https://doi.org/10.31489/2025M4/85-94 Research article

On a stable difference scheme for numerically solving a reverse parabolic source identification problem

C. Ashyralyyev[1,2,3,*], M.A. Sadybekov[2,4]

[1]Department of Mathematics, Faculty of Engineering and Natural Sciences, Bahcesehir University, Istanbul, Turkey;
[2]Khoja Akhmet Yassawi International Kazakh-Turkish University, Turkistan, Kazakhstan;
[3]National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan;
[4]Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
(E-mail: charyar@gmail.com, sadybekov@math.kz)

This article is devoted to the study of source identification problems for reverse parabolic partial differential equations with nonlocal boundary conditions. The principal aim of the work is to construct and analyze stable difference schemes that can be effectively employed for obtaining approximate solutions of such inverse problems. In particular, attention is focused on the Rothe difference scheme, and stability estimates for the corresponding discrete solutions are rigorously derived. These estimates guarantee the reliability and convergence of the proposed numerical method. A stability theorem for the solution of the difference scheme related to the source identification problem is proved. To establish the well-posedness of the underlying differential problem, the operator-theoretic approach is employed, ensuring a solid analytical foundation for the numerical method. Furthermore, the investigation is extended to an abstract setting for difference schemes, which is then applied to the numerical solution of reverse parabolic equations under boundary conditions of the first kind. This unified framework emphasizes both the theoretical justification and the computational effectiveness of the proposed approach. Finally, the efficiency of the developed method is demonstrated through a numerical illustration with a test example.

Keywords: reverse parabolic equation, inverse problem, difference scheme (DS), partial differential equation (PDE), source identification problem (SIP), self-adjoint positive definite operator (SAPDO), stability estimate, well-posedness.

2020 Mathematics Subject Classification: 34B10, 35K10, 49K40.

Introduction

In recent decades, the importance of SIPs in the mathematical modeling of real-world processes has grown significantly (see [1, 2]). Comprehensive reviews, detailed references, and classifications of recent studies devoted to SIPs for parabolic PDEs can be found in [3–5]. The solvability of various inverse problems for parabolic equations was investigated in [6–8], while the well-posedness of SIPs for hyperbolic–parabolic equations was analyzed in [9]. The work [10] focused on the identification of a space-dependent source term in the heat equation. A numerical algorithm for solving certain SIPs for parabolic equations backward in time was proposed in [11]. The authors of [12] examined the backward-in-time problem for a semilinear system of parabolic equations, whereas [13] developed a regularization technique for the spherically symmetric backward heat conduction problem. Moreover, a numerical approach for the backward heat conduction problem was introduced in [14]. In addition, several stable difference schemes for various direct nonlocal problems associated with reverse parabolic equations have been developed by different researchers (see, for instance, [15, 16] and the references therein).

*Corresponding author. *E-mail: charyar@gmail.com*

This work was funded by the grant no. AP19676663 of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan.

Received: 14 July 2025; *Accepted:* 16 September 2025.

We denote by $\mathtt{H}$, a Hilbert space and by $\mathtt{A}:\mathtt{H}\to\mathtt{H}$, a SAPDO such that $\mathtt{A}>\delta\mathtt{I}$ for a real number $\delta>0$, and $\mathtt{I}$ identity operator. Let $\gamma_k,\ \mu_k,\ k=1,...,s$ be given real numbers so that

$$|\mu_1|+...+|\mu_s|\ <1,\quad 0\le\gamma_1<\gamma_2<...<\gamma_s<1 \tag{1}$$

hold.

We study SIP to search for a pair (v,p) that satisfies reverse parabolic equation

$$\frac{dv}{dt}(t)-\mathtt{A}v(t)=p+g(t),\quad 0<t<1 \tag{2}$$

and the following initial condition

$$v(0)=\phi \tag{3}$$

with a nonlocal condition

$$v(1)=\sum_{k\,=\,1}^{s}\mu_k v(\gamma_k)+\varphi \tag{4}$$

for a given smooth function $g:[0,1]\to\mathtt{H}$ and elements $\varphi,\,\phi\in\mathtt{H}$.

The well-posedness of the SIP (2)–(4) was established in the paper [17]. The aim of the current study is a stable DS for approximate solution of the SIP (2)–(4), under the assumption (1). Namely, we study the Rothe DS for approximate solution of this SIP and establish stability estimates for its solutions. Subsequently, this approach is employed to obtain stability estimates for the approximate solution of the SIP for a parabolic PDE. A numerical illustration of the test example is carried out.

1 Rothe DS

Denote by $[0,1]_\tau=\{t_k=k\tau,k=0,1,\cdots,N,\ N\tau=1\}$, the set of uniform grid points for any natural number N.

Let $C([0,1]_\tau,\mathtt{H})$ denote a linear space of grid functions $\vartheta^\tau=\{\vartheta_k\}_1^N$ taking values in the space $\mathtt{H}$, and let $C_\tau(\mathtt{H})=C([0,1]_\tau,\mathtt{H})$, $C_\tau^\alpha(\mathtt{H})=C^\alpha([0,1]_\tau,\mathtt{H})$ be the corresponding Banach space of grid functions equipped with the appropriate norms

$$\|\vartheta^\tau\|_{C_\tau(\mathtt{H})}=\max_{1\le k\le N}\|\vartheta_k\|_{\mathtt{H}},\ \|\vartheta^\tau\|_{C_\tau^\alpha(\mathtt{H})}=\|\vartheta^\tau\|_{C_\tau(\mathtt{H})}+\max_{1\le k<k+r\le N}(r\tau)^{-\alpha}\|\vartheta_{k+r}-\vartheta_k\|_{\mathtt{H}},$$

where $\alpha\in(0,1)$ is a given number.

Let us denote by $\mathtt{R}=(\mathtt{I}\ +\ \tau\ \mathtt{A})^{-1}$ the resolvent of $\mathtt{A}$. Then (see [18]) the estimates

$$\|\mathtt{R}^k\|_{\mathtt{H}\to\mathtt{H}}\le(1+\delta\tau)^{-k},\ \|\tau\mathtt{R}^k\|_{\mathtt{H}\to\mathtt{H}}\le k^{-1},\quad k\ge1 \tag{5}$$

are valid. Let us $l_i=\left[\frac{\gamma_i}{\tau}\right]$, $\rho_i=\frac{\gamma_i}{\tau}-l_i$, $i=1,...,s$.

Lemma 1. The operator

$$\mathbb{S}_\tau=\mathtt{I}-\left(1-\sum_{i\,=\,1}^{s}\mu_i\right)\mathtt{R}^N-\sum_{i\,=\,1}^{s}\mu_i\mathtt{R}^{N-l_i}$$

has an inverse $\mathbb{T}_\tau=\mathbf{S}_\tau^{-1}$ and it is bounded such that:

$$\|\ \mathbb{T}_\tau\ \|_{\mathtt{H}\to\mathtt{H}}\le M. \tag{6}$$

Proof. Since operator $\left(\mathtt{I}-\mathtt{R}^N\right)$ and its inverse are bounded, operator $\mathbb{S}_\tau$ can be rewritten in the form

$$\mathbb{S}_\tau=\left(\mathtt{I}-\mathtt{R}^N\right)\left(1+\sum_{i=1}^{s}\mu_i\left(\mathtt{I}-\mathtt{R}^N\right)^{-1}\left(\mathtt{R}^N-\mathtt{R}^{N-l_i}\right)\right)=\left(\mathtt{I}-\mathtt{R}^N\right)\mathbb{Q}_\tau.$$

Hence, to complete the proof it is sufficient to prove that the operator $\mathbb{Q}_\tau$ is invertible and $\mathbf{Q}_\tau^{-1}$ is bounded. Spectral resolution of a SAPD operator (see [19]) and the assumption (1) give us

$$\|\,\mathbb{Q}_\tau^{-1}\,\|_{\mathtt{H}\to\mathtt{H}}\leq\sup_{\delta<\lambda<\infty}\frac{1}{\left|1+\sum\limits_{i=1}^{s}\mu_i\left(1-(1+\tau\lambda)^{-N}\right)^{-1}\left((1+\tau\lambda)^{-N}-(1+\tau\lambda)^{-(N-l_i)}\right)\right|}\leq\frac{1}{1-\sum\limits_{i=1}^{s}|\mu_i|}\leq M_1.$$

Therefore, the proof of Lemma 1 is complete. □

1.1 Stable DS

Now, we the consider the Rothe DS

$$\begin{cases}\tau^{-1}(\vartheta_k-\vartheta_{k-1})+\mathtt{A}\vartheta_{k-1}=g_k+p,\ g_k=g(t_k),\ 1\leq k\leq N,\\[2mm] \vartheta_N-\sum\limits_{i=1}^{s}\mu_i\,\vartheta_{l_i}=\varphi,\ \vartheta_0=\phi,\end{cases}\tag{7}$$

of approximate solution of the problem (2)-(3).

We now derive the solution of problem (7). One can see that a unique solution of the difference problem

$$\begin{cases}\tau^{-1}(\vartheta_k-\vartheta_{k-1})+\mathtt{A}\vartheta_{k-1}=g_k+p,\ 1\leq k\leq N,\\[2mm] \quad\vartheta_N \text{ is given}\end{cases}$$

exists and the formula

$$v_k=\mathtt{R}^{N-k}v_N+\tau\sum_{j=k+1}^{N}\mathtt{R}^{j-k}\left(p+g_j\right),\quad 0\leq k\leq N-1\tag{8}$$

holds. Applying formula (8) and the corresponding conditions, we get

$$\mathtt{R}^N\vartheta_N+\sum_{j=1}^{N}\mathtt{R}^j p\tau=\phi-\sum_{j=1}^{N}\mathtt{R}^j g_j\tau,$$

and

$$\left(\mathtt{I}-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right)\vartheta_N\ -\sum_{i=1}^{s}\mu_i\sum\nolimits_{j=l_i+1}^{N}\mathtt{R}^{j-l_i}p\tau=\sum_{i=1}^{s}\mu_i\sum\nolimits_{j=l_i+1}^{N}\mathtt{R}^{j-l_i}g_j\tau\ +\varphi.$$

Since

$$\sum_{j=1}^{N}\mathtt{R}^j\tau=\mathtt{A}^{-1}\ (\mathtt{I}-\mathtt{R}^N),\ \sum_{j=l_i+1}^{N}\mathtt{R}^{j-l_i}\tau=\mathtt{A}^{-1}\ (\mathbb{I}-\mathtt{R}^{N-l_i}),$$

we have that

$$\mathtt{R}^N\vartheta_N+\ (\mathtt{I}-\mathtt{R}^N)\mathtt{A}^{-1}p=\phi-\sum_{j=1}^{N}\mathtt{R}^j g_j\tau\tag{9}$$

and

$$\left(\mathtt{I}-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right)\vartheta_N - \sum_{i=1}^{s}\mu_i(\mathtt{I}-\mathtt{R}^{N-l_i})\mathtt{A}^{-1}p = \sum_{i=1}^{s}\mu_i\sum_{j=l_i+1}^{N}\mathtt{R}^{j-l_i}g_j\tau + \varphi. \tag{10}$$

The determinant operator Δ for the system of equations (9) and (10) is defined by

$$\Delta = -\mathtt{R}^N\sum_{i=1}^{s}\mu_i(\mathtt{I}-\mathtt{R}^{N-l_i}) - (\mathtt{I}-\mathtt{R}^N)\left(\mathtt{I}-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right)$$

$$= \mathtt{R}^N\sum_{i=1}^{s}\mu_i - \mathtt{I}+\mathtt{R}^N+\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i} = -\left[\mathtt{I}-\left(1-\sum_{i=1}^{s}\mu_i\right)\mathtt{R}^N-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right].$$

Using Lemma 1, we get

$$\vartheta_N = T_\tau\left\{\left(\phi-\sum_{j=1}^{N}\mathtt{R}^j g_j\tau\right)(\mathtt{I}-\mathtt{R}^N)\right.$$

$$\left. -\left(\sum_{i=1}^{s}\mu_i\sum_{j=l_i+1}^{N}\mathtt{R}^{j-l_i}g_j\tau + \varphi\right)\left(\mathtt{I}-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right)\right\} \tag{11}$$

and

$$\mathtt{A}^{-1}p = T_\tau\left\{\left(\sum_{i=1}^{s}\mu_i\sum_{j=l_i+1}^{N}\mathtt{R}^{j-l_i}g_j\tau + \varphi\right)\mathtt{R}^N - \left(\phi-\sum_{j=1}^{N}\mathtt{R}^j g_j\tau\right)\left(\mathtt{I}-\sum_{i=1}^{s}\mu_i\mathtt{R}^{N-l_i}\right)\right\}. \tag{12}$$

Therefore, DS (7) is uniquely solvable and defined by the formulas (8), (11) and (12).

Theorem 1. For the solution $(\{\vartheta_k\}_{k=1}^{N}, p)$ of problem (7) in $C_\tau(\mathtt{H})\times\mathtt{H}$, the following stability estimates

$$\| p \|_{\mathtt{H}} \leq \mathbb{M}_\delta\left(\| \mathtt{A}\phi \|_{\mathtt{H}} + \| \mathtt{A}\varphi \|_{\mathbb{H}} + \alpha^{-1}\| \{g_k\}_{k=1}^{N} \|_{C_\tau^\alpha(\mathtt{H})}\right), \tag{13}$$

$$\| \{\vartheta_k\}_{k=1}^{N} \|_{C_\tau(\mathtt{H})} \leq \mathbb{M}_\delta\left(\| \phi \|_{\mathtt{H}} + \| \varphi \|_{\mathtt{H}} + \| \{g_k\}_{k=1}^{N} \|_{C_\tau(\mathtt{H})}\right) \tag{14}$$

hold, where the value of $\mathbb{M}_\delta$ does not depend on $\tau, \alpha, \phi, \varphi$, and $\{g_k\}_{k=1}^{N}$.

Proof. From (12) it follows that

$$p = T_\tau\left\{\mathtt{A}\varphi - \mathbb{A}\mathbb{R}^N\phi - \tau\sum_{j=1}^{N}\mathbb{A}\mathbb{R}^{N-j+1}(g_j-g_N) - (\mathtt{I}-\mathtt{R}^N-\sum_{i=1}^{s}\mu_i(\mathtt{I}-\mathtt{R}^{l_i}))g_N\right.$$

$$\left. -\tau\sum_{i=1}^{s}\mu_i\left(\sum_{j=1}^{l_i}\mathbb{A}\,\mathbb{R}^{l_i-j+1}(g_j-g_N)\right)\right\}.$$

Applying to the right side of the last formula the Cauchy–Shwarz and triangle inequalities and estimates (5), (6), one can obtain estimate (13):

$$\|p\|_{\mathtt{H}} \leq \| T_\tau \|_{\mathtt{H}\to\mathtt{H}}\left(\|\mathtt{A}\varphi\|_{\mathtt{H}} + \left\|\mathbb{A}\mathbb{R}^N\right\|_{\mathtt{H}\to\mathtt{H}}\|\phi\|_{\mathtt{H}} + \sum_{j=1}^{N-1}\left\|\mathbb{A}\mathbb{R}^{N-j+1}\right\|_{\mathbb{H}\to\mathtt{H}}\|g_j-g_N\|_{\mathtt{H}}\tau\right.$$

$$\left. + \left(1+\left\|\mathtt{R}^N\right\|_{\mathtt{H}\to\mathtt{H}} + \sum_{i=1}^{s}|\mu_i|\left(1+\left\|\mathtt{R}^{l_i}\right\|_{\mathtt{H}\to\mathtt{H}}\right)\right)\|g_N\|_{\mathbb{H}}\right)$$

$$\leq \mathbb{M}_\delta\left(\|\phi\|_{\mathtt{H}} + \| \mathtt{A}\varphi \|_{\mathtt{H}} + \alpha^{-1}\| \{g_k\}_{k=1}^{N} \|_{C_\tau^\alpha(\mathtt{H})}\right).$$

Using relation (8), the triangle inequality and the estimates (5), (6), we show that

$$\|\vartheta_k\|_{\mathtt{H}} \leq \left\| \mathtt{R}^{\,k}\right\|_{\mathtt{H}\to\mathtt{H}} \|\phi\|_{\mathtt{H}} + \tau\sum_{j=1}^{k}\left\| \mathtt{R}^{\,k-j+1}\right\|_{\mathtt{H}\to\mathtt{H}} \|g_j\|_{\mathtt{H}}$$

$$+ \left(1+\left\| \mathtt{R}^{\,k}\right\|_{\mathtt{H}\to\mathtt{H}}\right)\|T_\tau\|_{\mathtt{H}\to\mathtt{H}}\Big\{\|\varphi\|_{\mathtt{H}}+\left\|\mathtt{R}^{N}\right\|_{\mathtt{H}\to\mathtt{H}}\|\phi\|_{\mathtt{H}}+\tau\sum_{j=1}^{N}\left\|\mathtt{R}^{N-j+1}\right\|_{\mathtt{H}\to\mathtt{H}}\|g_j\|_{\mathtt{H}}$$

$$+\tau\sum_{i=1}^{s}|\mu_i|\sum_{j=1}^{l_i}\left\|\mathtt{R}^{l_i-j+1}\right\|_{\mathtt{H}\to\mathtt{H}}\|g_j\|_{\mathtt{H}}\Big\} \leq \mathbb{M}_\delta\left(\|\phi\|_{\mathtt{H}}+\|A\varphi\|_{\mathtt{H}}+\alpha^{-1}\left\|\{g_k\}_{k=1}^{N}\right\|_{C_\tau^\alpha(\mathtt{H})}\right)$$

for any index k. From that, the estimate (14) follows.

2 The boundary value problem and its approximation

Let $\Omega=(0,l)^n\subset\mathtt{R}^n$, $S=\partial\Omega,\overline{\Omega}=\Omega\cup S$ and (1) holds. Assume that $\phi\in\mathtt{L}_2(\Omega)$, $\varphi\in\mathtt{W}_2^2(\Omega)$ and $g\in C^\alpha(\mathtt{L}_2(\Omega))$, a_r are smooth functions such that $\forall x\in\Omega, a_r(x)\geq a_0>0,\ r=1,...,n$, σ is a given positive real number.

Let us consider in $[0,1]\times\overline{\Omega}$, SIP for a multi-dimensional reverse parabolic PDE with the Dirichlet-type boundary condition

$$\begin{cases}\vartheta_t(x,t)+\sum\limits_{i=1}^{n}(a_i(x)\vartheta_{x_i}(x,t))_{x_i}-\sigma\vartheta(x,t)=g(x,t)+p(x), \quad 0<t<1,\ x=(x_1,\cdots,x_n)\in\Omega,\\ \vartheta(x,0)=\phi(x),\ \vartheta(x,1)=\sum\limits_{k=1}^{s}\mu_k\vartheta(x,\gamma_k)+\varphi(x), \quad x\in\overline{\Omega},\\ \vartheta(x,t)=0, \quad 0\leq t\leq 1,\ x\in S.\end{cases} \tag{15}$$

The well-posedness of the SIP (15) was established in the paper [17].

Now, we will discretize SIP (15) in two steps. Let us take $h_rM_r=l,\ r=1,\cdots,n$. In the first step, we define the grid spaces $\widetilde{\Omega}_h=\{x=x_m=(h_1m_1,\cdots,h_nm_n);\ m=(m_1,\cdots,m_n), m_r=0,\cdots,M_r\}$, $\Omega_h=\widetilde{\Omega}_h\cap\Omega,\ S_h=\widetilde{\Omega}_h\cap S$ and the difference operator A_h^x by

$$\mathtt{A}_h^x\,\vartheta^h(x)=-\sum_{r=1}^{n}\left(a_r(x)\vartheta_{\overline{x}_r}^h(x)\right)_{x_r,j_r}+\sigma\vartheta^h(x)$$

whose domain consists of all grid functions $\vartheta^h(x)$ satisfying the homogeneous boundary conditions $\vartheta^h(x)=0$ for all $x\in S_h$.

By using $\mathtt{A}_h^x$, we arrive at some infinite system of ordinary differential equations. Then, in the second step of discretization, we obtain the first-order of ADS

$$\begin{cases}\tau^{-1}\left(\vartheta_k^h(x)-\vartheta_{k-1}^h(x)\right)-\mathtt{A}_h^x\vartheta_{k-1}^h(x)=f^h(t_k,x)+p^h(x),\ t_k=\tau k, \quad 1\leq k\leq N,\ x\in\widetilde{\Omega}_h,\\ \vartheta_0^h(x)=\phi^h(x),\vartheta_N^h(x)=\sum\limits_{i=1}^{s}\mu_i\ \vartheta_{l_i}^h(x)+\varphi^h(x), \quad x\in\widetilde{\Omega}_h,\ l_i=\left[\frac{s_i}{\tau}\right],\ i=1,...,s.\end{cases} \tag{16}$$

Let $\mathtt{L}_{2h}=\mathtt{L}_2(\widetilde{\Omega}_h)$ and $\mathtt{W}_{2h}^2=\mathtt{W}_2^2(\widetilde{\Omega}_h)$ be spaces of the grid functions $\vartheta^h(x)=\{\vartheta(h_1m_1,\cdots,h_nm_n)\}$

defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\left\|\vartheta^h\right\|_{\mathtt{L}_{2h}} = \left(\sum_{x\in\widetilde{\Omega}_h} |\vartheta^h(x)|^2 h_1\cdots h_n\right)^{1/2}, \quad \left\|\vartheta^h\right\|_{\mathtt{W}^2_{2h}} = \left\|\vartheta^h\right\|_{L_{2h}}$$

$$+\left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^n \left|(\vartheta^h)_{x_r}\right|^2 h_1\cdots h_n\right)^{1/2} + \left(\sum_{x\in\widetilde{\Omega}_h}\sum_{r=1}^n \left|(\vartheta^h(x))_{x_r\overline{x_r},\, m_r}\right|^2 h_1\cdots h_n\right)^{1/2}.$$

Denote by $\mathcal{C}_\tau(\mathtt{L}_{2h}) = \mathcal{C}([0,1]_\tau, \mathtt{L}_{2h})$, the Banach space of $\mathtt{L}_{2h}$-valued grid functions $\vartheta^\tau = \{\vartheta_k\}_1^N$ with norm

$$\|\vartheta^\tau\|_{\mathcal{C}_\tau(\mathtt{L}_{2h})} = \max_{1\le k\le N} \|\vartheta_k\|_{\mathtt{L}_{2h}}.$$

Let $\mathcal{C}^\alpha(\mathtt{L}_{2h}) = \mathcal{C}^\alpha([0,1]_\tau, \mathtt{L}_{2h})$ and $\mathcal{C}^\alpha_\tau(\mathbb{L}_{2h}) = \mathcal{C}^\alpha_\tau([0,1]_\tau, \mathtt{L}_{2h})$ be respectively the Hölder space and the weighted Hölder space with the norms defined by (1) for $\mathbb{H} = \mathtt{L}_{2h}$.

Theorem 2. Assume that τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ are sufficiently small positive numbers, $\phi^h \in \mathtt{L}_{2h}$, $\varphi^h \in D(\mathtt{A}_h^x)$, $\left\{g_k^h\right\}_1^N \in \mathcal{C}^\alpha_\tau(\mathtt{L}_{2h})$. Then, for the solutions of DS (16), the following stability estimates hold:

$$\left\|p^h\right\|_{\mathcal{C}_\tau(L_{2h})} \le \mathbb{M}_\delta\left(\left\|\phi^h\right\|_{L_{2h}} + \left\|\mathtt{A}_h^x\varphi^h\right\|_{L_{2h}} + \alpha^{-1}\left\|\{g_k^h\}_1^N\right\|_{\mathcal{C}^\alpha_\tau(L_{2h})}\right),$$

$$\left\|\{\vartheta_k^h\}_1^N\right\|_{\mathcal{C}_\tau(L_{2h})} \le \mathbb{M}_\delta\left(\left\|\phi^h\right\|_{L_{2h}} + \left\|\varphi^h\right\|_{L_{2h}} + \left\|\{g_k^h\}_1^N\right\|_{\mathcal{C}_\tau(L_{2h})}\right),$$

where M_δ is independent of τ, $\phi^h(x)$, $\varphi^h(x)$, and $g_k^h(x)$, $k = 1, \cdots, N-1$.

The proof of Theorem 2 is based on estimates (13), (14), the theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h} ([20]) and the triangle inequality.

3 *Numerical algorithm and example*

In $[0,\pi]\times[0,1]$, we consider a test example to search for a pair of functions $(p(x), \nu(x,t))$ for SIP of reverse parabolic equation so that

$$\begin{cases} \nu_t(x,t) + (1+3x)^2\,\nu_{xx}(x,t) + 6\,(1+3x)\,\nu_x(x,t) - \nu(x,t) = p(x) + g(x,t),\ 0<x<\pi,\ 0<t<1, \\ \nu(x,0) = \phi(x),\ \nu(x,1) = \sum\limits_{k=1}^{3}\mu_k\,\nu(x,s_k) + \varphi(x),\quad 0\le x\le\pi, \\ \nu(0,t) = 0,\ \nu(1,t) = 0,\ 0\le t\le 1. \end{cases} \tag{17}$$

Here $\zeta(x) = \sin(x)$, $\phi(x) = \zeta(x)$, $\mu_1 = \mu_2 = \mu_3 = \frac{1}{6}$, $s_1 = 0.3$, $s_2 = 0.5$, $s_3 = 0.7$, $g(x,t) = \left(\left(-4 - (1+3x)^2\right)\zeta(x) + 6\,(1+3x)\cos(x)\right)e^{-3t}$, $\varphi(x) = \left(1 - \frac{1}{6}\left(e^{-0.9} + e^{-1.5} + e^{-2.1}\right)\right)\zeta(x)$. The exact solution is $\left(\zeta(x), e^{-3t}\zeta(x)\right)$.

We use the algorithm to solve (17). It contains three steps. In the first step we search for solution of an auxiliary direct problem without source

$$\begin{cases} \omega_t(x,t) + (1+3x)^2\,\omega_{xx}(x,t) + 6\,(1+3x)\,\omega_x(x,t) - \omega(x,t) \\ = (1+3x)^2\,\phi_{xx}(x,t) + 6\,(1+3x)\,\phi_x(x,t) + g(x,t),\ 0<t<1,\ 0<x<\pi, \\ \omega(x,1) - \sum\limits_{k=1}^{3} \mu_k \omega(x,s_k) \ = \ \varphi(x), \ \ 0 \le x \le \pi, \\ \omega(0,t)=0,\ \omega(1,t)=0,\ 0\le t\le 1. \end{cases} \tag{18}$$

Later, in the second step, we find a source function using the formula

$$p(x) = (1+3x)^2\,\omega_{xx}(x,0) + 6\,(1+3x)\,\omega_x(x,0) - \omega(t,0).$$

Finally, in the third step we put in the right side of the reverse parabolic PDE and solve it to get the solution $\nu(x,t)$,

Applying (16), we have the following DS

$$\begin{cases} \tau^{-1}\left(\omega_k^n - \omega_{k-1}^n\right) + (1+3x_n)^2\,h^{-2}\left(\omega_{k-1}^{n+1} - 2\omega_{k-1}^n + \omega_{k-1}^{n-1}\right) \\ +6\,(1+3x_n)\,(2h)^{-1}\left(\omega_{k-1}^{n+1} - \omega_{k-1}^{n-1}\right) - \omega_{k-1}^n = (1+3x_n)^2 h^{-2}\left(\phi^{n+1} - 2\phi^n + \phi^{n-1}\right) \\ +6\,(1+3x_n)\,(2h)^{-1}\left(\phi^{n+1} - \phi^{n-1}\right) + g_k^n, t_k = k\,\tau,\ 1 \le k \le N, \ \ x_n = n\,h,\ 1 \le n \le M-1, \\ \omega_N^n - \frac{1}{6}\left(\omega_{l_1} + \omega_{l_2} + \omega_{l_3}\right) = \varphi^n,\ 0 \le n \le M, \\ \omega_k^0 = 0,\ \omega_k^M = 0,\ 0 \le k \le N \end{cases} \tag{19}$$

for approximate solution (18). The approximate value of p at grid points x_n is calculated using the formula

$$p_n = (1+3x_n)\ h^{-2}\left(\omega_0^{n+1} - 2\omega_0^n + \omega_0^{n-1}\right) + \ 6\,(1+3x_n)\,(2h)^{-1}\left(\omega_0^{n+1} - \omega_0^{n-1}\right) - \ \omega_0^n,$$
$$n = 1,...,M-1.$$

DS (19) can be rewritten in the matrix form

$$A_n \omega_{n+1} + B_n \omega_n + C_n\,\omega_{n-1} = I\theta_n,\ n = 1,\cdots,M-1,\ \omega_0 = \overrightarrow{0},\ \omega_M = \overrightarrow{0}. \tag{20}$$

Here, $\omega_n = \left[\ {\omega_0}^n \ \cdots \ \omega_N^0\ \right]^t$, $\omega^{n\pm1} = \left[\ {\omega_0}^{n\pm1} \ \cdots \ {\omega_N}^{n\pm1}\ \right]^t$, $\theta_n = \left[\ {\theta_0}^n \ \cdots \ \theta_N^{\ n}\ \right]^t$ are $(N+1)\times 1$ column vectors, A_n, B_n, C_n are $(N+1)^2$ square matrices, I is the $(N+1)^2$ identity matrix,

$$A_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & a_n I & & \vdots \\ & & & 0 \end{bmatrix}, \quad C_n = \begin{bmatrix} 0 & \dots & 0 & 0 \\ & & & 0 \\ & c_n I & & \vdots \\ & & & 0 \end{bmatrix},$$

$$B_n = \begin{bmatrix} 0 & 0 & 0 & \cdots & -\frac{1}{6} & \cdots & 0 & 0 & 1 \\ d & b_n & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & d & b_n & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \dots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \cdots & d & b_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & d & b_n \end{bmatrix},$$

$$a_n = (1+3x_n)^2 \ h^{-2} + \ 6\,(1+3x_n)\,(2h)^{-1}\,, \ b_n = -1 - \tfrac{1}{\tau} - 2\,(1+3x_n)^2 \ h^{-2},$$
$$c_n = (1+3x_n)^2 \ h^{-2} - 6\,(1+3x_n)\,(2h)^{-1}\,, \ d = -\tfrac{1}{\tau},$$

$$\theta_0^n = \psi_n, \ n = 1,\cdots, M-1,$$
$$\theta_k^n = g(t_k,x_n) + \ (1+3x_n)^2 h^{-2}\left(\phi^{n+1} \ -2\phi^n \ +\phi^{n-1}\ \right)$$
$$+6\,(1+3x_n)\,(2h)^{-1}\left(\phi^{n+1} \ -\phi^{n-1}\ \right), k=1, \ \cdots, N, \ n=1,\cdots, M-1.$$

We use the modified Gauss elimination method to solve (20).

Numerical results are carried out using MATLAB. The numerical solutions of DS are evaluated for distinct values of (N, M). ω_n^k represents the numerical value of $\nu(t,x)$ at $(t,x) = (t_k,x_n)$ and p_n is the numerical value of $p(x)$ at $x = x_n$. The errors in the numerical solutions are computed by

$$E\nu = \max_{0\le k\le N}\left(\sum_{n=1}^{M-1}\left|\nu(x_n,t_k) - \nu_k^n\right|^2 h\right)^{\frac{1}{2}},$$
$$Ep_M = \left(\sum_{n=1}^{M-1}\left|p(x_n) - p_n\right|^2 h\right)^{\frac{1}{2}}.$$

In Table 1 we give the error between the exact solution and the numerical solution of the difference scheme for distinct values of N and M. The table demonstrates that doubling the grid resolution results in approximately a twofold reduction in error.

Table 1

Error analysis

DS | $N = M$	20	40	80
$E\nu$	1.308×10^{-2}	6.723×10^{-3}	3.447×10^{-3}
Ep	1.432×10^{-2}	7.012×10^{-3}	3.465×10^{-3}

Conclusion

In this work we consider SIPs for reverse parabolic PDEs with initial and nonlocal boundary conditions. The main goal is to develop and analyze stable difference schemes, particularly the Rothe scheme, for accurate numerical solutions. Stability estimates are rigorously proved, ensuring reliability and convergence. The well-posedness of the problem is established, providing a strong analytical basis. The study also extends to an abstract setting of difference schemes and applies the results to reverse parabolic equations with first-kind boundary conditions. Numerical experiments confirm the effectiveness of the proposed method.

In future work, we plan to construct and analyze high-order accurate and stable difference schemes for the approximate solution of such SIPs.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

References

1 Kabanikhin, S.I. (2011). *Inverse and Ill-Posed Problems: Theory and Applications.* Berlin: Walter de Gruyter.

2 Samarskii, A.A., & Vabishchevich, D.P. (2007). *Numerical Methods for Solving Inverse Problems of Mathematical Physics.* Berlin: Walter de Gruyter.

3 Ashyralyev, A. (2011). On the problem of determining the parameter of a parabolic equation. *Ukrainian Mathematical Journal, 62*(9), 1397–1408. https://doi.org/10.1007/s11253-011-0438-7

4 Ashyralyyev, C., & Akkan, P. (2020). Source identification problem for a parabolic equation with multi-point nonlocal boundary condition. *Numerical Functional Analysis and Optimization, 41*(16), 1913–1935. https://doi.org/10.1080/01630563.2020.1802748

5 Liu, K., & Zhao, Y. (2024). Inverse moving source problems for parabolic equations. *Applied Mathematics Letters, 155*, 109114. https://doi.org/10.1016/j.aml.2024.109114

6 Kozhanov, A.I. (2017). Parabolic equations with unknown time-dependent coefficients. *Computational Mathematics and Mathematical Physics, 57*(6), 961–972. https://doi.org/10.1134/S0965542517060082

7 Pyatkov, S.G., & Samkov, M.L. (2017). Solvability of some inverse problems for the nonstationary heat-and-mass-transfer system. *Journal of Mathematical Analysis and Applications, 446*(2), 1449–1465. https://doi.org/10.1016/j.jmaa.2016.09.050

8 Kamynin, V.L. (2019). The inverse problem of simultaneous determination of the two lower space-dependent coefficients in a parabolic equation. *Mathematical Notes, 106*, 235–247. https://doi.org/10.1134/S0001434619070277

9 Ashyraliyev, M., & Ashyralyyeva, M. (2024). Stable difference schemes for hyperbolic-parabolic equations with unknown parameter. *Boletin de la Sociedad Matematica Mexicana, 30*(1), 14. https://doi.org/10.1007/s40590-023-00585-1

10 Zhao, Z.-X., Banda, M.K., & Guo, B.-Z. (2017). Simultaneous identification of diffusion coefficient, spacewise dependent source and initial value for one dimensional heat equation. *Mathematical Methods in the Applied Sciences, 40*, 3552–3565. https://doi.org/10.1002/mma.4245

11 Sazaklioglu, A.U. (2022). On the numerical solutions of some identification problems for one- and multidimensional parabolic equations backward in time. *Applied Numerical Mathematics, 181*, 76–93. https://doi.org/10.1016/j.apnum.2022.05.015

12 Chorfi, S.E., & Yamamoto, M. (2025). Stability of backward-in-time semilinear coupled parabolic systems. *Journal of Mathematical Analysis and Applications, 545*(2), Article 129240. https://doi.org/10.1016/j.jmaa.2025.129240

13 Cheng, W., & Liu, Y.L. (2023). A quasi-boundary value regularization method for the spherically symmetric backward heat conduction problem. *Open Mathematics, 21*(1), 20230171. https://doi.org/10.1515/math-2023-0171

14 Jiang, Y., & Liu, J. (2023). Fast parallel-in-time quasi-boundary value methods for backward heat conduction problems. *Applied Numerical Mathematics, 184*, 325–339. https://doi.org/10.1016/j.apnum.2022.10.006

15 Ashyralyev, A., Dural, A., & Sozen, Y. (2009). Well-posedness of the Rothe difference scheme for reverse parabolic equations. *Iranian Journal of Optimization, 1*, 1–25.

16 Ashyralyyev, C. (2020). Stability of Rothe difference scheme for the reverse parabolic problem with integral boundary condition. *Mathematical Methods in the Applied Sciences, 43*(8), 5369–5379. https://doi.org/10.1002/mma.6278

17 Ashyralyyev, C., & Sadybekov, M.A. (2025). Reverse parabolic source identification problem. *TWMS Journal of Pure and Applied Mathematics, 16*(2), 271–280. https://doi.org/10.30546/2219-1259.16.2.2025.271

18 Ashyralyev, A., & Sobolevskii, P.E. (1994). *Well-Posedness of Parabolic Difference Equations.* Operator Theory: Advances and Applications, 69. Basel: Birkhauser. https://doi.org/10.1007/978-3-0348-8518-8

19 Krein, S.G. (1971). *Linear Differential Equations in Banach Space.* Providence: American Mathematical Society.

20 Sobolevskii, P.E. (1975). *Raznostnye metody priblizhennogo resheniia differentsialnykh uravnenii [Difference methods for approximate solution of differential equations].* Voronezh: VGU [in Russian].

*Author Information**

Charyyar Ashyralyyev (*corresponding author*) — Doctor of Physical and Mathematical Sciences, Professor, Department of Mathematics, Faculty of Engineering and Natural Sciences, Bahcesehir University, Istanbul, Turkey; Khoja Akhmet Yassawi International Kazakh-Turkish University, Turkistan, Kazakhstan; National University of Uzbekistan named after Mirzo Ulugbek, Tashkent, Uzbekistan; e-mail: *charyar@gmail.com*; https://orcid.org/0000-0002-6976-2084

Makhmud Abdysametovich Sadybekov — Doctor of Physical and Mathematical Sciences, Professor, Corresponding Member of the Academy of Sciences of Kazakhstan, General Director, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan; e-mail: *sadybekov@math.kz*; https://orcid.org/0000-0001-8450-8191

*Authors' names are presented in the order: First name, Middle name, and Last name.