{\partial\Omega} = \frac{\partial}{\partial n_x}\Delta_x^{p-1}K(\Delta_x^m w)\bigg|_{\partial\Omega};$$

b) for $m = 2p+1$

$$w\,|_{\partial\Omega} = K(\Delta_x^m w)|_{\partial\Omega},\ \ \frac{\partial}{\partial n_x}w\,\bigg|_{\partial\Omega} = \frac{\partial}{\partial n_x}K(\Delta_x^m w)\bigg|_{\partial\Omega}, \dots\dots\dots$$

$$\frac{\partial}{\partial n_x}\Delta_x^{p-1}w\,\bigg|_{\partial\Omega} = \frac{\partial}{\partial n_x}\Delta_x^{p-1}(K\Delta_x^m w)\bigg|_{\partial\Omega},\ \ \Delta_x^p w\,|_{\partial\Omega} = \Delta_x^p(K\Delta_x^m w)|_{\partial\Omega}.$$

In [11,12] the Fredholm property and index of the generalized Neumann problem containing powers of normal derivatives in the boundary conditions are investigated. The problems of solvability of various boundary value problems for differential-operator equations are studied in the works [13–19]. Applications of the Green function to problems in mechanics and physics can be found, in [20–24].

4 Example. General presentation of solutions of boundary value problems for biharmonic equations

As an example, we consider the following biharmonic equation

$$\Delta^2 u = f, \quad z = x + iy \in \Omega, \tag{4.1}$$

where f is a given function. This equation is often encountered in the study of two-dimensional problems of linear elasticity theory. Let us construct regular solutions of equation (4.1) in the two-dimensional region Ω of the plane of the complex variable $z = x = iy$. To find a particular solution $u_1(x,y)$ of equation (4.1), we adopt the notation $v = \Delta u_1$. The function u_1 will be a solution of equation (4.1) if $v(x,y)$ is a solution of the Poisson equation $\Delta v = f$.

The solution to this equation is given by the formula [25]

$$v \equiv 4\frac{\partial^2 u_1}{\partial z \partial \overline{z}} = \frac{1}{2\pi}\int_\Omega f(t)\log|t-z|d\xi d\eta, \quad t = \xi + i\eta.$$

Using the following obvious equality

$$\frac{\partial^2}{\partial z\partial\overline{z}}[(t-z)(\overline{t}-\overline{z})\log(t-z)(\overline{t}-\overline{z})] - 2 = 2\log|t-z|,$$

equation (4.1) can be written as

$$\frac{\partial^2}{\partial z\partial\overline{z}}[u_1 - \frac{1}{8\pi}\int_\Omega f(t)|t-z|^2\log|t-z|d\xi d\eta] = -\frac{1}{8\pi}\int_\Omega f(t)d\xi d\eta \equiv C = const.$$

Therefore,

$$u_1 = \frac{1}{8\pi}\int_\Omega f(t)|t-z|^2\log|t-z|d\xi d\eta + \Phi(z,\overline{z}),$$

where

$$\Phi(z,\overline{z}) = Cz\overline{z} + \varphi_1(z) + \varphi_2(\overline{z}),$$

and φ_1 and φ_2 are arbitrary analytic functions of the variables z and $\overline{z}$, respectively. Since Φ is a biharmonic function, the function

$$u_1 = \frac{1}{8\pi}\int_\Omega f(t)|t-z|^2\log|t-z|d\xi d\eta \tag{4.2}$$

can be taken as one of the particular solutions of equation (4.1).

If $u(x, y)$ is the desired solution of equation (4.1), then the function $w = u - u_1$ will be biharmonic, i.e.

$$\Delta^2 w = 0. \tag{4.3}$$

According to formula (85) from [25] the solution of equation (4.3) can be written in the form

$$w = \overline{z}\varphi(z) + z\overline{\varphi}(\overline{z}) + \psi(z) + \overline{\psi}(\overline{z}). \tag{4.4}$$

In the representation (4.4) of real biharmonic functions, the imaginary parts of the functions $\overline{z}\varphi(z)$ $\psi(z)$ are not present. Therefore, without loss of generality, we can assume that the analytic functions $\varphi(z)$ and $\psi(z)$ included in formula (4.4) at some point z_1 of the domain Ω satisfy the conditions

$$\varphi(z_1) = 0, \mathrm{Im}\varphi'(z_1) = 0 \tag{4.5}$$

and

$$\mathrm{Im}\psi'(z_1) = 0. \tag{4.6}$$

Thus we have proved the following theorem.

Theorem 4.1. a) To each pair of analytic functions $\varphi(z), \psi(z)$ formula (4.4) associates a well-defined biharmonic function $w(x, y)$. The converse statement is also true.

b) For each biharmonic function $w(x, y)$ there is a well-defined pair of analytic functions $\varphi(z)$, $\psi(z)$ satisfying conditions (4.5), (4.6) and $w(x, y)$ is represented by formula (4.4).

From this theorem we conclude that formula (4.4) gives a general representation of real biharmonic functions. Further, in view of the fact that

$$u = w + u_1,$$

on the basis of formulas (4.2) and (4.4) we arrive at the general complex representation of real solutions of equations (4.1):

$$u(x, y) = \frac{1}{8\pi}\int_\Omega f(t)|t - z|^2 \log|t - z| d\xi d\eta + \overline{z}\varphi(z) + z\overline{\varphi}(\overline{z}) + \psi(z) + \overline{\psi}(\overline{z}), \tag{4.7}$$

where $\varphi(z)$ and $\psi(z)$ are arbitrary analytic functions of a complex variable z satisfying conditions (4.5), (4.6). Formula (4.7) allows any linear problem for equation (4.1) to be reduced to the corresponding problem for biharmonic functions.

Conclusion

The studies carried out in this article are of significant importance in the theory of boundary value problems of linear and nonlinear partial differential equations, spectral theory, and the theory of numerical methods for approximate solutions of individual classes of boundary value problems for differential equations.

Thus, the object of our research was polyharmonic equations. Vekua's method is applicable for fixed boundary conditions on the surface of the domain, and the equation itself can change, i.e. minor terms can be added to the main equation. Otelbaev's method is applied for fixed equations, and the boundary conditions can change.

The problem is to determine the conditions under which both methods are applicable. This article is devoted to this problem and the applicability of the Vekua and Otelbaev methods. As an example, a biharmonic equation is given, which has an applied character in the theory of elasticity. A general complex representation of real solutions of the biharmonic equation is given in the form of formula (4.7).

Acknowledgments

This work was done with support of grant AP 19678182 of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

Author Contributions

N.M. Shynybayeva collected and analyzed data, and led manuscript preparation. P.Zh. Kozhobekova and M.T. Sabirzhanov assisted in data collection and analysis. B.D. Koshanov served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Vekua, I.N. (2014). *Generalized analytic functions.* Elsevier.

2 Otelbayev, M.O. (1979). K teorii obobshchennykh analiticheskikh funktsii Vekua [On the theory of generalized analytical functions of Vekua]. *Primenenenie metodov funktsionalnogo analiza i vychislitelnoi matematiki. Materialy shkoly-seminara – Application of methods of functional analysis and computational mathematics. School-seminar materials.* Novosibirsk, 80–99 [in Russian].

3 Kal'menov, T.S., Otelbaev, M., & Arepova, G.D. (2020). Bitsadze-Samarskii Boundary Conditions for an Elliptic-Parabolic Volume Potential with Smooth Matching. *Differential Equations, 56*(6), 740–755. https://doi.org/10.1134/S0012266120060075

4 Muskhelishvili, N.I., & Radok, J.R.M. (2008). *Singular integral equations: boundary problems of function theory and their application to mathematical physics.* Courier Corporation.

5 Koshanov, B.D. (2021). Green's functions and correct restrictions of the polyharmonic operator. *Journal of Mathematics, Mechanics and Computer Science, 109*(1), 34–54. https://doi.org/10.26577/JMMCS.2021.v109.i1.03

6 Koshanov, B.D. (2013). About solvability of boundary value problems for the nonhomogeneous polyharmonic equation in a ball. *Journal APAM, Advances in Pure and Applied Mathematics, 4*(4), 351–373. https://doi.org/10.1515/apam-2013-0022

7 Karachik, V. (2021). Dirichlet and Neumann boundary value problems for the polyharmonic equation in the unit ball. *Mathematics, 9*(16), 1907. https://doi.org/10.3390/math9161907

8 Karachik, V. (2023). Representation of the Green's Function of the Dirichlet Problem for the Polyharmonic Equation in the Ball. *Differential Equations, 59*, 1061–1074. https://doi.org/10.1134/S0012266123080050

9 Calderon, A.P., & Zygmund, A. (1952). On the existence of certain singular integrals. *Acta Math, 88*, 85–139. https://doi.org/10.1007/BF02392130

10 Sobolev, S.L. (1974). *Vvedenie v teoriiu kubaturnykh formul [Introduction to the theory of cubature formulas].* Moscow: Nauka [in Russian].

11 Koshanov, B.D., & Soldatov, A.P. (2018). On the Solvability of the Boundary Value Problems for the Elliptic Equation of High Order on a Plane. *Bulletin of the Karaganda University. Mathematics Series,* 3(91), 24–31. https://doi.org/10.31489/2018M3/24-30

12 Koshanov, B., & Soldatov, A. (2022). On Fredholm solvability and on the index of the generalized Neumann problem for an elliptic equation. *Complex Variables and Elliptic Equations, 12*(67), 2907–2923. https://doi.org/10.1080/17476933.2021.1958797

13 Kozhanov, A.I., Koshanov, B.D., Sultangazieva, Zh.B., Emir Kadyoglu, A.N., & Smatova, G.D. (2020). The spectral problem for nonclassical differential equations of the sixth order. *Bulletin of the Karaganda University. Mathematics Series,* 1(97), 79–86. https://doi.org/10.31489/2020M1/79-86

14 Koshanov, B.D., Kakharman, N., Segizbaeva, R.U., & Sultangazieva, Zh.B. (2022). Two theorems on estimates for solutions of one class of nonlinear equations in a finite-dimensional space. *Bulletin of the Karaganda University. Mathematics Series,* 3(107), 70–84. https://doi.org/10.31489/2022m3/70-84

15 Kanguzhin, B.E., & Koshanov, B.D. (2022). Uniqueness Criteria for Solving a Time Nonlocal Problem for a High-Order Differential Operator Equation $l(\cdot) - A$ with a Wave Operator with Displacement. *Simmetry, 14*(6), 1239. https://doi.org/10.3390/sym14061239

16 Kanguzhin, B.E., & Koshanov, B.D. (2023). Solution Uniqueness Criteria in a Time-Nonlocal Problem for the Operator Differential Equation with the Tricomi Operator. *Differential Equations, 59*(1), 1–12. https://doi.org/10.1134/S0012266123010019

17 Otelbaev, M., & Soldatov, A.P. (2021). Integral representations of vector functions based on the parametrix of first-order elliptic systems. *Computational Mathematics and Mathematical Physics, 61*, 964–973. https://doi.org/10.1134/S0965542521030143

18 Soldatov, A.P. (2022). On a Boundary Problem for a Fourth-Order Elliptic Equation on a Plane. *Computational Mathematics and Mathematical Physics, 62*, 599–607. https://doi.org/10.1134/S096554252204011X

19 Yuldashev, T.K., & Eshkuvatov, Y.F. (2022). On a Fredholm type integro-differential equations of fifth order with degenerate kernel. *Journal of Contemporary Applied Mathematics, 12*(1), 27–34.

20 Chung, F., & Zeng, J. (2023). Forest formulas of discrete Green's functions. *Journal of Graph Theory, 102*, 556–577. https://doi.org/10.1002/jgt.22887

21 Herrera, W.J., Vinck-Posada, H., & Paez, S.G. (2022). Green's functions in quantum mechanics courses. *American Journal of Physics, 90*, 763–769. https://doi.org/10.1119/5.0065733

22 Harkonen, V.J. (2022). Exact factorization of the many-body Green's function theory of electrons and nuclei. *Physical Review B, 106*. https://doi.org/10.1103/PhysRevB.106.205137

23 Dong, H., & Li, H. (2019). Optimal Estimates for the Conductivity Problem by Green's Function Method. *Archive for Rational Mechanics and Analysis, 231*, 1427–1453. https://doi.org/10.1007/s00205-018-1301-x

24 Grebenkov, D.S., & Traytak, S.D. (2019). Semi-analytical computation of Laplacian Green functions in three-dimensional domains with disconnected spherical boundaries. *Journal of Computational Physics, 379*, 91–117. https://doi.org/10.1016/j.jcp.2018.10.033

25 Bitsadze, A.V. (1984). *Osnovy teorii analiticheskikh funktsii komplesksnogo peremennogo [Fundamentals of the theory of complex variable analytic functions].* Moscow: Nauka [in Russian].

*Author Information**

Muzaffar Takhirovich Sabirzhanov — Doctor's student, Osh State University, Osh, Kyrgyzstan; e-mail: *smskg@bk.ru*; https://orcid.org/0009-0008-2431-7033

*The author's name is presented in the order: First, Middle and Last Names.

Bakytbek Danebekovich Koshanov (*corresponding author*) — Doctor of physical and mathematical sciences, Professor, Chief researcher at Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan; e-mail: *koshanov@math.kz*; https://orcid.org/0000-0002-0784-5183

Nazym Myrzabekkyzy Shynybayeva — Junior Researcher, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan; e-mail: *shynybayeva001@mail.ru*

Pardaz Zhumabaevna Kozhobekova — Doctor's student, Osh State University, Osh, Kyrgyzstan; e-mail: *pardaz@mail.ru*; https://orcid.org/0009-0002-3143-3442