https://doi.org/10.31489/2025M2/222-240 *Research article*

Variational method of numerical solution of the inverse problem of gas lift oil production process

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This paper proposes a constructive method for numerically solving direct and inverse problems arising in the gas-lift oil production process, which is described by a hyperbolic system of differential equations. To solve the direct problem, a second-order difference scheme is used, which ensures stability and accuracy of calculations in the space-time domain. The inverse problem is formulated as an optimal control problem, where the minimization of the objective functional is carried out using the gradient method. The calculation of the gradient of the objective function is based on the constructed adjoint problem using the Lagrange identity and the duality principle, which guarantees the mathematical rigor of the approach. Numerical experiments confirmed the efficiency of the proposed method for solving the inverse problem and optimizing the input parameters of the gas lift process. The adjoint problem contains valuable information about the solution of the direct problem, so the gradients of the functional are equal to the solution of the adjoint problem and its first derivative with respect to time at $t = 0$. Numerical calculations show that the values of the minimized functional decrease monotonically and remain bounded below. This means that the used iterative method converges. Additional conditions set at $T = 0$ for the direct problem are used to formulate the condition of the adjoint problem. The developed algorithm contributes to the development of the numerical implementation of the adjoint optimization method of the inverse problem for a hyperbolic equation. The problem of the type under study is of great practical importance and can be used to calculate the intensity of the gas lift process of oil production.

Keywords: gas lift process of oil production, hyperbolic equation, conjugate equation, inverse problem, optimal control, gradient method, numerical methods.

2020 Mathematics Subject Classification: 49N45.

Introduction

The gas lift process is an oil production technology in which gas is injected into the annular space of a well to reduce the density of the gas-liquid mixture (GLM), facilitating its rise to the surface.

The model proposed in [1–3] describes the dynamics of gas and gas-liquid mixtures in the designated regions using a system of hyperbolic partial differential equations. These equations take into account key parameters such as pressure P and gas volume flow rate Q, which significantly affect the transport of the mixture.

The method of lines allows to reduce a hyperbolic system of partial differential equations to a system of ordinary differential equations. Subsequently, the optimal control problem is formulated, the goal of which is to increase the volume of oil production with minimal gas costs. The control parameters in this case are the pressure or volume of the injected gas.

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This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (No. BR 27100483 "Development of predictive exploration technologies for identifying ore-prospective areas based on data analysis from the unified subsurface user platform "Minerals.gov.kz" using artificial intelligence and remote sensing methods").

Received: 16 October 2024; *Accepted:* 19 March 2025.

In [4], a mathematical method for optimizing oil recovery from gas-lift wells is presented. The main objective is to determine the economically feasible level of production and reduce operating costs by minimizing gas consumption. Based on the collected well performance data using the PIPESIM application, performance curves were modeled. A multi-criteria model was then developed for optimization.

In the study [5], a gas lift distribution method using genetic algorithms was studied to enhance oil recovery.

Modern methods for solving inverse problems in mathematical physics often involve formulating them as optimal control problems. In particular, such problems are formulated to control the behavior of objects described by partial differential equations. The goal is to transfer the system from one state to another by influencing its parameters. Such problems were first considered by J.L. Lions [6, 7].

The right side of the equation and the boundary conditions can act as control. The methods of adjoint equations are described in detail in the studies of V.I. Agoshkov [8].

Boundary control is a separate class of problems in which control is implemented through boundary conditions. In the works of V.A. Il'in and E.I. Moiseev [9], boundary control problems for string vibration equations were investigated, where control functions were constructed to transfer the system from the initial state to the final state in a certain time.

G.I. Marchuk [10] gave a definition of adjoint operators and equations, and described their application in mathematical modeling. Studies of linear operators in Hilbert spaces are widely covered in various monographs.

The method of fictitious regions for optimization problems was proposed in [11]. It allows constructing difference schemes in an extended region, which leads to the convergence of the solution to the desired one. Minimization of the Lagrange functional was carried out using the conjugate gradient method.

In [12], the Cauchy problem for the Helmholtz equation was investigated. The main attention was paid to reducing the problem to a boundary inverse problem. The proposed method was based on optimization using the Landweber and Nesterov methods.

In the article by A.V. Arguchintsev and V.P. Poplevko [13], the problem of optimal control of a system of semilinear hyperbolic equations with boundary conditions specified through ordinary differential equations with delay was considered.

The work [14] considers the development of methods for solving optimal control problems in the class of smooth control actions, taking into account the constraints characteristic of inverse problems of mathematical physics.

In the work of N.M. Temirbekov [15], the Bublov-Galerkin projection method with bases in the form of Legendre wavelets was used to solve the Fredholm integral equation of the first kind. In the Galerkin method, Legendre wavelets are used as basis functions and a system of linear algebraic equations is obtained to determine the expansion coefficients. This system of linear algebraic equations is solved by the conjugate gradient method.

Inverse problems of various types occur in many areas, including everyday practice. The papers [16, 17] are devoted to the study of the application of numerical methods to solving problems related to acoustic equations, with a special emphasis on problems that have significant practical significance, for example, in the field of medical imaging and theoretical acoustics.

In the study [18], an optimization method was proposed for solving the Cauchy problem, which is based on the minimization of a functional that includes both the problem data and the regularization term. The use of numerical schemes within this method allowed stabilizing the solution process and minimizing the influence of errors. This method is one of the first numerical methods for solving boundary inverse problems and demonstrates significant advantages in the context of ill-posed problems.

Paper [19] presents the construction of an algorithm and a numerical solution of the inverse problem for the acoustics equation using the gradient method. The authors investigate the correctness of the problem statement, propose a numerical solution method, and analyze the convergence and accuracy of the solutions obtained. The main focus is on the application of gradient optimization methods to recover unknown parameters in an acoustic model. The results show the effectiveness of the proposed algorithm, as well as the possibility of its use in practical problems of acoustic analysis.

The paper [20] considers numerical modeling for improved prediction of the transport of pollutants in the atmosphere of industrial regions. The authors are developing a mathematical model and a numerical algorithm that allows taking into account the complex dynamic processes of diffusion and convection of pollutants in the air.

The purpose of this article is to develop a finite-difference method for solving direct and inverse initial-boundary value problems for a hyperbolic equation with discontinuous and rapidly changing coefficients. Given additional conditions on the solution and its first derivative with respect to time at $T = 0$, it is necessary to determine the initial conditions on the solution and its first derivative at $t = 0$.

For this purpose, the inverse problem is considered as a variational one and minimization of the functional by the gradient method leads to a conjugate retrospective problem. An algorithm for the numerical implementation of the gradient method is developed. Numerous calculations of the problem are given. The results show the convergence of the iterative process. The values of the functional decrease monotonically, and the initial conditions of the direct problem are restored in accordance with the values of the additional conditions, which corresponds to the physical formulation of the problem.

1 Statement of the direct and inverse problem

The mathematical model of the operation of a gas lift well is described by the following system of linear differential equations [1–3]:

$$\frac{\partial P}{\partial t} = -\frac{c^2}{\overline{F}} \cdot \frac{\partial Q}{\partial x}, \tag{1}$$

$$\frac{\partial Q}{\partial t} = -\overline{F} \cdot \frac{\partial P}{\partial x} - 2a \cdot Q, \quad t \geq 0,\ x \in (0, 2l), \tag{2}$$

where t is time, x is a coordinate along the well depth, P is pressure, Q is volumetric gas flow rate, $\overline{F}$ is a cross-sectional area of the well, c is speed of sound in liquid, l is well depth, a is a coefficient depending on input parameters.

Parameters c, $\overline{F}$, and a may have different values in different parts of the well.

The model parameters depend on the well section:

$$c = \begin{cases} c_1, & x \in (0, l), \\ c_2, & x \in (l, 2l), \end{cases} \qquad \overline{F} = \begin{cases} \overline{F}_1, & x \in (0, l), \\ \overline{F}_2, & x \in (l, 2l), \end{cases} \qquad a = \begin{cases} a_1, & x \in (0, l), \\ a_2, & x \in (l, 2l). \end{cases}$$

Initial conditions:

$$P(0, x) = P^0(x), \quad Q(0, x) = Q^0(x), \tag{3}$$

where $P^0(x)$ and $Q^0(x)$ are the initial distribution of pressure and volumetric flow rate of gas.

Boundary conditions:

The following conditions are set at the boundaries of the domain:

1. At the wellhead ($x = 0$):

$$P(t,0) = P_0(t), \quad Q(t,0) = Q_0(t). \tag{4}$$

2. At the domain boundary ($x = l$):

$$P(t,l+0) = P(t,l-0) + P_{res}(t), \quad Q(t,l+0) = Q(t,l-0) + Q_{res}(t), \tag{5}$$

where $P_{res}(t)$ and $Q_{res}(t)$ are the pressure and flow rate of gas from the formation.

3. At the well outlet ($x = 2l$):

$$P(t,2l) = P_{out}(t), \quad Q(t,2l) = Q_{out}(t), \tag{6}$$

where Q_{res} is volumetric flow rate from the reservoir, P_{res} is reservoir pressure, $P^0(x)$ is initial gas pressure, $Q^0(x)$ is initial gas volumetric flow rate, $P_{out}(t)$ is outlet pressure, $Q_{out}(t)$ is volumetric flow rate at the outlet.

To transform the system of first-order hyperbolic equations (1), (2) into one second-order equation:

1. Calculate the derivative with respect to x from (1):

$$\frac{\partial^2 P}{\partial t \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial t}\right) = \frac{\partial}{\partial x}\left(-\frac{c^2}{\overline{F}} \cdot \frac{\partial Q}{\partial x}\right) = -\frac{c^2}{\overline{F}} \cdot \frac{\partial^2 Q}{\partial x^2}. \tag{7}$$

2. We calculate the derivative with respect to t from equation (2):

$$\frac{\partial^2 Q}{\partial t^2} = -\overline{F} \cdot \frac{\partial^2 P}{\partial t \partial x} - 2a \cdot \frac{\partial Q}{\partial t}. \tag{8}$$

3. Substitute (7) into (8):

$$\frac{\partial^2 Q}{\partial t^2} + 2a\frac{\partial Q}{\partial t} = c^2 \frac{\partial^2 Q}{\partial x^2}, \quad x \in (0,2l), \quad t \in (0,T). \tag{9}$$

Initial and boundary conditions (3)–(6) remain unchanged

$$Q(0,x) = Q^0(x), \quad \frac{\partial Q}{\partial t}(0,x) = G^0(x), \quad x \in [0,2l], \tag{10}$$

$$Q(t,0) = Q_0(t), \quad Q(t,2l) = Q_{out}(t), \quad t \in [0,T], \tag{11}$$

$$Q(t,l+0) = Q(t,l-0) + Q_{res}(t), \quad t \in [0,T], \tag{12}$$

where $G^0(x)$ is the initial velocity of gas displacement.

$G^0(x)$ is expressed as follows:

$$G^0(x) = -\overline{F} \cdot \frac{\partial P^0(x)}{\partial x} - 2aQ^0(x).$$

The direct problem is to find the function $Q(t,x)$ based on the given functions: $Q^0(x)$, $G^0(x)$, $Q_0(t)$, $Q_{res}(t)$, $Q_{out}(t)$.

The inverse problem is formulated as finding the initial velocity of gas displacement $G^0(x)$ based on the known parameters of the direct problem (9)–(12) and the additional condition:

$$Q(T,x) = Q^{(1)}(x), \quad \frac{\partial Q}{\partial t}(T,x) = Q^{(2)}(x), \quad x \in [0,2l]. \tag{13}$$

Solving the inverse problem involves using the method conjugate equations and minimization of the objective functional, which allows us to restore $G^0(x)$, guaranteeing the fulfillment of all conditions of the problem.

Now we will reduce the equation (9) to an invariant form to get rid of the first-order derivative.

We make the following notation $Q(t,x) = V(t,x)\cdot e^{\alpha t}$ and substitute into (9) and initial boundary conditions (10)–(12)

$$\frac{\partial^2 V}{\partial t^2} + (2\alpha + 2a)\cdot\frac{\partial V}{\partial t} + \left(\alpha^2 + 2a\cdot\alpha\right)V = c^2\frac{\partial^2 V}{\partial x^2}.$$

We have the initial $H^0(x) = G^0(x) + \alpha\cdot Q^0(x)$, boundary $V_0(t) = Q_0(t)\cdot e^{-\alpha t}$, $V_{out}(t) = Q_{out}(t)\cdot e^{-\alpha t}$, $V_{res}(t) = Q_{res}(t)\cdot e^{-\alpha t}$ and additional conditions $V(T,x) = V^{(1)}(x)$.

We equate the coefficient $\alpha = -a$.

Then problem (9)–(12), (13) is transformed into the following problem

$$\frac{\partial^2 V}{\partial t^2} - a^2 V = c^2\frac{\partial^2 V}{\partial x^2}, \quad x\in(0,2l), \quad t\in(0,T), \tag{14}$$

with the initial conditions

$$V(0,x) = Q^0(x), \quad \frac{\partial V}{\partial t}(0,x) = H^0(x), \quad x\in[0,2l] \tag{15}$$

and the boundary conditions respectively

$$V(t,0) = V_0(t), \quad V(t,2l) = V_{out}(t), \quad t\in[0,T], \tag{16}$$

$$V(t,l+0) = V(t,l-0) + V_{res}(t), \quad t\in[0,T], \tag{17}$$

where $H^0(x) = G^0(x) + a\cdot Q^0(x)$, $V_0(t) = Q_0(t)\cdot e^{at}$, $V_{out}(t) = Q_{out}(t)\cdot e^{at}$, $V_{res}(t) = Q_{res}(t)\cdot e^{at}$.

In this direct problem, we need to find $V(t,x)$ given the functions $H^0(x)$, $Q^0(x)$, $V_0(t)$, $V_{res}(t)$, $V_{out}(t)$.

We have additional conditions

$$V(T,x) = V^{(1)}(x) \quad \text{at} \quad t = T, \ x\in[0,2l], \tag{18}$$

$$\frac{\partial V}{\partial t}(T,x) = V^{(2)}(x) \quad \text{at} \quad t = T, \ x\in[0,2l]. \tag{19}$$

In the inverse problem, it is necessary to find $H^0(x)$ from the retrospective problem (14), (16), (17) with the additional condition (18).

2 *Statement of the variational problem*

The inverse problem of mathematical physics is often reduced to the optimal control problem, which allows it to be solved using variational methods. In this case, it is required to find the initial values $Q^0(x)$ and $H^0(x)$ belonging to the space $W_2^2(0,2l)$, such that the solutions of problem (14)–(17), (18), (19) are as close as possible to the given value $V^{(1)}(x)$ at $t = T$.

To evaluate how closely the solution $V(T,x)$ matches the desired value $V^{(1)}(x)$, we define the objective functional:

$$J\left(H^0,Q^0\right) = \int_0^{2l}\left[V\left(T,x;Q^0(x)\right) - V^{(1)}(x)\right]^2 dx + \int_0^{2l}\left[V_t\left(T,x;H^0(x)\right) - V^{(2)}(x)\right]^2 dx \to \min, \tag{20}$$

Here the notations $V\left(T,x;Q^0\left(x\right)\right)$ and $V_t\left(T,x;H^0\left(x\right)\right)$ means the dependence of the solution $V\left(T,x\right)$ to problem (14)–(17), (18) on the initial conditions.

The goal is to find functions $Q^0(x)$ and $H^0\left(x\right)$ that minimize the objective functional $J\left(Q^0,H^0\right)$:

$$J\left(Q^0,H^0\right)\to\min,\quad Q^0(x)\in G_{ad}^{(1)},\,H^0(x)\in G_{ad}^{(2)},$$

where $G_{ad}^{(1)}\subset W_2^1(0,2l),G_{ad}^{(2)}\subset W_2^2(0,2l)$ is the set of admissible values.

We minimize the functional using the gradient method [21].

To minimize the objective functional $J(Q^0,H^0)$, the iterative gradient method is used. The value of the function $Q^0(x)$, $H^0(x)$ is updated at each iteration using the following formula:

$$Q_{n+1}^0=Q_n^0-\alpha\cdot J'\left(Q_n^0\right),\quad H_{n+1}^0=H_n^0-\alpha\cdot J'\left(H_n^0\right),\quad n=0,1,2,\ldots,$$

where $Q_n^0(x)$, $H_n^0(x)$ are approximations at the n-th iteration; α is a relaxation parameter. The relaxation parameter determines different gradient methods, and its choice is important.

$J'(Q_n^0(x))$ is a gradient of the functional $J(Q^0)$ with respect to $Q^0(x)$, $J'(H_n^0(x))$ is a gradient of the functional $J(H^0)$ with respect to $H^0(x)$.

Let us define the first variation of the entire functional (20)

$$\delta J\left(Q^0,H^0\right)=J\left(Q^0+\delta Q^0,\,H^0+\delta H^0\right)-J\left(Q^0,H^0\right)=$$

$$=\int_0^{2l}\left[V\left(T,x;Q^0\left(x\right)+\delta Q^0\right)-V^{(1)}\left(x\right)\right]^2dx+\int_0^{2l}\left[V_t\left(T,x;H^0\left(x\right)+\delta H\right)-V^{(2)}\left(x\right)\right]^2dx-$$

$$-\int_0^{2l}\left[V\left(T,x;Q^0\right)-V^{(1)}\left(x\right)\right]^2dx-\int_0^{2l}\left[V_t\left(T,x;H^0\left(x\right)\right)-V^{(2)}\left(x\right)\right]^2dx.$$

Considering that

$$V\left(T,x;Q^0\left(x\right)+\delta Q^0\right)=V\left(T,x;Q^0\left(x\right)\right)+\delta V\left(T,x;\delta Q^0\left(x\right)\right),$$

$$V_t\left(T,x;H^0\left(x\right)+\delta H^0\right)=V_t\left(T,x;H^0\left(x\right)\right)+\delta V_t\left(T,x;\delta H^0\left(x\right)\right).$$

We have

$$\begin{aligned}\delta J\left(Q^0,H^0\right)\approx&\int_0^{2l}\delta V\left(T,x;\delta Q^0\left(x\right)\right)\cdot2\cdot\left[V\left(T,x;Q^0\left(x\right)\right)-V^{(1)}\left(x\right)\right]dx+\\&+\int_0^{2l}\delta V_t\left(T,x;\delta H^0\left(x\right)\right)\cdot2\cdot\left[V_t\left(T,x;H^0\left(x\right)\right)-V^{(2)}\left(x\right)\right]dx.\end{aligned}\qquad(21)$$

Due to the smallness of the terms containing $\delta V^2\left(T,x;\delta Q^0\left(x\right)\right)$, $\left[\dfrac{\partial\delta V}{\partial t}\left(T,x;\delta H^0\left(x\right)\right)\right]^2$ they can be neglected.

On the other hand, in accordance with the definition of the Frechet derivative, the equality is satisfied:

$$\delta J\left(Q^{0}\right) =< J'Q^{0}, \delta Q^{0} >, \quad \delta J\left(H^{0}\right) =< J'H^{0}, \delta H^{0} > . \tag{22}$$

Let us introduce the following notations:

$$\widetilde{V}\,(t,x){=}V\big(t,x;H^{0}+\delta H^{0}\big), \qquad V\,(t,x)=V\,\big(t,x;H^{0}\big)\,, \qquad \delta V\,(t,x)=\widetilde{V}\,(t,x)-V\,(t,x)\,.$$

Let us consider the perturbed problem corresponding to problem (14)–(17):

$$\frac{\partial^{2}\widetilde{V}}{\partial t^{2}}-a^{2}\widetilde{V}=c^{2}\frac{\partial^{2}\widetilde{V}}{\partial x^{2}}, \qquad x\in(0,2l)\,, \quad t\in(0,T), \tag{23}$$

with initial conditions:

$$\widetilde{V}\,(0,x)=Q^{0}\,(x)+\delta Q^{0}\,(x)\,, \quad \frac{\partial\widetilde{V}}{\partial t}(0,x)=H^{0}\,(x)+\delta H^{0}, \quad x\in[0,2l]\,, \tag{24}$$

boundary conditions:

$$\widetilde{V}\,(t,0){=}V_{0}\,(t), \quad \widetilde{V}\,(t,2l){=}V_{out}\,(t), \quad t{\in}\,[0,T], \tag{25}$$

and the consistency condition:

$$\widetilde{V}\,(t,l+0)=\widetilde{V}\,(t,l-0)+V_{res}\,(t)\,, \quad t\in[0,T]\,. \tag{26}$$

Let us subtract problem (23)–(26) from problem (14)–(17) to obtain the equation for $\delta V\left(T,x;\delta H^{0}\right)$:

$$\frac{\partial^{2}\delta V}{\partial t^{2}}-a^{2}\delta V=c^{2}\frac{\partial^{2}\delta V}{\partial x^{2}}, \quad x\in(0,2l)\,, \quad t\in(0,T), \tag{27}$$

$$\delta V\,(0,x)=\delta Q^{0}\,(x)\,, \quad \frac{\partial\delta V}{\partial t}(0,x)=\delta H^{0}\,(x)\,, \quad x\in[0,2l]\,, \tag{28}$$

$$\delta V\,(t,0)=0, \;\; \delta V\,(t,2l)=0, \quad t\in[0,T]\,, \tag{29}$$

$$\delta V\,(t,l-0)=\delta V\,(t,l+0)\,, \quad t\in[0,T]\,. \tag{30}$$

Let us consider an expression that is identically equal to zero obtained from (27) by multiplying by the still unknown function $V^{*}\,(t,x)$ and integrating over t and over x.

$$(A\delta V,V^{*})=\int\limits_{0}^{T}\int\limits_{0}^{2l}\left[\frac{\partial^{2}\delta V}{\partial t^{2}}-a^{2}\delta V-c^{2}\frac{\partial^{2}\delta V}{\partial x^{2}}\right]\cdot V^{*}dxdt\equiv0,$$

where $AV=\dfrac{\partial^{2}V}{\partial t^{2}}-a^{2}V-c^{2}\dfrac{\partial^{2}V}{\partial x^{2}}$.

Let's perform integration by parts of this expression:

$$\int\limits_{0}^{T}\int\limits_{0}^{2l}\left[\frac{\partial^{2}\delta V}{\partial t^{2}}-a^{2}\delta V-c^{2}\frac{\partial^{2}\delta V}{\partial x^{2}}\right]\cdot V^{*}dxdt=$$

$$= \int_0^{2l} \left[\frac{\partial \delta V}{\partial t} \cdot V^* \Big|_0^T - \int_0^T \frac{\partial \delta V}{\partial t} \cdot \frac{\partial V^*}{\partial t} dt \right] dx - a^2 \int_0^T \int_0^{2l} \delta V \cdot V^* dx dt -$$

$$- c^2 \int_0^T \left[\frac{\partial \delta V}{\partial x} V^* \Big|_0^{2l} - \int_0^{2l} \frac{\partial \delta V}{\partial x} \cdot \frac{\partial V^*}{\partial x} dx \right] dt.$$

We again perform integration by parts for the original expression:

$$(A\delta V, V^*) = \int_0^T \int_0^{2l} \left[\frac{\partial^2 V^*}{\partial t^2} - a^2 V^* - c^2 \frac{\partial^2 V^*}{\partial x^2} \right] \delta V dx dt + \int_0^{2l} \left[\frac{\partial \delta V}{\partial t} (T, x) V^* (T, x) - \right.$$

$$\left. - \frac{\partial \delta V}{\partial t} (0, x) V^* (0, x) - \delta V (T, x) \frac{\partial V^*}{\partial t} (T, x) + \delta V (0, x) \frac{\partial V^*}{\partial t} (0, x) \right] dx -$$

$$- c^2 \int_0^T \left[- \frac{\partial \delta V}{\partial x} (t, 0) V^* (t, 0) + \frac{\partial \delta V}{\partial x} (t, 2l) V^* (t, 2l) + \right.$$

$$\left. + \delta V (t, 0) \frac{\partial V^*}{\partial x} (t, 0) - \delta V (t, 2l) \frac{\partial V^*}{\partial x} (t, 2l) \right] dt.$$

Taking into account the boundary conditions (28)–(30), we write the expression as follows:

$$(\delta V, A^* V^*) = \int_0^T \int_0^{2l} \left[\frac{\partial^2 V^*}{\partial t^2} - a^2 V^* - c^2 \frac{\partial^2 V^*}{\partial x^2} \right] \delta V dx dt + \int_0^{2l} \left[\frac{\partial \delta V}{\partial t} (T, x) V^* (T, x) - \right.$$

$$\left. - \frac{\partial \delta V}{\partial t} (0, x) V^* (0, x) - \delta V (T, x) \frac{\partial V^*}{\partial t} (T, x) + \delta V (0, x) \frac{\partial V^*}{\partial t} (0, x) \right] dx -$$

$$- c^2 \int_0^T \left[- \frac{\partial \delta V}{\partial x} (t, 0) V^* (t, 0) + \frac{\partial \delta V}{\partial x} (t, 2l) V^* (t, 2l) \right] dt. \tag{31}$$

To satisfy the Lagrange identity, the requirements that all non-integral terms be equal to zero, as well as the conditions for the variations of the functional (21) and the Frechet derivatives (22), lead to the following conjugate problem.

$$A^* V^* = \frac{\partial^2 V^*}{\partial t^2} - a^2 V^* - c^2 \frac{\partial^2 V^*}{\partial x^2} = 0, \tag{32}$$

$$V^* (T, x) = 2 \left[V \left(T, x; Q^0 \right) - V^{(1)} (x) \right], \quad \frac{\partial V^*}{\partial t} (T, x) = 2 \left[V_t \left(T, x; H^0 \right) - V^{(2)} (x) \right], \tag{33}$$

$$V^* (t, 0) = 0, \quad V^* (t, 2l) = 0. \tag{34}$$

These arguments on the expression (31) lead to the following lemma.

Lemma. Let $Q^0, \ Q^0 + \delta Q^0 \in G_{ad}^{(1)}, \quad H^0, \ H^0 + \delta H^0 \in G_{ad}^{(2)}$ be given elements.

If $V\left(t,x;Q^0\left(x\right)\right)$ is a solution to problem (14)–(17), and $V^*\left(t,x;Q^0\right)$ is a solution to adjoint problem (32)–(34), then the following identity holds

$$\begin{gathered}\int\limits_0^{2l}\delta\frac{\partial\delta V}{\partial t}\left(T,x\right)V^*\left(T,x\right)dx-\int\limits_0^{2l}\frac{\partial\delta V}{\partial t}\left(0,x\right)V^*\left(0,x\right)dx=\\=\int\limits_0^{2l}\delta V\left(T,x\right)\frac{\partial V^*}{\partial t}\left(T,x\right)dx-\int\limits_0^{2l}\delta V\left(0,x\right)\frac{\partial V^*}{\partial t}\left(0,x\right)dx.\end{gathered}\tag{35}$$

From condition (35), taking into account boundary conditions (28), (33) and the definition of the Frechet derivative (22), we obtain the form

$$J'\left(Q^0\right)=-V^*\left(0,x\right),\quad J'\left(H^0\right)=-\frac{\partial V^*}{\partial t}\left(0,x\right).\tag{36}$$

Adjoint problem (32)–(34), the formulas for the gradients of the functional (36) follow from the Lagrange principle:

$$\left(A\delta V,V^*\right)=\left(\delta V,A^*V^*\right)$$

and is called the principle of duality.

3 Algorithm for solving a variational problem

1. Set the initial approximation $Q_0^0(x)$, $H_0^0(x)$.
2. Assume that Q_n^0 and H_n^0 are already known, then solve direct problem (14)–(17).
3. We calculate the value of the functional

$$J\left(Q_n^0,H_n^0\right)=\int\limits_0^{2l}\left[V\left(T,x;Q_n^0\right)-V^{(1)}\left(x\right)\right]^2dx+\int\limits_0^{2l}\left[\frac{\partial V}{\partial t}\left(T,x;H^0\left(x\right)\right)-V^{(2)}\left(x\right)\right]^2dx.$$

4. If the current value of the functional $J\left(Q_n^0,H_n^0\right)$ is not small enough, then we solve adjoint problem (32)–(34).
5. Calculate the gradient of functional (36).
6. We calculate the next approximation

$$Q_{n+1}^0=Q_n^0-\alpha\cdot J'\left(Q_n^0\right),\quad H_{n+1}^0=H_n^0-\alpha\cdot J'\left(H_n^0\right).$$

7. We return to step 2 using the updated Q_{n+1}^0, H_{n+1}^0.

4 Numerical solution of the direct problem

4.1 Scheme for solving the direct problem

We approximate the problem (14)–(17) using a uniform grid. Let N_t be the number of grid nodes in time on the interval $[0,\ T]$, and N_x be the number of grid nodes in space on the interval $[0,\ 2l]$.

Let us construct in the domain $\Omega=((0,2l)\times(0,T))$ a regular grid ω_h with steps $\tau=T/N_t$, $h=2l/N_x$, where N_t, N_x are positive integers.

In the grid $\overline{\omega}_{h\tau}=\left\{x=ih,\quad t=k\tau,\quad i=\overline{0,\quad N_x},\quad k=\overline{0,\quad N_t}\right\}$, we formulate the difference problem. Accordingly, we write the approximation of problem (14)–(17) as follows: we replace the derivatives included in equation (14) by the formulas

$$\frac{\partial^2 V}{\partial t^2} \sim y_{\bar{t}t}^k = \frac{y_i^{k+1} - 2y_i^k + y_i^{k-1}}{\tau^2},$$

$$\frac{\partial^2 V}{\partial x^2} \sim \Lambda y^k = y_{\bar{x}x}^k = \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{h^2}.$$

Let's consider a family of schemes with weights

$$y_{\bar{t}t}^k = c^2 \Lambda \left(\sigma y^{n+1} + (1-2\sigma)\, y^n + \sigma y^{n-1}\right) + a^2 \cdot y_i^n, \tag{37}$$

$$y_0 = y_0\,(t)\,, y_{Nx} = y_{out}\,(t)\,, \quad t \in [0, T]\,,$$
$$y^0\,(x) = Q^0\,(x)\,, \quad y_t\,(0,\, x) = \tilde{H}^0\,(x)\,.$$

The boundary conditions and the first initial condition on the grid $\omega_{h\tau}$ are satisfied exactly. We choose $\tilde{H}^0\,(x)$ so that the approximation error $\tilde{H}^0\,(x) - \dfrac{\partial V\,(0,\, x)}{\partial t}$ is $O\left(\tau^2\right)$.

Thus, the difference problem is posed. To determine y_i^{n+1}, we obtain the boundary value problem

$$c^2\sigma \cdot \gamma^2 \cdot \left(y_{i+1}^{n+1} + y_{i-1}^{n+1}\right) - \left(1 + 2c^2\sigma\gamma^2\right) y_i^{n+1} = -F_i, \quad 0 < i < N_x, \tag{38}$$

$$y_0 = V_0(t^{t+1}), \quad y_{Nx} = V_{out}(t^{t+1}), \quad \gamma = \frac{\tau}{h}, \tag{39}$$

$$F_i = \left(2y_i^n - y_i^{n-1}\right) + a^2\tau^2 y_i^n + c^2\tau^2\,(1-2\sigma) \cdot y_{\bar{x}x}^n + c^2\tau^2\sigma \cdot \gamma^2 y_{\bar{x}x}^{n-1},$$

$$F_i = \left(2 + a^2\tau^2\right) y_i^n - y_i^{n-1} + c^2\tau^2\,(1-2\sigma) \cdot y_{\bar{x}x}^n + c^2\tau^2 y_{\bar{x}x}^{n-1}. \tag{40}$$

Problem (38)–(40) is solved by the sweep method. The sweep is stable for $\sigma > 0$.

The approximation error for scheme (38) will be $\psi = O\left(\tau^2 + h^2\right)$, provided that the second initial $V_t\,(0, x) = \tilde{H}^0\,(x)$ is approximated by the second order. If we put

$$\tilde{H}_0\,(x) = H_0\,(x) + 0,5\tau \cdot \left(a^2 Q_0\,(x) + c^2\,(Q_0\,(x))''\right),$$

then $\tilde{H}_0\,(x) - V_t\,(x) = O\left(\tau^2\right)$.

Next

$$y^0 = Q^0\,(x_i)\,, \quad y_i^1 = y_i^0 + h \cdot \tilde{H}_0\,(x_i)\,, \quad i = 1,\; ...,\, N_x.$$

4.2 Scheme of the solution of the adjoint problem.

Analogously to the direct one, we approximate the problem (34)–(36) using a uniform grid. Accordingly, we write the approximation of the problem (34)–(36) as follows:

$$y_{\bar{t}t}^* = c^2 \Lambda \left(\sigma y^{*n+1} + (1-2\sigma)\, y^{*n} + \sigma y^{*n-1}\right) + a^2 \cdot y_i^{*n}, \tag{41}$$

$$y_i^{*N_t} = 0, \tag{42}$$

$$y_0^{*n} = 0, y_{N_x}^{*n} = 0, \quad n = N_t - 1, \quad N_t - 2, ..., 1, \tag{43}$$

where $V(x) = 2\left(V\left(T, x; H^0\right) - V^{(1)}(x)\right)$.

To determine y_i^{*n-1} from (41), we obtain a three-layer difference problem

$$c^2\sigma\Lambda y^{*n-1} - y^{*n-1}/\tau^2 = \frac{-ay_i^{*n+1} - 2y_i^n}{\tau^2} - c^2\Lambda\left(\sigma y^{*n+1} + (1-2\sigma)\, y^{*n}\right) + a^2 \cdot y_i^{*n}. \qquad (44)$$

Multiplying (44) by τ^2 and expanding the difference operator Λ, we have

$$c^2\sigma \cdot \gamma^2 \cdot \left(y_{i+1}^{*n-1} + y_{i-1}^{*n-1}\right) - \left(1 + 2c^2\sigma\gamma^2\right) y_i^{*n-1} = -F_i, \quad 0 < i < N_x, \qquad (45)$$

where $F_i = \left(2y_i^n - y_i^{n-1}\right) + a^2\tau^2 y_i^{*n} + c^2\tau^2\sigma y^{*n+1} + c^2\tau^2\left(1-2\sigma\right) y^{*n}$.

Problem (45) with conditions (42), (43) is also solved by the sweep method [22]. From (45) it is clear that the sweep is stable for $\sigma > 0$.

5 Numerical solution of the problem

5.1 Computational experiment

The following initial data were set for the computational experiment: $Q_0 = 0.21$ (m^3/s) is volumetric flow rate of the injected gas, $\bar{P}_0 = 1.0355$ is initial pressure distribution, $\overline{P}_{pl} = 0.2195$ is formation pressure, $Q_{pl} = 0.001$ (m^3/s) is volumetric flow rate from the formation, $\overline{P}_{out} = 1$ is outlet pressure, $Q_{out} = 0.2$ (m^3/s) is volumetric flow rate at the outlet, $\bar{l} = 1$ is well depth, $\lambda_1 = 0.01$ is hydraulic resistance in the ring, $\lambda_2 = 0.23$ is hydraulic resistance in the well, $d_1 \approx 0.1353$ (m) is effective diameter of the well annular space, $d_2 = 0.073$ (m) is diameter of the inner well, $\rho_1 = 0.75$ (kg/m^3) is gas density, $\rho_2 = 700$ (kg/m^3) is oil density, $g = 9.8$ (m/s^2) is acceleration due to gravity, $c_1 = 331$ (m/s) is C-speed of sound in the annular space, $c_2 = 850$ (m/s) is speed of sound in the well, $r_1 = \dfrac{d_1}{2}$ (m) is radii of the annular space, $r_2 = \dfrac{d_2}{2}$ (m) is radii of the inner well, $F_1 = \pi r_1^2$ (m^2) is cross-sectional area of the annular space of the well, $F_2 = \pi r_2^2$ (m^2) is cross-sectional area of the inner well, $w_1 = \dfrac{Q_0}{F_1\rho_1}$ (m/s) is averaged over the cross-section velocity of the mixture in the annulus, $w_2 = \dfrac{Q_0}{F_2\rho_2}$ (m/s) is averaged over the cross-section velocity of the mixture in the well; coefficients in the ring and the inner well: $a_1 = \dfrac{g}{2w_1} + \dfrac{\lambda_1 w_1}{4d_1}$, $a_2 = \dfrac{g}{2w_2} + \dfrac{\lambda_2 w_2}{4d_2}$.

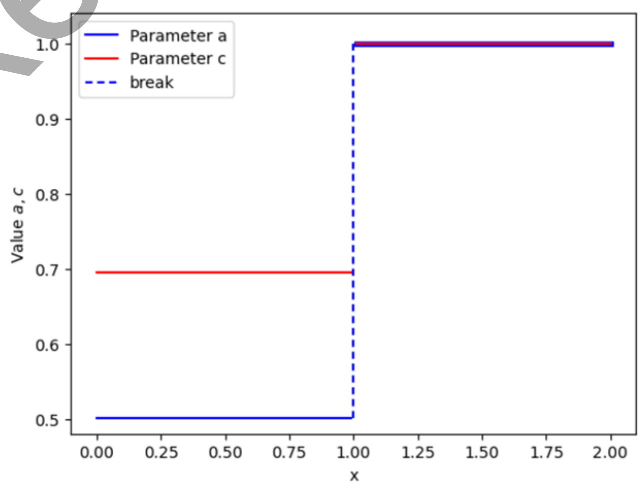

Figure 1. Graphs of functions a and c

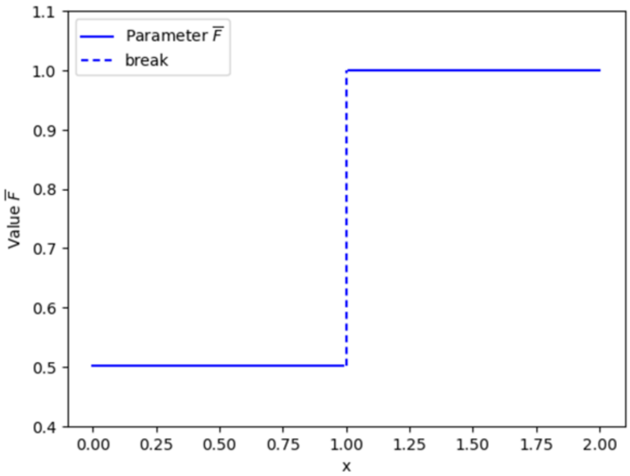

Figure 2. Graph of functions $\overline{F}$

The graphs of the functions a, c and $\overline{F}$ are shown in Figures 1 and 2. They show the change in these parameters along the wellbore depth and reflect the physical conditions in the annular space and in the wellbore.

We computed the test problem using the following parameters:
$l = 1$, $T = 0.001$, $N_x = 21$, $N_t = 21$, grid steps $h = \frac{2l}{N_x}$ and $\tau = \frac{T}{N_t}$, gradient descent step $\alpha = 0.09$.

The initial conditions $Q^0(x)$ and $P^0(x)$ for the direct problem were specified as linear functions:

$$Q^0(x) = Q_0 + \frac{Q_{out} - Q_0}{2l} \cdot x,$$

$$P^0(x) = P_0 + \frac{P_{out} - P_0}{2l} \cdot x.$$

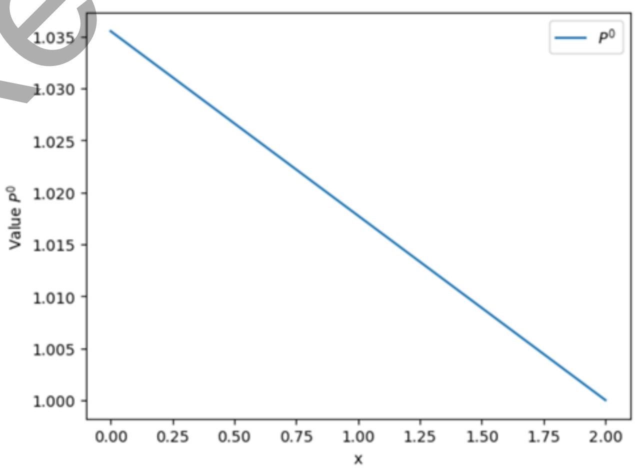

Figure 3. Graph of functions $P^0\,(x)$

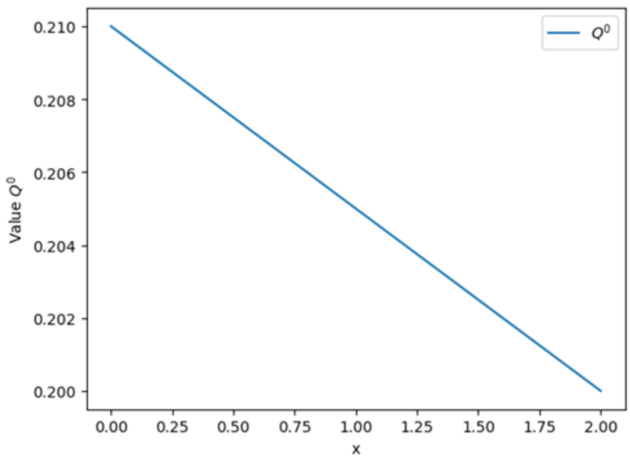

Figure 4. Graph of functions $Q^0\left(x\right)$

Figures 3 and 4 show the graphs of the function $P^0\left(x\right)$, $Q^0\left(x\right)$.

The additional condition for the inverse problem was set as follows:

$$V^{(1)}(T,x) = \left(-x^2 + q\cdot x + r\right)\cdot e^{aT},$$

$$V^{(2)}(T,x) = a\cdot\left(-x^2 + q\cdot x + r\right)\cdot e^{aT},$$

where $q = \dfrac{Q_{out} - Q_0 + 4l^2}{2l}$, $r = Q_0$.

For the numerical solution of the direct and adjoint problem in the difference schemes (37), (41), the weight coefficient σ is chosen equal to 1. The program is implemented in Python 3.13.2. The library for working with multidimensional arrays numpy and matplotlib.pyplot were used to output graphs.

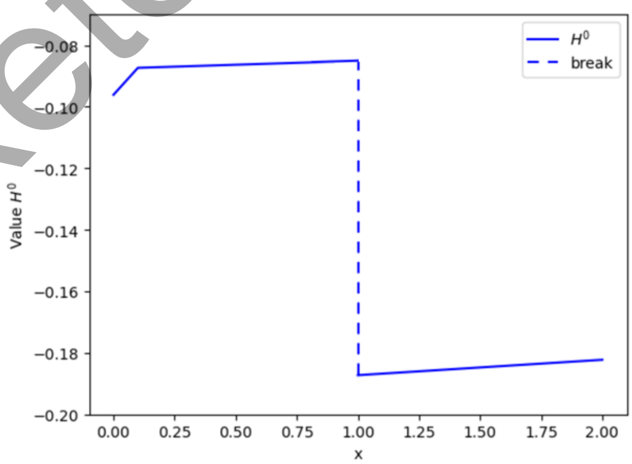

Figure 5. Initial approximation $H_0\left(x\right)$

The initial approximation of the sought function H_n^0 is given in the form of piecewise constant functions as shown in Figure 5. The graph shows a stepwise distribution associated with a sharp change in the functions a in the middle of the region. This is due to the fact that the values of the physical parameters at depth l change when moving from the annular space of the outer well to the production well.

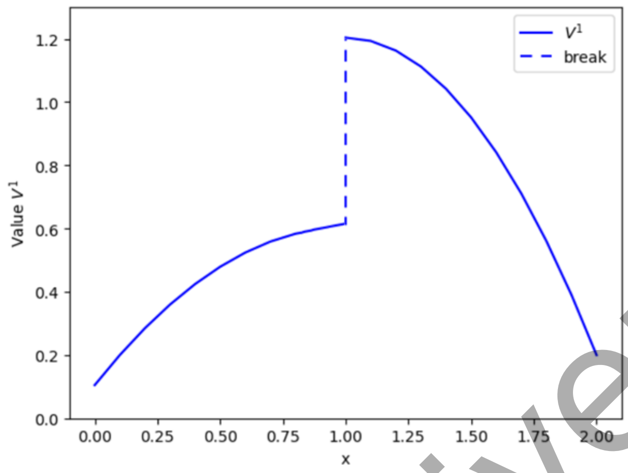

Figure 6. Graph of functions $V^{(1)}(x)$

Figure 6 shows a graph of the change in the values of $V^{(1)}(x)$.

In the iterative process, the value of the functional J decreases monotonically and reaches the value $\|J_n - J_{n-1}\| \le \varepsilon$, $\varepsilon = 1 \times 10^{-7}$ at $n = 66$ iterations. The graph of the decrease in the value of the functional is shown in Figure 7.

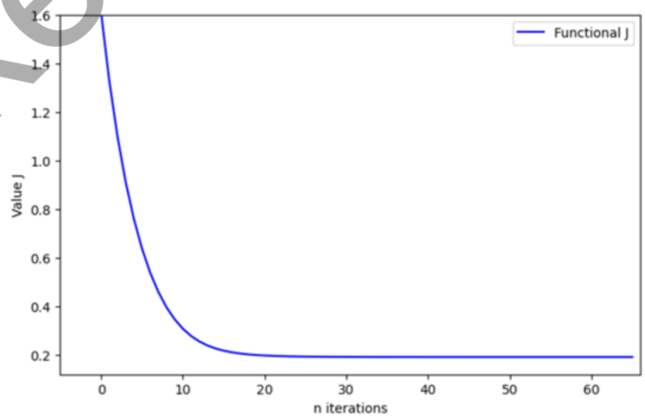

Figure 7. Graph of decreasing functional J

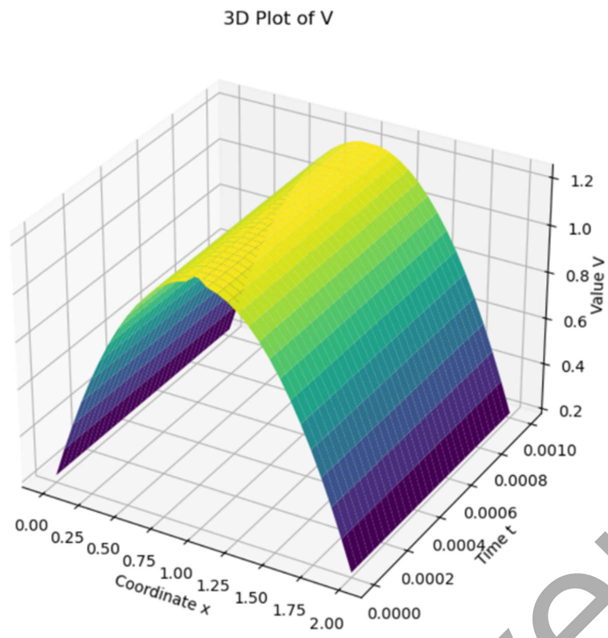

Figure 8. Graph of the solution of the direct problem of functions $V\,(t,x)$

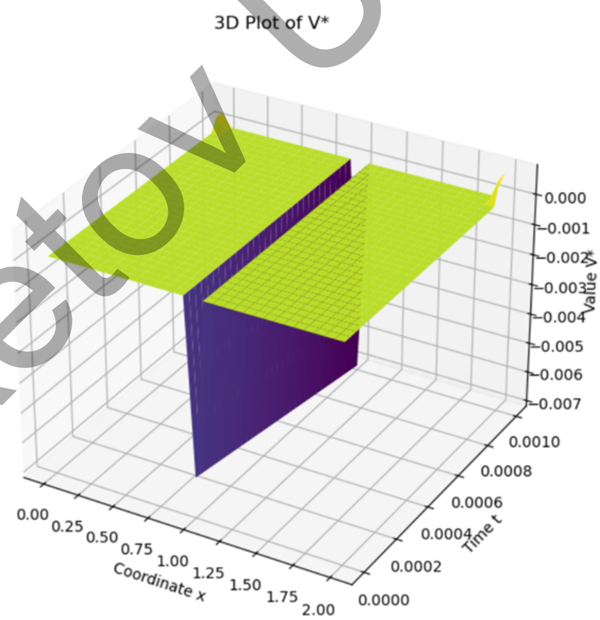

Figure 9. Graph of the solution of the conjugate problem of functions $V^*\,(t,x)$

Figures 8 and 9 show three-dimensional graphs of the functions V and V^*.

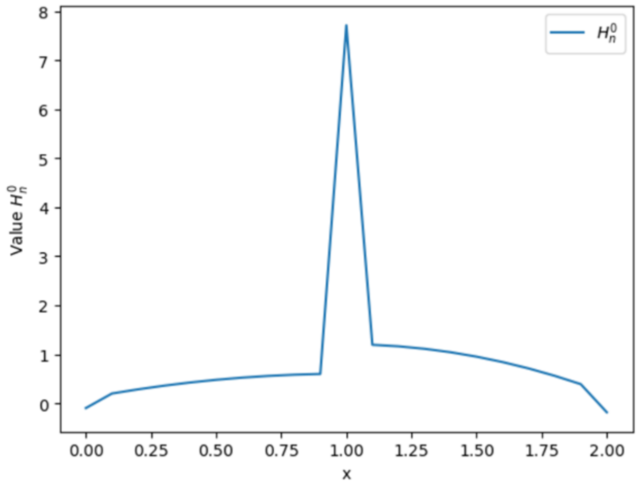

Figure 10. H_n^0 graphs

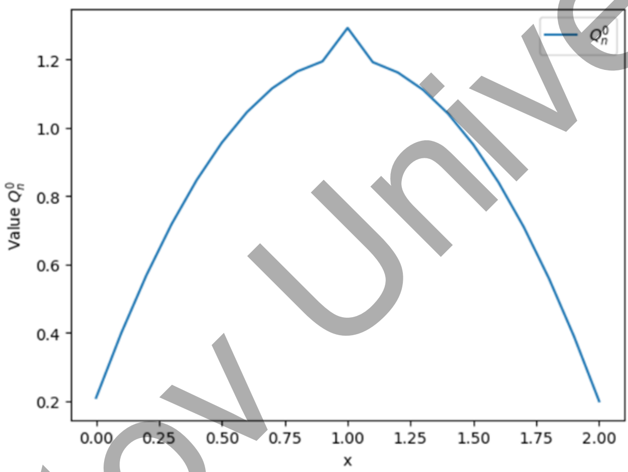

Figure 11. Q_n^0 graphs

Figures 10 and 11 show the graphs of the desired functions H_n^0 and Q_n^0 for 66 iterations.

It is known that the adjoint problem carries valuable information about the solution of the direct problem. This property is confirmed by numerical calculations, since the gradients of the functional for determining the initial conditions of the direct problem at each iteration were chosen as the solution of the adjoint problem at $t = 0$, i.e.,

$$J'\left(H_n^0\right) = V^*\left(0, x\right), J'\left(Q_n^0\right) = \frac{\partial V^*}{\partial t}\left(0, x\right).$$

The numerical calculations performed confirm the effectiveness of the proposed algorithm for modeling the gas lift process of oil production.

Conclusion

In this paper, we present a numerical method for solving direct and inverse problems associated with the gas-lift oil production process using the adjoint equation method. The mathematical model of the process is represented by a hyperbolic equation. The inverse problem is formulated as a problem of restoring the initial condition based on additional information about the solution at $t = T$. To solve it, an optimal control method is used, including minimization of the objective functional using the gradient method, where the gradient of the functional is determined through the solution of the adjoint retrospective problem.

The numerical experiment results demonstrate the effectiveness of the proposed method for inverse problems in the gas-lift process. The plotted graphs demonstrate that with the correct choice of parameters, such as the gradient descent step, it is possible to achieve high accuracy of the solution in a relatively small number of iterations.

The proposed method has practical value, as it enables optimization of gas-lift well operating parameters, reducing gas injection costs while maximizing oil recovery. In the future, the method can be expanded to analyze the influence of various boundary conditions and nonlinear effects, and also adapted to multidimensional models of the gas-lift process, which will expand its scope of application in the oil industry.

Acknowledgments

This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (No. BR 27100483 "Development of predictive exploration technologies for identifying ore-prospective areas based on data analysis from the unified subsurface user platform "Minerals.gov.kz" using artificial intelligence and remote sensing methods").

Author Contributions

N.M. Temirbekov collected and analyzed data, and led manuscript preparation. A.K. Turarov assisted in data collection and analysis. N.M. Temirbekov served as the principal investigator of the research grant and supervised the research process. All authors participated in the revision of the manuscript and approved the final submission.

Conflict of Interest

The authors declare no conflict of interest.

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