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On a Problem of Heat Equation with Fractional Load

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Abstract—In the paper, the solvability problems of an nonhomogeneous boundary value problem in the first quadrant for a fractionally loaded heat equation are studied. Feature of this problem is that, firstly, the loaded term is presented in the form of the Caputo fractional derivative with respect to the spatial variable, secondly, the order of the derivative in the loaded term is less than the order of the differential part and, thirdly, the point of load is moving. The problem is reduced to the Volterra integral equation of the second kind, the kernel of which contains the generalized hypergeometric series. The kernel of the obtained integral equation is estimated and it is shown that the kernel of the equation has a weak singularity (under certain restrictions on the load), this is the basis for the statement that the loaded term in the equation is a weak perturbation of its differential part. In addition, the limiting cases of the order of the fractional derivative are considered. It is proved that there is continuity in the order of the fractional derivative.

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1. INTRODUCTION

The constant interest in studying loaded differential equations is explained both by the expanding number of their applications, and by the fact that loaded equations constitute a special class of equations with their own specific problems. The most general definition of a loaded equation was first given by Nakhushev [1–3]. In the monograph [1], the author gives concepts and a detailed classification of various loaded equations: loaded differential, loaded integral, loaded integro-differential, loaded functional equations, and their numerous applications to biology problems.

The study of the generalized solvability of nonhomogeneous boundary value problems for loaded differential equations in Sobolev s paces was carried out in [4, 5]. In addition, a review of loaded equations is given in [4, 6]. In the monograph [7], boundary problems are studied for a loaded heat conduction operator in bounded and unbounded domains when the order of the derivative in the loaded term is equal to or higher than the order of the differential part of the equation.

In recent years, interest has grown in the study of fractional differential equations in which an unknown function is contained under the sign of a fractional derivative. This is due both to the development of the theory of fractional integration and differentiation, as well as applications of the apparatus of fractional integration and differentiation in various fields of science [8–13]. Some authors [14, 15] studied loaded differential equations that contain fractional derivatives of the traces of an unknown function with respect to the time variable, but the order of the derivative in the loaded term is strictly less than the corresponding order of the differential part of the equation and the load point is fixed, that is, is motionless.

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For practical applications, the definition of Caputo fractional derivative is significant. It differs from the Riemann–Liouville definition in that the function, first, the function is differentiated with the smallest integer order exceeding some non-integer order, and then the result is integrated with the order, which is their difference. Of interest are the boundary value problems for the loaded heat equation when the loaded term is represented in the form of a Caputo fractional derivative [16, 17]:

$$cD_{a,t}^{\beta}f(t) = \frac{1}{\Gamma(n-\beta)}\int\limits_{0}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}}d\tau; \quad \beta, \quad a \in \Re, \quad n-1<\beta<n,$$

when $a=0$ and $n=1$, so $0<\beta<1$, that is

$$cD_{0,t}^{\beta}f(t) = \frac{1}{\Gamma(1-\beta)}\int\limits_{0}^{t}\frac{f^{/}(\tau)}{(t-\tau)^{\beta}}d\tau.$$

and the load point moves at a variable velocity.

2. STATEMENT OF THE PROBLEM

We consider the problem in a domain $Q=\{(x,t): x>0, t>0\}$

$$u_t - u_{xx} + \lambda\left\{D_{0,x}^{\beta}u\left(x,t\right)\right\}\big|_{x=\gamma(t)} = f\left(x,t\right), \tag{1}$$

$$u\left(x,0\right)=0, \quad u\left(0,t\right)=0, \quad 0<\beta<1, \quad n=1. \tag{2}$$

We introduce the notation

$$\mu\left(t\right) = \left\{D_{0,x}^{\beta}u\left(x,t\right)\right\}\big|_{x=\gamma(t)} = \frac{1}{\Gamma\left(1-\beta\right)}\int\limits_{0}^{x}\frac{u_{\xi}\left(\xi,t\right)}{\left(x-\xi\right)^{\beta}}d\xi. \tag{3}$$

3. REDUCING THE PROBLEM TO AN INTEGRAL EQUATION

We invert the differential part of the problem (1)–(2) [18]

$$u\left(x,t\right) = -\lambda\int\limits_{0}^{t}\int\limits_{0}^{+\infty}G\left(x,\xi,t-\tau\right)\mu\left(\tau\right)d\xi d\tau + \int\limits_{0}^{t}\int\limits_{0}^{+\infty}G\left(x,\xi,t-\tau\right)f\left(\xi,\tau\right)d\xi d\tau,$$

where

$$G\left(x,\xi,t\right) = \frac{1}{2\sqrt{\pi t}}\left\{\exp\left(-\frac{\left(x-\xi\right)^{2}}{4t}\right) - \exp\left(-\frac{\left(x+\xi\right)^{2}}{4t}\right)\right\}.$$

Taking into account the ratio

$$\int_{0}^{+\infty}G\left(x,\xi,t-\tau\right)d\xi = erf\left(\frac{x}{2\sqrt{t-\tau}}\right)$$

we get the following representation of the solution to the problem (1)-(2)

$$u\left(x,t\right) = -\lambda\int_{0}^{t}erf\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu\left(\tau\right)d\tau + f_1\left(x,t\right), \tag{4}$$

where

$$f_1\left(x,t\right) = \int\limits_{0}^{t}\int\limits_{0}^{+\infty}G\left(x,\xi,t-\tau\right)f\left(\xi,\tau\right)d\xi d\tau.$$

Taking into account the notation (3), equality (4) can be rewritten in the form

$$\mu(t) = -\lambda \left\{ D_{0,x}^{\beta} \left[\int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}} \right) \mu(\tau)\, d\tau \right] \right\} |_{x=\gamma(t)} + f_2(t), \tag{5}$$

where

$$f_2(t) = \frac{1}{\Gamma(1-\beta)} \int_0^x \frac{\frac{\partial f_1(\xi,\tau)}{\partial \xi}}{(x-\xi)^{\beta}} d\xi \,|_{x=\gamma(t)}$$

We calculate the fractional derivative:

$$D_{0,x}^{\beta} \left[\int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}} \right) \mu(\tau)\, d\tau \right] = \frac{1}{\Gamma(1-\beta)} \int_0^x \frac{1}{(x-\xi)^{\beta}} \frac{\partial}{\partial \xi} \left[\int_0^t erf\left(\frac{\xi}{2\sqrt{t-\tau}} \right) \mu(\tau)\, d\tau \right] d\xi$$

$$= \frac{1}{\sqrt{\pi}\Gamma(1-\beta)} \int_0^x \frac{1}{(x-\xi)^{\beta}} \left(\int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}} \exp\left(-\frac{\xi^2}{4(t-\tau)} \right) d\tau \right) d\xi$$

$$= \frac{1}{\sqrt{\pi}\Gamma(1-\beta)} \int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}} \left[\int_0^x \frac{1}{(x-\xi)^{\beta}} \exp\left(-\frac{\xi^2}{4(t-\tau)} \right) d\xi \right] d\tau.$$

We have

$$D_{0,x}^{\beta} \left[\int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}} \right) \mu(\tau)\, d\tau \right] = \frac{1}{\sqrt{\pi}\Gamma(1-\beta)} \int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}} \mathrm{I}(x,t,\tau,\beta)\, d\tau, \tag{6}$$

where

$$\mathrm{I}(x,t,\tau,\beta) = \int_0^x \frac{1}{(x-\xi)^{\beta}} \exp\left(-\frac{\xi^2}{4(t-\tau)} \right) d\xi.$$

From formula 3.478(3) [19], the following equality follows

$$\mathrm{I}(x,t,\tau,\beta) = \mathrm{B}(1-\beta;1)\, x^{1-\beta}\, {}_2F_2\left(\frac{1}{2}, 1; \frac{2-\beta}{2}, \frac{3-\beta}{2}; -\frac{x^2}{4(t-\tau)} \right), \tag{7}$$

where $\mathrm{B}(1-\beta;1) = \frac{\Gamma(1-\beta)}{\Gamma(2-\beta)}$. Substituting (7) into equality (6), we obtain

$$D_{0,x}^{\beta} \left[\int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}} \right) \mu(\tau)\, d\tau \right]$$

$$= \frac{x^{1-\beta}}{\sqrt{\pi}\Gamma(2-\beta)} \int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}} {}_2F_2\left(\frac{1}{2}, 1; \frac{2-\beta}{2}, \frac{3-\beta}{2}; -\frac{x^2}{4(t-\tau)} \right).$$

When $x = \gamma(t)$ we have

$$D_{0,x}^{\beta} \left[\int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}} \right) \mu(\tau)\, d\tau \right] |_{x=\gamma(t)} = -\frac{(\gamma(t))^{1-\beta}}{\sqrt{\pi}\Gamma(2-\beta)} \int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}}$$

$$\times\, {}_2F_2\left(\frac{1}{2}, 1; \frac{2-\beta}{2}, \frac{3-\beta}{2}; -\frac{\gamma^2(t)}{4(t-\tau)} \right) d\tau. \tag{8}$$

In view of (8), equation (5) can be rewritten in the form

$$\mu(t) + \lambda \int_0^t K_\beta(t,\tau)\,\mu(\tau)\,d\tau = f_2(t), \tag{9}$$

where

$$K_\beta(t,\tau) = \frac{(\gamma(t))^{1-\beta}}{\sqrt{\pi}\Gamma(2-\beta)} \frac{1}{\sqrt{t-\tau}}\, {}_2F_2\left(\frac{1}{2}, 1; \frac{2-\beta}{2}, \frac{3-\beta}{2}; -\frac{\gamma^2(t)}{4(t-\tau)}\right). \tag{10}$$

Here ${}_2F_2(a_1, a_2; b_1, b_2; z) = \sum_{\kappa=0}^{\infty} \frac{(a_1)_\kappa (a_2)_\kappa}{(b_1)_\kappa (b_2)_\kappa} \frac{z^\kappa}{\kappa!}$, where $(a)_\kappa = \frac{\Gamma(a+\kappa)}{\Gamma(a)}$ is the Pohammer symbol. Therefore $(1)_\kappa = \kappa!$. Then in (10) we have

$${}_2F_2\left(\frac{1}{2}, 1; \frac{2-\beta}{2}, \frac{3-\beta}{2}; -\frac{\gamma^2(t)}{4(t-\tau)}\right) = \sum_{\kappa=0}^{\infty} \frac{\left(\frac{1}{2}\right)_\kappa}{\left(\frac{2-\beta}{2}\right)_\kappa \left(\frac{3-\beta}{2}\right)_\kappa} \left(\frac{(\gamma(t))^2}{4(t-\tau)}\right)^\kappa.$$

4. LIMIT CASES β

4.1. $\beta = 0$

Then from [9] we have $D_{0,x}^0 f(x) = f(x)$. Problem (1)–(2) will take the form

$$u_t - u_{xx} + \lambda u(\gamma(t), t) = f(x,t), \tag{11}$$

$$u(x,0) = 0, \quad u(0,t) = 0. \tag{12}$$

We denote $\mu(t) = u(\gamma(t), t)$. Following [7], we invert the differential part in the problem (11), (12)

$$u(x,t) = -\lambda \int_0^t \int_0^{+\infty} \mu(\tau)\, G(x,\xi,t-\tau)\,d\xi d\tau + \int_0^t \int_0^{+\infty} f(\xi,\tau)\, G(x,\xi,t-\tau)\,d\xi d\tau. \tag{13}$$

Taking into account the ratio

$$\int_0^{+\infty} G(x,\xi,t-\tau)\,d\xi = erf\left(\frac{x}{2\sqrt{t-\tau}}\right) \tag{14}$$

equality (13) will take the form

$$u(x,t) = -\lambda \int_0^t \mu(\tau)\, erf\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau + f_1(x,t), \tag{15}$$

where

$$f_1(x,t) = \int_0^t \int_0^{+\infty} f(\xi,\tau)\, G(x,\xi,t-\tau)\,d\xi d\tau.$$

.

By virtue of the notation $\mu(t) = u(\gamma(t), t)$ from the equality (15) when $x = \gamma(t)$ we get an integral equation for an unknown function $\mu(t)$

$$\mu(t) + \lambda \int_0^t erf\left(\frac{\gamma(t)}{2\sqrt{t-\tau}}\right) \mu(\tau)\,d\tau = f_2(t),$$

where $f_2(t) = f_1(\gamma(t), t)$. On the other hand, in equality (6) we take the limit when $\beta \to 0+0$:

$$\lim_{\beta\to 0+0} D_{0,x}^{\beta}\left[\int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(t)\,d\tau\right] = \lim_{\beta\to 0+0}\left\{\frac{1}{\sqrt{\pi}\Gamma(1-\beta)}\int_0^t \frac{\mu(t)}{\sqrt{t-\tau}}\mathrm{I}(x,t,\tau,\beta)\,d\tau\right\},$$

where

$$\mathrm{I}(x,t,\tau,\beta) = \int_0^x \frac{1}{(x-\xi)^{\beta}}\exp\left(-\frac{\xi^2}{4(t-\tau)}\right)d\xi.$$

The function under the limit sign is a definite and continuous for $\beta = 0$. Therefore, we can pass to the limit under the sign of the integral:

$$\lim_{\beta\to 0+0} D_{0,x}^{\beta}\left[\int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau)\,d\tau\right] = \frac{1}{\sqrt{\pi}}\int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}}\left(\int_0^x \exp\left(-\frac{\xi^2}{4(t-\tau)}\right)d\xi\right)d\tau$$

$$= \left\|z = \frac{\xi}{2\sqrt{t-\tau}}, dz = \frac{d\xi}{2\sqrt{t-\tau}}\right\| = \frac{1}{\sqrt{\pi}}\int_0^t \frac{\mu(\tau)}{\sqrt{t-\tau}}2\sqrt{t-\tau}\left(\int_0^{\frac{x}{2\sqrt{t-\tau}}} e^{-z^2}dz\right)d\tau$$

$$= \int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau)\,d\tau.$$

Finally, when $x = \gamma(t)$ we get

$$K_0(t,\tau) = \lim_{\beta\to 0+0} D_{0,x}^{\beta}\left[\int_0^x erf\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau)\,d\tau\right]\Bigg|_{x=\gamma(t)} = \int_0^t erf\left(\frac{\gamma(t)}{2\sqrt{t-\tau}}\right)\mu(\tau)\,d\tau.$$

Then equation (9) when $\beta \to 0+0$ and $x = \gamma(t)$ takes the form

$$\mu(t) + \lambda\int_0^t K_0(t,\tau)\mu(\tau)\,d\tau = f_2(t), \tag{16}$$

where $K_0(t,\tau) = erf\left(\frac{\gamma(t)}{2\sqrt{t-\tau}}\right)$, $f_2(t) = f_1(\gamma(t); t)$.

Remark 1. *Equation (16) can be obtained by taking the limit of (10) when* $\beta \to 0+0$:

$$K_0(t,\tau) = \lim_{\beta\to 0+0} K_{\beta}(t,\tau) = \lim_{\beta\to 0+0}\left\{\frac{\gamma(t)^{1-\beta}}{\sqrt{\pi}\Gamma(2-\beta)\sqrt{t-\tau}}{}_2F_2\left(\frac{1}{2},1;\frac{2-\beta}{2},\frac{3-\beta}{2},-\frac{(\gamma(t))^2}{4(t-\tau)}\right)\right\}$$

$$= \frac{\gamma(t)}{\sqrt{\pi(t-\tau)}}{}_2F_2\left(\frac{1}{2},1;1,\frac{3}{2};-\frac{(\gamma(t))^2}{4(t-\tau)}\right).$$

Because from formula 5.2.8(8) [20]

$$F_2\left(\frac{1}{2},1;1,\frac{3}{2},-z\right) = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k}\frac{(-z)^k}{k!} = \sum_{k=0}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k+\frac{3}{2}\right)}\frac{(-z)^k}{k!}$$

$$= \sum_{k=0}^{\infty}\frac{\frac{1}{2}}{k+\frac{1}{2}}\frac{(-z)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\frac{z^k}{k!} = \frac{1}{\sqrt{z}}\sum_{k=1}^{\infty}\frac{(-1)^k(\sqrt{z})^{2k+1}}{(2k+1)\,k!} = \frac{1}{\sqrt{z}}\frac{\sqrt{\pi}}{2}erf\left(\sqrt{z}\right),$$

then for $z = \frac{(\gamma(t))^2}{4(t-\tau)}$ *we have*

$$K_0(t,\tau) = \frac{\gamma(t)}{\sqrt{\pi(t-\tau)}} \frac{\sqrt{\pi}}{2} \frac{2\sqrt{t-\tau}}{\gamma(t)} erf\left(\frac{\gamma(t)}{2\sqrt{t-\tau}}\right) = erf\left(\frac{\gamma(t)}{2\sqrt{t-\tau}}\right).$$

Then equation (9) will take the form (16) when $\beta \to 0+0$.

4.2. $\beta = 1$

Then [9] $D^1_{0,x} f(x) = f'(x)$. Problem (1)–(2) will take the form

$$u_t - u_{xx} + \lambda u_x(x,t)\big|_{x=\gamma(t)} = f(x,t), \tag{17}$$

$$u(x,0) = 0, \quad u(0,t) = 0. \tag{18}$$

We denote $\mu(t) = u_x(x,\gamma(t))$. We invert the differential part in problem (17), (18):

$$u(x,t) = -\lambda \int\limits_0^t \int\limits_0^{+\infty} \mu(\tau) G(x,\xi,t-\tau) d\xi d\tau + \int\limits_0^t \int\limits_0^{+\infty} f(E,\tau) G(x,\xi,t-\tau) d\xi d\tau.$$

In view of relation (14), the last equality can be rewritten in the form

$$u(x,t) = -\lambda \int\limits_0^t \mu(\tau)\, erf\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau + f_1(x,t), \tag{19}$$

where

$$f_1(x,t) = \int\limits_0^t \int\limits_0^{+\infty} f(\xi,\tau) G(x,\xi,t-\tau) d\xi d\tau.$$

Equality (19) is differentiable with respect to x, given that

$$\frac{\partial}{\partial x}\left(erf\left(\frac{x}{2\sqrt{t-\tau}}\right)\right) = \frac{x}{\sqrt{\pi(t-\tau)}} \exp\left(-\frac{x^2}{4(t-\tau)}\right).$$

Further, substituting $x = \gamma(t)$ and taking into account the designation $\mu(t) = u_x(\gamma(t);t)$ from equality (19) we obtain the Volterra integral equation of the second kind:

$$\mu(t) + \lambda \int\limits_0^t K_1(t,\tau) \mu(\tau) d\tau = f_2(t), \tag{20}$$

where

$$K_1(t,\tau) = \frac{1}{\sqrt{\pi(t-\tau)}} \exp\left(-\frac{\gamma^2(t)}{4(t-\tau)}\right), \quad f_2(t) = f_1(\gamma(t),t).$$

On the other hand, in equality (6) we cannot take the limit at $\beta \to 1-0$, i.e. to find

$$\lim_{\beta \to 1-0} D^\beta_{0,x}\left[\int\limits_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right) \mu(\tau) d\tau\right]$$

because when passing to the limit, we obtain the uncertainty of the form (0∞). Indeed, by virtue of the formula 3.478(3) from [20] we have

$$\lim_{\beta \to 1-0} \mathrm{I}(x,t,\tau,\beta) = \int\limits_0^x \frac{1}{x-\xi} \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\xi$$

$$= \mathrm{B}\,(0,2)\, x\,{}_2F_2\left(1,\frac{3}{2};1,\frac{3}{2};-\frac{x^2}{4\,(t-\tau)}\right) = \Gamma\,(0)\, x \exp\left(-\frac{x^2}{4\,(t-\tau)}\right).$$

$$\lim_{\beta\to 1-0} \mathrm{I}\,(x,t,\tau,\beta) = \Gamma\,(0)\, x \exp\left(-\frac{x^2}{4\,(t-\tau)}\right). \tag{21}$$

Since $\lim\limits_{z\to 0+0} \Gamma\,(z) = \infty$, then taking into account (21) we get the uncertainty when taking the limit in equality (6). Therefore, equality (6) requires further transformation. To do this, we make the limit transition in equality (10), which is an equivalent transformation to equality (6):

$$\lim_{\beta\to 1-0} K_\beta\,(t,\tau) = \lim_{\beta\to 1-0}\left\{\frac{(\gamma\,(t))^{1-\beta}}{\sqrt{\pi}\Gamma\,(2-\beta)}\frac{1}{\sqrt{t-\tau}}\,{}_2F_2\left(\frac{1}{2},1;\frac{2-\beta}{2},\frac{3-\beta}{2};-\frac{(\gamma\,(t))^2}{4\,(t-\tau)}\right)\right\}$$

$$= \frac{1}{\sqrt{\pi\,(t-\tau)}}\,{}_2F_2\left(\frac{1}{2},1;\frac{1}{2},1;-\frac{(\gamma\,(t))^2}{4\,(t-\tau)}\right) = \frac{1}{\sqrt{\pi\,(t-\tau)}}\exp\left(-\frac{(\gamma\,(t))^2}{4\,(t-\tau)}\right).$$

So,

$$\lim_{\beta\to 1-0} K_\beta\,(t,\tau) = \frac{1}{\sqrt{\pi\,(t-\tau)}}\exp\left(-\frac{(\gamma\,(t))^2}{4\,(t-\tau)}\right). \tag{22}$$

Obviously, the limit (22) coincides with the kernel of the integral equation (20). Thus, there is continuity in order of the fractional derivative when $0<\beta<1$.

5. ESTIMATION OF THE KERNEL OF THE INTEGRAL EQUATION

We study the kernel of integral equation (9) K_β, which has singularities when $\tau = t$ and $t = 0$. Direct study of the kernel K_β is difficult, since the kernel of the integral equation under study contains a generalized hypergeometric series. We find $\int\limits_0^t K_\beta\,(t,\tau)\,d\tau$ and calculate $\lim\limits_{t\to 0}\int\limits_0^t K_\beta\,(t,\tau)\,d\tau$.

We consider the kernel of integral equation (9) K_β in the form of representation (6) when $x=\gamma\,(t)$:

$$K_\beta\,(t,\tau) = \frac{1}{\sqrt{\pi}\Gamma\,(1-\beta)\sqrt{t-\tau}}\int\limits_0^{\gamma(t)}\frac{1}{(\gamma\,(t)-\xi)^\beta}\exp\left(-\frac{\xi^2}{4\,(t-\tau)}\right)d\xi.$$

We calculate $\int\limits_0^t K_\beta\,(t,\tau)\,d\tau$.

$$\int\limits_0^t K_\beta\,(t,\tau)\,d\tau = \frac{1}{\sqrt{\pi}\Gamma\,(1-\beta)}\int\limits_0^t\frac{1}{\sqrt{t-\tau}}\left[\int\limits_0^{\gamma(t)}\frac{1}{(\gamma\,(t)-\xi)^\beta}\exp\left(-\frac{\xi^2}{4\,(t-\tau)}\right)d\xi\right]d\tau$$

$$= \frac{1}{\sqrt{\pi}\Gamma\,(1-\beta)}\int\limits_0^{\gamma(t)}\frac{1}{(\gamma\,(t)-\xi)^\beta}\left[\int\limits_0^t\frac{1}{\sqrt{t-\tau}}\exp\left(-\frac{\xi^2}{4\,(t-\tau)}\right)d\tau\right]d\xi.$$

So,

$$\int\limits_0^t K_\beta\,(t,\tau)\,d\tau = \frac{1}{\sqrt{\pi}\Gamma\,(1-\beta)}\int_0^{\gamma(t)}\frac{1}{(\gamma\,(t)-\xi)^\beta}J\,(t,\tau)\,d\xi, \tag{23}$$

where

$$J\,(t,\tau) = \int\limits_0^t\frac{1}{\sqrt{t-\tau}}\exp\left(-\frac{\xi^2}{4\,(t-\tau)}\right)d\tau.$$

We will introduce a replacement: $z = \frac{\xi}{2\sqrt{t-\tau}}$. Then $J(t,\tau) = \xi \int\limits_{\frac{\xi}{2\sqrt{t}}}^{+\infty} \frac{1}{z^2} e^{-z^2} dz$. From the formula 3.461 (5) [19]

$$\int\limits_{u}^{+\infty} e^{-\mu^2 x^2} \frac{dx}{x^2} = \frac{1}{u} e^{-\mu^2 u^2} - \mu\sqrt{\pi} erfc(u\mu), \quad |\arg\mu| < \frac{\pi}{4}, \quad u > 0$$

when $u = \dfrac{\xi}{2\sqrt{t}}$, $\mu = 1$ we get $J(t,\tau) = 2\sqrt{t} e^{-\frac{\xi^2}{4t}} - \xi\sqrt{\pi} erfc\left(\frac{\xi}{2\sqrt{t}}\right)$. Substituting $J(t,\tau) = 2\sqrt{t} e^{-\frac{\xi^2}{4t}} - \xi\sqrt{\pi} erfc\left(\frac{\xi}{2\sqrt{t}}\right)$ into (23), we obtain

$$\int\limits_0^t K_\beta(t,\tau)\, d\tau = \frac{1}{\sqrt{\pi}\Gamma(1-\beta)} \int\limits_0^{\gamma(t)} \frac{1}{(\gamma(t)-\xi)^\beta} \left[2\sqrt{t} e^{-\frac{\xi^2}{4t}} - \xi\sqrt{\pi} erfc\left(\frac{\xi}{2\sqrt{t}}\right)\right] d\xi$$

or

$$\int\limits_0^t K_\beta(t,\tau)\, d\tau = \frac{2\sqrt{t}}{\sqrt{\pi}\Gamma(1-\beta)} J_1(t,\beta) - \frac{1}{\Gamma(1-\beta)} J_2(t,\beta), \tag{24}$$

$$J_1(t,\beta) = \int\limits_0^{\gamma(t)} \frac{1}{(\gamma(t)-\xi)^\beta} e^{-\frac{\xi^2}{4t}} d\xi, \quad J_2(t,\beta) = \int\limits_0^{\gamma(t)} \frac{\xi}{(\gamma(t)-\xi)^\beta} erfc\left(\frac{\xi}{2\sqrt{t}}\right) d\xi.$$

We calculate the integrals $J_1(t,\beta)$ and $J_2(t,\beta)$. For calculation $J_1(t,\beta)$, we apply the formula 3.478 (3) [19] at $n = 2$, $u = \gamma(t)$, $\nu = 1$, $\mu = 1-\beta$.

Then

$$J_1(t,\beta) = \mathrm{B}(1-\beta, 1)(\gamma(t))^{1-\beta} {}_2F_2\left(\frac{1}{2}, 1; \frac{2-\beta}{2}, \frac{3-\beta}{2}; -\frac{(\gamma(t))^2}{4t}\right)$$

$$= \frac{\Gamma(1-\beta)}{\Gamma(2-\beta)}(\gamma(t))^{1-\beta} {}_2F_2\left(1, \frac{3}{2}; \frac{3-\beta}{2}, \frac{4-\beta}{2}; -\frac{(\gamma(t))^2}{4t}\right). \tag{25}$$

For calculation $J_2(t,\beta)$, we apply the formula 2.8.2(3) [21].

We obtain

$$J_2(t,\beta) = -\frac{2}{2\sqrt{\pi t}}(\gamma(t))^{3-\beta} \mathrm{B}(3, 1-\beta) {}_3F_3\left(\frac{3}{2}, 2, \frac{1}{2}; \frac{4-\beta}{2}, \frac{5-\beta}{2}, \frac{3}{2}; -\frac{(\gamma(t))^2}{4t}\right)$$

$$+ (\gamma(t))^{2-\beta} \mathrm{B}\left(2, \frac{1}{4}\right) = -\frac{\Gamma(1-\beta)}{\Gamma(4-\beta)} \frac{(\gamma(t))^{3-\beta}}{\sqrt{\pi t}} {}_2F_2\left(\frac{1}{2}, 2; \frac{4-\beta}{2}, \frac{5-\beta}{2}; -\frac{(\gamma(t))^2}{4t}\right)$$

$$+ \frac{\Gamma(1-\beta)}{\Gamma(3-\beta)}(\gamma(t))^{2-\beta}. \tag{26}$$

The expressions (25) and (26) for $J_1(t,\beta)$ and $J_2(t,\beta)$ we substitute in (24):

$$\int\limits_0^t K_\beta(t,\tau)\, d\tau = \frac{2\sqrt{t}}{\sqrt{\pi}\Gamma(1-\beta)} \frac{\Gamma(1-\beta)}{\Gamma(2-\beta)}(\gamma(t))^{1-\beta} {}_2F_2\left(1, \frac{3}{2}; \frac{3-\beta}{2}, \frac{4-\beta}{2}; -\frac{\gamma^2(t)}{4t}\right)$$

$$+ \frac{1}{\Gamma(1-\beta)} \frac{\Gamma(1-\beta)}{\Gamma(4-\beta)} \frac{(\gamma(t))^{3-\beta}}{\sqrt{\pi t}} {}_2F_2\left(\frac{1}{2}, 2; \frac{4-\beta}{2}, \frac{5-\beta}{2}; -\frac{(\gamma(t))^2}{4t}\right) - \frac{1}{\Gamma(3-\beta)}$$

$$\times\left(\gamma\left(t\right)\right)^{2-\beta}=\frac{2\sqrt{t}}{\sqrt{\pi}\Gamma\left(2-\beta\right)}\left(\gamma\left(t\right)\right)^{1-\beta}{}_2F_2\left(1,\frac{3}{2};\frac{3-\beta}{2},\frac{4-\beta}{2};-\frac{\left(\gamma\left(t\right)\right)^2}{4t}\right)$$

$$+\frac{1}{\Gamma\left(4-\beta\right)\sqrt{\pi t}}\left(\gamma\left(t\right)\right)^{3-\beta}{}_2F_2\left(\frac{1}{2},2;\frac{4-\beta}{2},\frac{5-\beta}{2};-\frac{\left(\gamma\left(t\right)\right)^2}{4t}\right)-\frac{1}{\Gamma\left(3-\beta\right)}\left(\gamma\left(t\right)\right)^{2-\beta}$$

$$=\frac{2\sqrt{t}}{\sqrt{\pi}\Gamma\left(2-\beta\right)}\left(\gamma\left(t\right)\right)^{1-\beta}\sum_{\kappa=0}^{\infty}\frac{\left(1\right)_\kappa\left(\frac{3}{2}\right)_\kappa\left(-1\right)^\kappa}{\left(\frac{3-\beta}{2}\right)_\kappa\left(\frac{4-\beta}{2}\right)_\kappa4^\kappa\kappa!}\left(\gamma\left(t\right)\right)^{2\kappa}+\frac{1}{\Gamma\left(4-\beta\right)}$$

$$\times\frac{1}{\sqrt{\pi t}}\left(\gamma\left(t\right)\right)^{3-\beta}\sum_{\kappa=0}^{\infty}\frac{\left(\frac{1}{2}\right)_\kappa\left(2\right)_\kappa\left(-1\right)^\kappa}{\left(\frac{4-\beta}{2}\right)_\kappa\left(\frac{5-\beta}{2}\right)_\kappa4^\kappa\kappa!}\left(\gamma\left(t\right)\right)^{2\kappa}-\frac{1}{\Gamma\left(3-\beta\right)}\left(\gamma\left(t\right)\right)^{2-\beta}. \tag{27}$$

Considering formula II.2 [22], we have

$$\left(1\right)_\kappa=\kappa!,\quad\left(2\right)_\kappa=\left(\kappa+1\right)!,\quad\left(\frac{1}{2}\right)_\kappa=2^{-2\kappa}\frac{\left(2\kappa\right)!}{\kappa!},\quad\left(\frac{3}{2}\right)_\kappa=2^{-2\kappa}\frac{\left(2\kappa+1\right)!}{\kappa!},$$

$$\left(\frac{3-\beta}{2}\right)_\kappa\left(\frac{4-\beta}{2}\right)_\kappa2^{2\kappa}=\left(3-\beta\right)_{2\kappa},\quad\left(\frac{4-\beta}{2}\right)_\kappa\left(\frac{5-\beta}{2}\right)_\kappa2^{2\kappa}=\left(4-\beta\right)_{2\kappa}.$$

Then equality (27) can be rewritten in the form

$$\int\limits_0^t K_\beta\left(t,\tau\right)d\tau=\frac{2\sqrt{t}}{\sqrt{\pi}\Gamma\left(2-\beta\right)}\left(\gamma\left(t\right)\right)^{1-\beta}\sum_{\kappa=0}^{\infty}\frac{2^{-2\kappa}\left(2\kappa+1\right)!\left(-1\right)^\kappa}{\left(3-\beta\right)_{2\kappa}\kappa!}\left(\gamma\left(t\right)\right)^{2\kappa}$$

$$+\Gamma\left(4-\beta\right)\sqrt{\pi t}\left(\gamma\left(t\right)\right)^{3-\beta}\sum_{\kappa=0}^{\infty}\frac{2^{-2\kappa}\left(2\kappa\right)!\left(\kappa+1\right)\left(-1\right)^\kappa}{\left(4-\beta\right)_{2\kappa}\kappa!}\left(\gamma\left(t\right)\right)^{2\kappa}-\frac{1}{\Gamma\left(3-\beta\right)}\left(\gamma\left(t\right)\right)^{2-\beta}.$$

Considering formula II.2 [22], we have

$$\Gamma\left(4-\beta\right)=\frac{\left(3-\beta\right)_{2\kappa}\frac{\Gamma\left(4-\beta+2\kappa\right)}{\Gamma\left(3-\beta+2\kappa\right)}}{\frac{\Gamma\left(4-\beta\right)}{\Gamma\left(3-\beta\right)}}=\frac{\left(3-\beta\right)_{2\kappa}\left(3-\beta+2\kappa\right)}{3-\beta}.$$

Then last equality can be rewritten in the form

$$\int\limits_0^t K_\beta\left(t,\tau\right)d\tau=\sum_{\kappa=0}^{\infty}\frac{2\sqrt{t}\left(2\kappa\right)!\left(2\kappa+1\right)\left(-1\right)^\kappa}{\sqrt{\pi}\Gamma\left(2-\beta\right)4^\kappa\left(3-\beta\right)_{2\kappa}\kappa!}\left(\gamma\left(t\right)\right)^{2\kappa+1-\beta}$$

$$+\sum_{\kappa=0}^{\infty}\frac{\left(2\kappa\right)!\left(\kappa+1\right)\left(-1\right)^\kappa}{\sqrt{\pi}\left(2-\beta\right)\Gamma\left(2-\beta\right)\sqrt{t}4^\kappa\left(3-\beta+2\kappa\right)\left(3-\beta\right)_{2\kappa}\kappa!}\left(\gamma\left(t\right)\right)^{2\kappa+3-\beta}$$

$$-\frac{1}{\Gamma\left(3-\beta\right)}\left(\gamma\left(t\right)\right)^{2-\beta}=\sum_{\kappa=0}^{\infty}\frac{\left(2\kappa\right)!\left(-1\right)^\kappa}{\sqrt{\pi}\Gamma\left(2-\beta\right)4^\kappa\left(3-\beta\right)_{2\kappa}\kappa!}\left(\gamma\left(t\right)\right)^{2\kappa+1-\beta}$$

$$\times\left(2\left(2\kappa+1\right)\sqrt{t}+\frac{\kappa+1}{\left(2-\beta\right)\left(3-\beta+2\kappa\right)\sqrt{t}}\gamma^2\left(t\right)\right)-\frac{1}{\Gamma\left(3-\beta\right)}\left(\gamma\left(t\right)\right)^{2-\beta}.$$

So,

$$\int\limits_0^t K_\beta\left(t,\tau\right)d\tau=\frac{1}{\sqrt{\pi}\Gamma\left(2-\beta\right)}\sum_{\kappa=0}^{\infty}\frac{\left(2\kappa\right)!\left(-1\right)^\kappa}{4^\kappa\kappa!\left(3-\beta\right)_{2\kappa}}\frac{\left(\gamma\left(t\right)\right)^{2\kappa+1-\beta}}{\sqrt{t}}$$

$$\times\left(2\left(2\kappa+1\right)t+\frac{\kappa+1}{\left(2-\beta\right)\left(3-\beta+2\kappa\right)}\gamma^2\left(t\right)\right)-\frac{1}{\Gamma\left(3-\beta\right)}\left(\gamma\left(t\right)\right)^{2-\beta}. \tag{28}$$

We will take $\gamma(t) \sim t^{\omega}$ when $t \to 0$. From equality (28) we have

$$\int_0^t K_\beta(t,\tau)\,d\tau = \frac{1}{\sqrt{\pi}\Gamma(2-\beta)} \sum_{\kappa=0}^{\infty} \frac{(2\kappa)!\,(-1)^{\kappa}}{4^{\kappa}\kappa!\,(3-\beta)_{2\kappa}} t^{\omega(2\kappa+1-\beta)-\frac{1}{2}}$$

$$\times \left(2(2\kappa+1)\,t + \frac{\kappa+1}{(2-\beta)(3-\beta+2\kappa)} t^{2\omega}\right) - \frac{t^{\omega(2-\beta)}}{\Gamma(3-\beta)}. \quad (29)$$

For the convergence of $\lim\limits_{t\to 0}\int_0^t K_\beta(t,\tau)\,d\tau$ the following conditions must be met:

$$\begin{cases} \omega(2\kappa+1-\beta) - \frac{1}{2} > 0, \quad \kappa = 0,1,2,\ldots \\ 2\omega > 0, \\ \omega(2-\beta) > 0. \end{cases}$$

Since $0 < \beta < 1$, then the last system of inequalities will be equivalent to the inequality: $\omega > \frac{1}{2(1-\beta)}$. So, $\lim\limits_{t\to 0}\int\limits_0^t K_\beta(t,\tau)\,d\tau = 0$, if

$$\gamma(t) \sim t^{\omega} \quad t \to 0, \quad \omega > \frac{1}{2(1-\beta)}. \quad (30)$$

When $\omega = \dfrac{1}{2}$ (29) takes the form

$$\int_0^t K_\beta(t,\tau)\,d\tau = \frac{1}{\sqrt{\pi}\Gamma(2-\beta)} \sum_{\kappa=0}^{\infty} \frac{(2\kappa)!\,(-1)^{\kappa}}{4^{\kappa}\kappa!\,(3-\beta)_{2\kappa}} t^{\frac{2(\kappa+1)-\beta}{2}}$$

$$\times \left(2(2\kappa+1) + \frac{\kappa+1}{(2-\beta)(3-\beta+2\kappa)}\right) - \frac{t^{\frac{2-\beta}{2}}}{\Gamma(3-\beta)}.$$

Hence it is obvious that

$$\lim_{t\to 0}\int_0^t K_\beta(t,\tau)\,d\tau = 0, \quad 0 < \beta < 1, \quad \gamma(t) \sim \sqrt{t} \quad t \to 0. \quad (31)$$

Let $\omega = \dfrac{1}{2(1-\beta)}$. Then (29) can be rewritten in the form

$$\int_0^t K_\beta(t,\tau)\,d\tau = \frac{1}{\sqrt{\pi}\Gamma(2-\beta)} \sum_{\kappa=0}^{\infty} \frac{(2\kappa)!\,(-1)^{\kappa}}{4^{\kappa}\kappa!\,(3-\beta)_{2\kappa}} t^{\frac{\kappa}{1-\beta}}$$

$$\times \left(2(2\kappa+1)\,t + \frac{\kappa+1}{(2-\beta)(3-\beta+2\kappa)} t^{\frac{1}{1-\beta}}\right) - \frac{1}{\Gamma(3-\beta)} t^{\frac{2-\beta}{2(1-\beta)}}. \quad (32)$$

Hence we have

$$\lim_{t\to 0}\int_0^t K_\beta(t,\tau)\,d\tau = 0, \quad \omega = \frac{1}{2(1-\beta)}, \quad \gamma(t) \sim t^{\omega}, \quad t \to 0.$$

Thus, from (30), (31) and (32) we obtain when $0 < \beta < 1$, $\omega \geq \frac{1}{2(1-\beta)}$ or $\omega = \frac{1}{2}$; $\lim\limits_{t\to 0}\int\limits_0^t K_\beta(t,\tau)\,d\tau = 0$, if $\gamma(t) \sim t^{\omega}$. Consider the limiting cases β.

I. $\beta = 0$. From formula 1.5.1(8) [21] we have $K_0(t,\tau) = erf\left(\frac{\gamma(t)}{2\sqrt{t-\tau}}\right)$;

$$\int_0^t K_0(t,\tau)\,d\tau = \int_0^t erf\left(\frac{\gamma(t)}{2\sqrt{t-\tau}}\right)d\tau = \left\|z = \frac{\gamma(t)}{2\sqrt{t-\tau}}, t-\tau = \frac{\gamma^2(t)}{4z^2}, d\tau = \frac{\gamma^2(t)}{4z^3}dz\right\|$$

$$= \frac{\gamma^2(t)}{4}\int_{\frac{\gamma(t)}{2\sqrt{t}}}^{\infty} z^{-3} erf(z)\,dz = \frac{1}{2}erf\left(\frac{\gamma(t)}{2\sqrt{t}}\right) + \frac{\gamma(t)\sqrt{t}}{2\sqrt{\pi}}\exp\left(-\frac{\gamma^2(t)}{4t}\right) - \frac{1}{4}\gamma^2(t)\,erfc\left(\frac{\gamma(t)}{2\sqrt{t}}\right).$$

So,

$$\int_0^t K_0(t,\tau)\,d\tau = \frac{1}{2}erf\left(\frac{\gamma(t)}{2\sqrt{t}}\right) + \frac{\gamma(t)\sqrt{t}}{2\sqrt{\pi}}\exp\left(-\frac{\gamma^2(t)}{4t}\right) - \frac{1}{4}\gamma^2(t)\,erfc\left(\frac{\gamma(t)}{2\sqrt{t}}\right). \tag{33}$$

Let $\gamma(t) \sim t^\omega$ when $t \to 0$. Then from (33) we have

$$\lim_{t\to 0}\int_0^t K_0(t,\tau)\,d\tau = \lim_{t\to 0}\left\{\frac{1}{2}erf\left(\frac{1}{2}t^{\omega-\frac{1}{2}}\right)\right.$$
$$\left. + \frac{1}{2\sqrt{\pi}}t^{\omega+\frac{1}{2}}\exp\left(-\frac{1}{4}t^{2\omega-1}\right) - \frac{1}{4}t^{2\omega}erfc\left(\frac{1}{2}t^{\omega-\frac{1}{2}}\right)\right\}. \tag{34}$$

Since

$$\lim_{t\to 0} erf\left(\frac{1}{2}t^{\omega-\frac{1}{2}}\right) = \begin{cases} 1; & \text{if} \quad \omega < \frac{1}{2}, \\ erf\left(\frac{1}{2}\right); & \text{if} \quad \omega = \frac{1}{2}, \\ 0; & \text{if} \quad \omega > \frac{1}{2}. \end{cases}$$

Then

$$\lim_{t\to 0} t^{\omega+\frac{1}{2}}\exp\left(-\frac{1}{4}t^{2\omega-1}\right) = \lim_{t\to 0}\frac{t^{\omega+\frac{1}{2}}}{\exp\left(\frac{1}{4}t^{2\omega-1}\right)} = 0 \quad \forall\omega.$$

By virtue of decomposition $erfz$ for any small z [23] when $\omega > 0$ $\lim_{t\to 0} t^{2\omega} erfc\left(\frac{1}{2}t^{\omega-\frac{1}{2}}\right) = 0$, because $\lim_{t\to 0} t^{2\omega} = 0$ and $erfc\left(\frac{1}{2}t^{\omega-\frac{1}{2}}\right)$ is the bounded function. Then from (34), taking into account the limit relations, we obtain: at $\omega > \frac{1}{2}$, if $\gamma(t) \sim t^\omega$ when $t \to 0$, then $\lim_{t\to 0}\int_0^t K_0(t,\tau)\,d\tau = 0$.

For $\omega = \frac{1}{2}$ from (34) we have

$$\lim_{t\to 0}\int_0^t K_0(t,\tau)\,d\tau = \lim_{t\to 0}\left\{\frac{1}{2}erf\frac{1}{2} + \frac{1}{2\sqrt{\pi}e^{\frac{1}{4}}}t - \frac{1}{4}t\ erfc\left(\frac{1}{2}\right)\right\} = \frac{1}{2}erf\frac{1}{2}.$$

So at $\omega > \frac{1}{2}$, if $\gamma(t) \sim t^\omega$ when $t \to 0$, then $\lim_{t\to 0}\int_0^t K_0(t,\tau)\,d\tau = 0$

II. $\beta = 1$. $K_1(t,\tau) = \frac{1}{\sqrt{\pi(t-\tau)}}\exp\left(-\frac{\gamma^2(t)}{4(t-\tau)}\right)$. Then, taking into account formula 3.461(1) from [19], we get

$$\int_0^t K_1(t,\tau)\,d\tau = \int_0^t \frac{1}{\sqrt{\pi(t-\tau)}}\exp\left(-\frac{\gamma^2(t)}{4(t-\tau)}\right)d\tau$$

$$= \left\| z = \frac{\gamma(t)}{2\sqrt{t-\tau}}, \sqrt{t-\tau} = \frac{\gamma(t)}{2z}, d\tau = \frac{\gamma^2(t)}{4z^3} dz \right\|$$

$$= \frac{\gamma(t)}{2\sqrt{\pi}} \int\limits_{\frac{\gamma(t)}{2\sqrt{t}}}^{+\infty} \frac{1}{z^2} e^{-z^2} dz = \frac{\gamma(t)}{2\sqrt{\pi}} \left[-\sqrt{\pi} erfc\left(\frac{\gamma(t)}{2\sqrt{t}}\right) + \frac{2\sqrt{t}}{\gamma(t)} \exp\left(-\frac{\gamma^2(t)}{4t}\right) \right]$$

$$= \frac{\sqrt{t}}{\sqrt{\pi}} \exp\left(-\frac{\gamma^2(t)}{4t}\right) - \frac{1}{2}\gamma(t)\, erfc\left(\frac{\gamma(t)}{2\sqrt{t}}\right).$$

Then if $\gamma(t) \sim t^{\omega}$ when $t \to 0$, at $\omega > 0$ we obtain

$$\lim_{t\to 0} \int\limits_0^t K_1(t,\tau)\, d\tau = \lim_{t\to 0} \left\{ \frac{\sqrt{t}}{\sqrt{\pi}} \exp\left(-\frac{1}{4} t^{2\omega-1}\right) - \frac{1}{2} t^{\omega}\ erfc\left(\frac{1}{2} t^{\omega-\frac{1}{2}}\right) \right\} = 0.$$

Summarizing, we obtain if $\gamma(t) \sim t^{\omega}$ when $t \to 0$ at $\omega > 0$: $\lim\limits_{t\to 0} \int\limits_0^t K_{\beta}(t,\tau)\, d\tau = 0$ at $\omega \geq \frac{1}{2}$ and $0 < \beta \leq 1$ or at $\omega > \frac{1}{2}$ and $\beta = 0$.

6. MAIN RESULT

Theorem. *Integral equation (9) with kernel of the form (10) at* $\omega \geq \dfrac{1}{2}$ *and* $0 < \beta \leq 1$ *or at* $\omega > \dfrac{1}{2}$ *and* $\beta = 0$ *and* $\gamma(t) \sim t^{\omega}$ *when* $t \to 0$ *uniquely solvable in the class of continuous functions for any continuous right-hand side.*

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