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Universal elements of unitriangular matrices groups

The following theorems are proved for a matrix g from the group of unitriangular matrices over a commutative and associative ring K of finite dimension of greater than three with unity: 1) if the matrix g is universal then all of its elements are on the first collateral diagonal except extreme ones are nonzero; 2) if all elements of the first collateral diagonal of the matrix g , with the possible exception of the last element are reversible in K , then g is universal; 3) if the ring K is Euclidean and has no reversible elements except trivial ones, then it follows from the universality of the matrix g that all the elements of its first collateral diagonal, except the extreme ones, are reversible in K .

Keywords: unitriangular matrix group, commutator, commutant, universal element, ring, euclidean ring, associative ring.

We denote the group of all upper unitriangular matrices over a commutative associative ring K with unity by $UT_n(K)$. Its commutant $UT'_n(K)$ consists of all matrices with the first zero collateral diagonal (see [1], for example).

In paper of A. Bier [2], it is proved that every element of commutant $UT'_n(\mathbb{F})$ is a commutator in the case of a field \mathbb{F} of characteristic zero, i.e. for each element f of $UT'_n(\mathbb{F})$ the equation of the form

$$[x_1, x_2] = f, \quad (1)$$

is always solvable in the group $UT_n(\mathbb{F})$, where $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$ is the commutator of variables x_1 and x_2 .

This result is significantly enhanced in the paper of N.S. Bahta [3], where it is proved that any element $f \in UT'_n(K)$ can be represented in the form $[g, x]$, where g is a fixed element of the group $UT_n(K)$. Any element having the first collateral diagonal consisting of units can be taken as a element g . In addition, in [4] similar results were obtained for the members of lower central row of the group $UT_n(K)$ (see also [5]). Paper [6] provides an overview of results on the solvability of equations in groups which mentions the results discussed.

The concept of universal element belongs to V. Roman'kov. In papers of A. Konyrkhanova [7, 8] some necessary and sufficient conditions of an universality of an element of groups $UT_n(\mathbb{F})$ and $UT_n(\mathbb{Z})$, were first obtained, where \mathbb{F} — is arbitrary field and \mathbb{Z} — is a ring of integers.

In this paper necessary and sufficient conditions for the element universality for unitriangular matrices group of arbitrary finite dimension over a commutative associative ring with unity and over Euclidean ring are obtained.

The element g of group G is called *universal*, if the equation

$$[g, x] = f, \quad (2)$$

is solvable for any element f from the commutant G' of group G .

If $n = 2$, then $G \cong UT_2(K) \cong K^+$, i.e. G is Abelian group, therefore $G' = E$, where E is identity matrix. Hence, every element of G is universal. Therefore, in the further groups $UT_n(K)$ of dimension $n \geq 3$ are considered.

Lemma 1. If φ is automorphism of the group G , then the element $g \in G$ is universal if and only if its image $\varphi(g)$ is universal.

Proof. Necessity. Let g be universal in G . Then equation (2) is solvable in G for any element f from the commutant G' . Hence, we obtain

$$[\varphi(g), \varphi(x)] = \varphi(f). \quad (3)$$

Since, the automorphism φ maps the commutant G' into G' , then the equality

$$\{\varphi(f) | f \in G'\} = G'$$

holds. From this and (3) it follows that the element $\varphi(g)$ is universal. The necessity is proved.

Sufficiency. Let $\varphi(g)$ be universal in G , where φ is automorphism of the group G . Then the equation

$$[\varphi(g), x] = f \tag{4}$$

is solvable in G for all $f \in G'$. Since φ^{-1} is also the automorphism of the group G' , then we have from (4):

$$[\varphi^{-1}(\varphi(g)), \varphi^{-1}(x)] = \varphi^{-1}(f);$$

i.e.

$$[g, \varphi^{-1}(x)] = \varphi^{-1}(f).$$

From this and in virtue of the equality $\{\varphi^{-1}(f) | f \in G'\} = G'$ we obtain that g is universal in G . The lemma is proved.

Let K be an associative commutative ring with unity. We introduce the following matrices from $UT_n(K)$:

$$g = \begin{pmatrix} 1 & g_{12} & g_{13} & \dots & g_{1,n-1} & g_{1n} \\ 0 & 1 & g_{23} & \dots & g_{2,n-1} & g_{2n} \\ 0 & 0 & 1 & \dots & g_{3,n-1} & g_{3n} \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 & g_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, g^* = \begin{pmatrix} 1 & 1 & g_{13}^* & \dots & g_{1,n-1}^* & g_{1n}^* \\ 0 & 1 & 1 & \dots & g_{2,n-1}^* & g_{2n}^* \\ 0 & 0 & 1 & \dots & g_{3,n-1}^* & g_{3n}^* \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 & \varepsilon \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \tag{5}$$

where $\varepsilon = 1$, if $g_{n-1,n} \neq 0$, and $\varepsilon = 0$, otherwise, and the elements $g_{i,i+1} \neq 0$ for $i = 1, 2, \dots, n - 2$, and all elements of the first collateral diagonal are equal to 1 in g^* with the possible exception of ε .

We denote by $diag_i a$ (where $i = 1, 2, \dots, n$) a diagonal matrix of the group of all triangular matrices $T_n(K)$, in which elements on the i -th place of the main diagonal equal $a \in K \setminus 0$, and the other elements of the main diagonal are units.

Theorem 1. Let matrix g of the group of all unitriangular matrices $UT_n(K)$ over a commutative associative ring K with unity such that all of its elements are on the first collateral diagonal with the possible exception of the last element are reversible in K . Then there exists a matrix g^* from $UT_n(K)$ of the form (5) conjugate with g in the group $T_n(K)$, and such that the following statement is true: matrix g is universal if and only if g^* is universal.

Proof. Let $g \in UT_n(K)$ is defined as in (5), and elements $g_{12}, g_{23}, \dots, g_{n-2,n-1}$ are reversible in ring K . Let us take a diagonal matrix $a_1 = diag_2 g_{12}^{-1}$. Then direct calculations yield the following equation:

$$a_1^{-1} g a_1 = \begin{pmatrix} 1 & 1 & g_{13} & g_{14} & \dots & g_{1,n-1} & g_{1n} \\ 0 & 1 & g_{12}g_{23} & g_{12}g_{24} & \dots & g_{12}g_{2,n-1} & g_{12}g_{2n} \\ 0 & 0 & 1 & g_{34} & \dots & g_{3,n-1} & g_{3n} \\ 0 & 0 & 0 & 1 & \dots & g_{4,n-1} & g_{4n} \\ \vdots & \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & 1 & g_{n-1,n} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \rightleftharpoons g_1. \tag{6}$$

Let $a_2 = diag_3 g_{23}^{-1} g_{12}^{-1}$. Then we have from (6):

$$a_2^{-1} g_1 a_2 = \begin{pmatrix} 1 & 1 & * & * & \dots & * \\ 0 & 1 & 1 & * & \dots & * \\ 0 & 0 & 1 & g_{12}g_{23}g_{34} & \dots & * \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \dots & g_{n-1,n} \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \rightleftharpoons g_2,$$

where $*$ represents some elements of the ring K . Thus, conjugating the matrix g with appropriate product of matrices of the form $a_1 \cdot a_2 \cdot \dots \cdot a_{i-1}$, $i = 1, 2, \dots, n - 1$, we obtain a matrix of the form g^* from (5).

Matrices a_{i-1} are triangular, i.e. $a_{i-1} \in T_n(K)$. Conjugates of elements $g \in UT_n(K)$ by the product of matrices $a_1 \cdot a_2 \cdot \dots \cdot a_{i-1}$ in $T_n(K)$ are automorphisms of the group $UT_n(K)$. From this and Lemma 1 follows that the matrix g is universal if and only if g^* is universal. The theorem is proved.

Theorem 2. The matrix $g \in UT_3(K)$, where K is a commutative associative ring with a unity, is universal if and only if g_{12} and g_{23} are coprime.

Proof. Necessity. Let g be universal. Then the equation (2) has a solution for any element $f \in UT'_n(K)$. Direct calculations yield the following equation

$$[g, x] = \begin{pmatrix} 1 & 0 & g_{12}x_{23} - x_{12}g_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since g is a universal element, the commutator $[g, x]_{13}$ must take any value. Hence the equation (2) is solvable, and therefore g_{12} and g_{23} are coprime.

Sufficiency. Let elements g_{12} and g_{23} of the matrix g are coprime. Let us prove that the equation (2) is solvable in $UT_3(K)$, i.e. for any $f \in UT'_3(K)$ the equation

$$g_{12}x_{23} - x_{12}g_{23} = f_{13} \quad (7)$$

is solvable.

Since g_{12} and g_{23} are coprime, then there exist elements $u, v \in K$, such that

$$g_{12}u + g_{23}v = 1.$$

It follows that the values $x_{23} = f_{13}u$ and $x_{12} = -f_{13}v$ are the solution of the equation (7). Hence, g is a universal element of the group $UT_3(K)$.

The theorem is proved.

Corollary 1. There exists an algorithm which determines its universality by any matrix $g \in UT_3(\mathbb{Z})$.

Hereinafter we assume that $n > 3$.

Theorem 3 (a necessary condition for the universality). Let K be associative commutative ring with a unity. Then from universality of the element $g \in UT_n(K)$, $n > 3$ follows that

$$g_{i-1,i} \neq 0, 2 < i < n.$$

Proof. Let the matrix g be universal in $UT_n(K)$. Then for any matrix $f \in UT'_n(K)$ the following system of equations has a solution in $UT_n(K)$:

$$\begin{cases} f_{13} = g_{12}x_{23} - x_{12}g_{23}; \\ f_{24} = g_{23}x_{34} - x_{23}g_{34}; \\ f_{35} = g_{34}x_{45} - x_{34}g_{45}; \\ \dots \\ f_{n-2,n} = g_{n-2,n-1}x_{n-1,n} - x_{n-2,n-1}g_{n-1,n}. \end{cases} \quad (8)$$

Assume the contrary, i.e. $g_{i-1,i} = 0$ for some $i \in \{3, 4, \dots, n-1\}$. Then we have from (8):

$$\begin{cases} f_{i-2,i} = x_{i-1,i}g_{i-2,i-1}; \\ f_{i-1,i+1} = -x_{i-1,i}g_{i,i+1}. \end{cases} \quad (9)$$

Let us prove that in this case $g_{i-2,i-1} \neq 0$. Indeed, otherwise, it follows from the first equation of the system (9) that for any matrix $f \in UT'_n(K)$ the equality $f_{i-2,i} = 0$ is true. Since there exist an element f in $UT'_n(K)$ such that $f_{i-2,i} \neq 0$, it implies a contradiction. Similarly $g_{i,i+1} \neq 0$. Thus,

$$g_{i-2,i-1} \neq 0, g_{i,i+1} \neq 0.$$

From this and (9) follows that for any matrix f the following statement holds:

$$\text{if } f_{i-2,i} \neq 0, \text{ then } f_{i-1,i+1} \neq 0. \quad (10)$$

Since, f is any matrix in $UT'_n(K)$, there exists, for example, a matrix:

$$f = \begin{pmatrix} 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \in UT'_n(K),$$

for which the condition (10) is false. This is a contradiction. Consequently, $g_{i-1,i} \neq 0$. The theorem is proved.

Theorem 4 (a sufficient condition for the universality). Let K be an associative commutative ring with a unity 1. If all the elements of the first collateral diagonal of the matrix $g \in UT_n(K)$, $n \geq 3$, with the possible exception of the last element are reversible in K , then g is universal.

Proof. By Theorem 1, we can assume that all the elements of the first collateral diagonal matrix g , except perhaps the last one, are equal to 1. To prove the theorem, we need to solve the equation (2) for any matrix f from the commutant $UT'_n(K)$. It follows that the first collateral diagonal of the matrix x is defined by the following system of equations:

$$\begin{cases} f_{13} = x_{23} - x_{12}; \\ f_{24} = x_{34} - x_{23}; \\ \dots \\ f_{n-3,n-1} = x_{n-2,n-1} - x_{n-3,n-2}; \\ f_{n-2,n} = x_{n-1,n} - x_{n-2,n-1} \cdot g_{n-1,n}, \end{cases} \quad (11)$$

If we assume that $x_{12} = 0$, we find x_{23} , from the first equation and we find x_{34} , from the second equation, etc. Thus, the values of the first collateral diagonal of the matrix x are defined.

The second collateral diagonal of the matrix x is defined by the system of equations:

$$\begin{cases} f_{14} = x_{24} - x_{13} + b_{14}; \\ f_{25} = x_{35} - x_{24} + b_{25}; \\ f_{36} = x_{46} - x_{35} + b_{36}; \\ \dots \\ f_{n-3,n} = x_{n-2,n} - x_{n-3,n-1} \cdot g_{n-1,n} + b_{n-3,n}, \end{cases}$$

where $b_{i,i+3}$ are some constants. Assuming that $x_{13} = 0$, one can determine the values of $x_{i,i+2}$, $i = 2, 3, \dots, n-2$, similar to the determination of the values $x_{i,i+1}$ of a system (11). Continuing similarly, we find a solution of equation (2). The theorem is proved.

Similarly we can prove the following theorem

Theorem 5 (a sufficient condition for the universality). Let K be an associative commutative ring with a unity 1. If all the elements of the first collateral diagonal of the matrix $g \in UT_n(K)$, $n \geq 3$, with the possible exception of the first element are reversible in K , then g is universal.

Corollary 2. If all the elements of the first collateral diagonal of the matrix $g \in UT_n(\mathbb{Z})$, $n \geq 3$, with the possible exception of the first or last element are equal to 1 or -1 , then g is universal.

Lemma 2. Assume that there are no reversible elements in a Euclidean ring E except 1 and -1 . Then for any non-zero elements g_1, g_2 in E the following equivalence holds:

elements g_1, g_2 do not have a common divisor except 1 if and only if there exist $u_1, u_2 \in E$ such that:

$$g_1 u_1 + g_2 u_2 = 1. \quad (12)$$

Proof. Assume that g_1, g_2 do not have common divisors except 1. Then the greatest common divisor is $(g_1, g_2) = 1$. Hence, by the Euclidean algorithm, there exist $u_1, u_2 \in E$ such that (12) holds.

Let (12) be true. Assume the contrary, i.e.

$$g_\varepsilon = d \cdot v_\varepsilon, d \neq 1,$$

where $\varepsilon = 1, 2$. Then we have from (12) that

$$d(u_1 v_1 + u_2 v_2) = 1.$$

Hence d is reversible and $d \neq 1$. This is a contradiction. The lemma is proved.

Theorem 6 (a necessity condition for the universality). Let E be an Euclidean ring having no reversible elements except 1 and -1 . Then, it follows from the universality of the matrix $g \in UT_n(E)$, $n > 3$ that all the elements of its first collateral diagonal are equal to ± 1 , with the possible exception of extreme ones, i.e.

$$|g_{i-1,i}| = 1, 2 < i < n. \quad (13)$$

Proof. Let g be universal in $G \simeq UT_n(E)$. Then for any matrix $f \in UT_n(E)$ the equation (2) is solvable in G . Hence, the elements of the first collateral diagonal of the matrix x are the solution of the system (8). Let us

first establish that all the elements $g_{i-1,i}, 2 < i < n$, are non-zero. Assume the contrary, i.e. $g_{i-1,i} = 0$ for some i . Then the two successive equations of the system (8) containing $g_{i-1,i}$ are as follows:

$$\begin{cases} f_{i-2,i} = -x_{i-1,i}g_{i-2,i-1}; \\ f_{i-1,i+1} = x_{i-1,i}g_{i,i+1}. \end{cases}$$

Since the left-hand sides of these equations can take any values, for example, a value of 1, then

$$g_{i-2,i-1} \neq 0, g_{i,i+1} \neq 0$$

for any $2 < i < n$. Assuming $f_{i-3,i-1} = 1$ in (8), we obtain

$$x_{i-2,i-1}g_{i-3,i-2} - x_{i-3,i-2}g_{i-2,i-1} = 1, \quad (14)$$

where $3 < i < n$. Since the ring E is Euclidean, then it follows from (14) that $g_{i-2,i-1}$ and $g_{i-3,i-2}$ are coprime when $3 < i < n$. Let us prove the validity of (13). Two consecutive equations of the system (8) of the form:

$$\begin{cases} x_{i-1,i}g_{i-2,i-1} - x_{i-2,i-1}g_{i-1,i} = f_{i-2,i}; \\ x_{i,i+1}g_{i-1,i} - x_{i-1,i}g_{i,i+1} = f_{i-1,i+1}, \end{cases} \quad (15)$$

$2 < i < n$, have solutions for any values of $f_{i-1,i+1}$. Let $f_{i-1,i+1} = 0$. Then we have from (15):

$$x_{i-1,i} \cdot g_{i,i+1} = x_{i,i+1} \cdot g_{i-1,i}. \quad (16)$$

Since $g_{i,i+1}$ and $g_{i-1,i}$ are coprime when $2 < i < n$, then by lemma 2, they have no common divisors. From this and (16) follows that $x_{i,i+1}$ is divided by $g_{i,i+1}$, i.e.

$$x_{i,i+1} = d \cdot g_{i,i+1}.$$

Similarly we have

$$x_{i-1,i} = d \cdot g_{i-1,i}.$$

We obtain from this and from the first equation of the system (15) that

$$g_{i-1,i}(d \cdot g_{i-2,i-1} - x_{i-2,i-1}) = f_{i-2,i}. \quad (17)$$

If we assume that $f_{i-2,i} = 1$, then it follows from (17) that the element $g_{i-1,i}$ is reversible. Then we have $|g_{i-1,i}| = 1$ by assumption of the theorem.

Corollary 3. Let R be a ring of integers or a ring of polynomials $\mathbb{Z}[x]$ over a ring of integers \mathbb{Z} . Then, from the universality of the matrix g of $UT_n(R)$ follows that all the elements of its first collateral diagonal except possible extreme ones, are equal to 1 or -1 .

It is known that if E is an Euclidean ring, where there are no reversible elements except 1 and -1 , then the ring of polynomials $E(x)$ is the same Euclidean ring.

Corollary 4. Let E be an Euclidean ring, where there are no reversible elements except 1 and -1 . Then from the universality of matrices g in $UT_n(E(x))$ over the ring of polynomials of one variable follows that all the elements of the first collateral are equal to 1 or -1 except possible extreme ones.

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References

- 1 Каргаполов М.И. Основы теории групп / М.И. Каргаполов, Ю.И. Мерзляков. — М.: Наука, 1982. — 288 с.
- 2 Bier A. The width of verbal subgroups in the group of unitriangular matrices over a field / A. Bier // Int. Journal Alg. Comput. — 2012. — Vol. 22. — No. 3. — P. 21–41.
- 3 Бахта Н.С. О представимости коммутанта группы $UT(n, K)$ множеством значений функции одной переменной / Н.С. Бахта // Вестн. Омского ун-та. — 2012. — № 2. — С. 44–46.

- 4 Бахта Н.С. О представимости членов нижнего центрального ряда группы $UT(n, K)$ множеством значений функции одной переменной / Н.С. Бахта // Вестн. Омского ун-та. — 2013. — № 4. — С. 13–15.
- 5 Меньшов А.В. О p -разрешимости некоторых регулярных уравнений над p -группой Гейзенберга / А.В. Меньшов, В.А. Романьков // Вестн. Омского ун-та. — 2014. — № 3. — С. 11–14.
- 6 Roman'kov V. Equations over groups / V. Roman'kov // Groups Complexity Cryptology. — 2012. — Vol. 4. — No. 2. — P. 191–240.
- 7 Конырханова А.А. Универсальные элементы групп унитарных матриц над полем / А.А. Конырханова // Вестн. Омского ун-та. — 2015. — № 4(78). — С. 18–20.
- 8 Конырханова А.А. Универсальные элементы групп унитарных матриц над кольцом целых чисел / А.А. Конырханова // Вестн. Омского ун-та. — 2016. — № 2. — С. 11–13.

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Униүшбұрышты матрицалар топтарының әмбебап элементтері

Элементтері бірлік коммутативті және ассоциативті K сақинадан алынған ақырлы өлшемі үштен артық униүшбұрышты матрицалар тобының g матрицасы үшін келесі теоремалар дәлелденген: 1) егер g матрицасы әмбебап болса, онда оның бірінші қосалқы диагоналінің шеткі элементтерінен басқалары нөлден өзгеше; 2) егер g матрицасының бірінші қосалқы диагоналінің соңғы элементінен басқасы K -да қайтымды болса, онда g әмбебап; 3) егер K сақинасы евклидтік болса және оның тривиалды элементтерден басқа қайтымды элементтері болмаса, онда g матрицасының әмбебаптығынан оның бірінші қосалқы диагоналінің шеткі элементтерден басқа барлық элементтері K -да қайтымды болатыны шығады.

Кілт сөздер: униүшбұрышты матрицалар тобы, коммутатор, коммутант, әмбебап элемент, сақина, евклидтік сақина.

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Универсальные элементы групп унитарных матриц

Для матрицы g из группы унитарных матриц данной конечной размерности, больше трех, над коммутативным и ассоциативным кольцом K с единицей доказаны следующие теоремы: 1) если матрица g универсальна, то все её элементы первой побочной диагонали, кроме крайних, отличны от нуля; 2) если все элементы первой побочной диагонали матрицы g , кроме, возможно, последнего, обратимы в K , то g универсальна; 3) если кольцо K евклидово и не имеет обратимых элементов, кроме тривиальных, то из универсальности матрицы g следует, что все элементы её первой побочной диагонали, кроме крайних, обратимы в K .

Ключевые слова: группа унитарных матриц, коммутатор, коммутант, универсальный элемент, кольцо, евклидово кольцо, ассоциативное кольцо.

References

- 1 Kargaplov, M.I., Merzlyakov, Yu.I. (1982). *Osnovy teorii hrupp [Foundations of groups theory]*. Moscow: Nauka [in Russian].
- 2 Bier, A. (2012). The width of verbal subgroups in the group of unitriangular matrices over a field. *Int. Journal Alg. Comput.*, Vol. 22, 3, 21–41.
- 3 Bahta, N.S. (2012). O predstavimosti kommutanta hruppy $UT(n, K)$ mnozhestvom znachenii funksii odnoi peremennoi [On the representability of the commutator of group $UT(n, K)$ by the set of values of one variable]. *Vestnik Omskoho universiteta – Bulletin of the Omsk University*, 2, 44–46 [in Russian].

- 4 Bahta, N.S. (2013). O predstavimosti chlenov nizhneho tsentralnoho riada hruppy $UT(n, K)$ mnozhestvom znachenii funktsii odnoi peremennoi [On representability of members of the low central series of the group $UT(n, K)$ by values of one-variable function]. *Vestnik Omskoho universiteta – Bulletin of the Omsk University*, 4, 13–15 [in Russian].
- 5 Men'shov, A.V., Roman'kov, V.A. (2014). O p -razreshimosti nekotorykh rehuliarnykh uravnenii nad p -hruppoi Heizenberha [On p -solvability of some regular equations over a Heisenberg p -group]. *Vestnik Omskoho universiteta – Bulletin of the Omsk University*, 3, 11–14 [in Russian].
- 6 Roman'kov, V. (2012). Equations over groups. *Groups Complexity Cryptology*, Vol. 4, 2, 191–240 [in Russian].
- 7 Konyrkhanova, A.A. (2015). Universalnye elementy hrupp unitreuholnykh matrits nad polem [Universal elements of the groups of unitriangular matrices over a field]. *Vestnik Omskoho universiteta – Bulletin of the Omsk University*, 4(78), 18–20 [in Russian].
- 8 Konyrkhanova, A.A. (2016). Universalnye elementy hrupp unitreuholnykh matrits nad koltsom tselykh chisel [Universal elements of unitriangular matrices groups over ring of integers]. *Vestnik Omskoho universiteta – Bulletin of the Omsk University*, 2, 11–13 [in Russian].

РЕПОЗИТОРИЙ КАРГУ