

Model companion properties of some theories

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The class \mathcal{K} of algebraic systems of signature σ is called a formula-definable class if there exists an algebraic system A of signature σ such that for any algebraic system B of signature σ it is $B \in \mathcal{K}$ if and only if $Th(B) \cdot Th(A) = Th(A)$. The paper shows that the formula-definable class of algebraic systems is idempotently formula-definable and is an axiomatizable class of algebraic systems. Any variety of algebraic systems is an idempotently formula-definite class. If the class \mathcal{K} of all existentially closed algebraic systems of a theory T is formula-definable, then a theory of the class \mathcal{K} is a model companion of the theory T . Also, in the paper the examples of some theories on the properties of formula-definability, pseudofiniteness and smoothly approximability of their model companion were discussed.

Keywords: model companion, pseudofinite theory, formula-definable class, smoothly approximated structure.

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Introduction

In the literature on model theory and universal algebra, after the theorem of Feferman S., Vaught R.L. [1], the product of complete theories is considered in various articles. In particular, in [2], it is shown that the product of two stable (superstable, ω -stable) theories will be a stable (superstable, ω -stable) theory, that is, the set of all stable (superstable, ω -stable) theories with the operation of the product of theories is a commutative semigroup.

A. Robinson introduced the definition of a model companion for a theory [3]. In articles by various authors, results are obtained regarding the existence of a model companion for a theory. In particular, in [4], there is the following criterion for the existence of a model companion for inductive theories.

Theorem 1. (P. Eklof, G. Sabbagh [4]) Let T be an inductive theory. Then T has a model companion T' if and only if the class of existentially closed models of a theory T is elementary.

Various properties of model companions from different points of view have been studied in the works of [5–7]. Pseudofinite models and ω -categorical smoothly approximated models were considered in [8–12].

1 Background information

Let us give the necessary definitions and known results on the theory of models and universal algebra. For brevity, by the word model, we mean an algebraic system.

Let L be a countable language of first-order signature σ . For any model A of language L , we denote by $Th(A)$ the set of all sentences (bounded formulas) of language L that are true in model A , that is,

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$Th(A)$ the complete theory of model A . For models B, A of language L , the notation $B \equiv A$ means $Th(B) = Th(A)$.

For a class \mathcal{K} (we assume that all classes are abstract, that is, closed with respect to isomorphism), $Th(\mathcal{K})$ is the set of complete theories of all models of class \mathcal{K} . $Th(L)$ is the set of all complete theories of the language L . Since the language L is countable, the power is $|Th(L)| \leq 2^\omega$. If \mathcal{K} is an axiomatizable class of models of a language L , then $Th(\mathcal{K})$ is the theory of class \mathcal{K} .

Definition 1. [13] A class \mathcal{K} of models of signature σ is called a *formula-definable class* if there exists a model A of signature σ such that for any model B of signature σ , $B \in \mathcal{K}$ if and only if $Th(B) \cdot Th(A) = Th(A)$. The model A is then called the determinant of the class \mathcal{K} , and if $Th(A) \cdot Th(A) = Th(A)$, then the class \mathcal{K} is called *idempotently formula-definable*.

Preliminary results in this direction were obtained in works [14–16].

Definition 2. If $S \subset Th(L)$, then $M(S)$ is the class of all models of all theories from S . We call the set S of theories axiomatizable if $M(S)$ is an axiomatizable class. A class \mathcal{K} of models is called *inductive* if $Th(\mathcal{K})$ is an inductive theory, that is, $Th(\mathcal{K})$ is a $\forall\exists$ -theory. Not every set of theories is axiomatizable.

Theorem 2. (S. Feferman, R. Vaught [1]) Filtered products and direct products of models of a language L preserve elementary equivalence.

This theorem allows us to introduce the product operation $Th(A) \cdot Th(B) \Leftrightarrow Th(A \times B)$, (the symbol \Leftrightarrow means by definition), the direct product $\prod_{i \in I} T_i$ of complete theories $T_i, i \in I \Leftrightarrow Th(\prod_{i \in I} T_i)$, the ultraproduct $\prod_{i \in I} T_i / D$ of complete theories $T_i, i \in I$ by ultrafilter D over set $I \Leftrightarrow Th(\prod_{i \in I} T_i / D)$, the ultradegree T^I / D of complete theory T by ultrafilter D over set $I \Leftrightarrow Th(\prod_{i \in I} T_i / D)$, where $T_i = T$ for all $i \in I$.

We assume that the direct product of models is the direct product of a non-empty set of models. The direct product of an empty set of models is a trivial model.

It is clear that $S \subset Th(L)$ is axiomatizable if and only if S is closed with respect to ultraproducts of theories.

A theory T is called an *idempotent* theory if $T \cdot T = T$. A model A is called an *idempotent model* if $Th(A \times A) = Th(A)$.

The set $Th(L)$ with the operation \cdot product of theories is a commutative semigroup with identity (we will not take much into account the theory of the trivial model, although, of course, it is a neutral element for the operation \cdot).

Subsemigroups of semigroups $\langle Th(L); \cdot \rangle$ we call them semigroups of complete theories.

Definition 3. [17] A set $S \subset Th(L)$ is called a *formula-definable set of theories* if there is a theory $T \in Th(L)$ such that for any theory $T_1 \in Th(L)$ it holds, $T_1 \in S$ if and only if $T_1 \cdot T = T$. The theory T , in this case, is called the determinant of the set S . If the determinant of the set S is an idempotent theory T , then S is called an idempotent formula-definable set of theories, and T in this case is called the idempotent determinant of the set S .

It is clear that the class of models \mathcal{K} is formula-definable if and only if $Th(\mathcal{K})$ is formula-definable. Furthermore, the class of models \mathcal{K} is idempotent formula-definable if and only if $Th(\mathcal{K})$ is idempotent formula-definable.

In proving the results of the article, we will use the following theorems:

Theorem 3. (J. Keisler [18]) For any model A and any ultrafilter D over I , $A \equiv A^I / D$.

Theorem 4. (J. Keisler [18]) By any sentence ϕ there is a number n such that for any index set I and any models $A_i, i \in I$, there is a subset J in I that contains at most n elements, and for any $V, J \subset V \subset I$, $\prod_{i \in V} A_i \models \phi$ if and only if $\prod_{i \in I} A_i \models \phi$.

Theorem 5. (S. Feferman – R. Vaught [1]) For any two sets of models $\{A_i | i \in I\}$, $\{B_i | i \in I\}$ and for any ultrafilter D on I , $\prod_{i \in I} (A_i \times B_i) / D \cong \prod_{i \in I} A_i / D \times \prod_{i \in I} B_i / D$.

Theorem 6. (F. Galvin, J. Weinstein [19]) Let A, B, C be models of the language L . If $A \times B \times C \equiv A$, then $A \times B \equiv A$.

2 Formula-definable semigroups of complete theories

This section presents the results obtained on formula-definable semigroups of complete theories [14] and formula-definable classes of models.

Let T^n mean $\prod_{i \in I} T_i$, where $|I| = n$, $T_i = T$, for all $i \in I$, and T^I mean $\prod_{i \in I} T_i$, where $T_i = T$ for all $i \in I$.

Lemma 2.1. For any theory $T \in Th(L)$ it holds

- 1) $T^I / D = T$ for any ultrafilter D over the set I .
- 2) If T is an idempotent theory, then $T^I = T$ for any set I .

Proof. 1) $T^I / D = T$. To prove it, you should use the fact that $T^I / D \Leftrightarrow Th(\prod_{i \in I} T_i / D)$, where $T_i = T$ for all $i \in I$ and apply Theorem 4, relying on Theorem 3.

2) Let T be an idempotent theory. It is clear that for any finite n , $T^n = T$.

Let I be an infinite set. And for some sentence $\phi \in T$, sentence $\phi \notin T^I$, then by Theorem 5, this contradicts the fact that for all finite m greater than a sufficiently large n , $\phi \in T^m = T$ holds. This means $T^I = T$.

Lemma 2.2. For any two sets of complete theories $\{T_i | i \in I\}$ and $\{T'_i | i \in I\}$ and for any ultrafilter D on I , $\prod_{i \in I} (T_i \cdot T'_i) / D = \prod_{i \in I} T_i / D \cdot \prod_{i \in I} T'_i / D$.

Proof. Follows directly from Theorem 6, relying on Theorem 3.

Lemma 2.3. Let T_1, T_2, T_3 be complete theories. If $T_1 \cdot T_2 \cdot T_3 = T_3$, then $T_1 \cdot T_3 = T_3$.

Proof. Follows from Theorem 7, based on Theorem 3.

Theorem 7. The formula-definable set of complete theories S is closed under finite, arbitrary direct products of theories.

Proof. Let the theory T be the determinant of the set S . The finite closedness of S with respect to the product is beyond doubt due to the associativity and commutativity of the direct product of theories.

Let $\{T_i | i \in I\} \subset S$ be an infinite set. If T is an idempotent theory, which means $T \in S$, then to prove the infinite closedness of S with respect to the product, one should use the same reasoning as in the proof of Lemma 2.1.

If the determinant of $T \notin \{T_i | i \in I\}$, then consider the set $\{T_i | i \in I\} \cup \{T\}$. Let for some sentence $\phi \in T$, sentence $\phi \notin \prod_{i \in I} T_i \cdot T$, then by Theorem 5, there exists a finite $J \subset I$ such that for any $V, J \subseteq V \subseteq I$, $\phi \notin \prod_{i \in I} T_i \cdot T$. However, this contradicts the fact that for all finite $V, J \subseteq V \subseteq I$ and the power V is greater than a sufficiently large n , $\phi \in \prod_{i \in I} T_i \cdot T$ holds.

Corollary 2.1. The formula-definable class of models \mathcal{K} is closed under finite, arbitrary direct products of models. Its set of complete theories $Th(\mathcal{K})$ is also closed with respect to finite, arbitrary direct products of theories.

Lemma 2.4. The set of complete theories, closed under arbitrary direct products of theories, contains an idempotent theory $T' \in S$ such that for each theory $T \in S$, the following holds: $T \cdot T' = T'$.

Proof. Let us take the direct product of all theories from S , that is $\prod_{T \in S} T$. Since S is closed with respect to arbitrary direct products of theories, then $\prod_{T \in S} T \in S$. (In general, $|S| \leq 2^\omega$). Due to the closedness of S , the product $\prod_{T \in S} T \cdot \prod_{T \in S} T \in S$. This means there is a theory $T' \in S$ and $\prod_{T \in S} T \cdot \prod_{T \in S} T = T'$, which is present in both products. Now applying Lemma 2.3, we obtain that for any theory $T \in S$, the following holds: $T \cdot T' = T'$, including $T' \cdot T' = T'$.

Corollary 2.2. The class of models \mathcal{K} , which is closed with respect to arbitrary direct products of models, contains an idempotent model $A \in \mathcal{K}$ such that for each model $B \in \mathcal{K}$, $Th(B \times A) = Th(A)$ holds.

Theorem 8. A formula-definable set of complete theories S is an idempotent formula-definable set of theories. And the idempotent determinant of the set S is unique.

Proof. Let T^* be the determinant of the set S . By Theorem 7, S is closed under arbitrary direct products of theories. By Lemma 2.4, there is an idempotent theory $T' \in S$ such that for any theory $T \in S$, the following holds: $T \cdot T' = T'$. Now, if for some complete theory $T_1 \notin S$, $T_1 \cdot T' = T'$, then since $T_1 \cdot T' \cdot T^* = T^*$, then by Lemma 2.3, $T_1 \cdot T^* = T^*$ holds. That is, $T_1 \in S$. We have a contradiction. This means that the theory T' is an idempotent determinant of the set S .

There is only one idempotent determinant for S . Indeed, if there are two idempotent determinants T_1 and T_2 for S , then since $T_1 \in S$ and $T_2 \in S$ we have $T_1 = T_1 \cdot T_2 = T_2$.

Corollary 2.3. A formula-definable class of models of complete theories S is an idempotent formula-definable class of models.

Theorem 9. A formula-definable set of complete theories S is an axiomatizable set of complete theories.

Proof. Let $\{T_i | i \in I\} \subseteq S$ and $\prod_{i \in I} T_i / D$ be the ultraproduct of theories over the ultrafilter D over I . Using Lemmas 2.1 and 2.2, we obtain $\prod_{i \in I} T_i / D \cdot T = \prod_{i \in I} T_i / D \cdot T^I / D = \prod_{i \in I} (T_i \cdot T) / D = T$. This means that S is closed under the ultraproduct of theories, that is, S is an axiomatizable set of theories.

Corollary 2.4. A formula-definable class of models is an axiomatizable class.

Theorem 10. Each variety V is an idempotent formula-definable class of models.

Proof. The variety V is closed under arbitrary direct products. This means that $Th(V)$ is closed under the product of complete theories. Then, by Lemma 2.4, there is an idempotent theory $T \in Th(V)$ such that for any model $B \in V$, $Th(B) \cdot T = T$. Let A be a model of a theory T , then A is an idempotent model, and for any model $B \in V$, it is true $Th(B \times A) = Th(A)$. Since $T \in Th(V)$, then in model A , the truths are all the identities that define the variety V . Therefore, if $B \notin V$, then $Th(B \times A) \neq Th(A)$. This means that the variety V is an idempotent formula-definable class of models.

3 Some examples of theories with a model companion

Here, we study examples of some theories and their model companions for fulfilling formula-definable, pseudofinite and smoothly approximable properties. In what follows, T is not necessarily a complete theory.

Definition 4. (model companion of theory [3]) Theory T_1 is called a *model companion* of theory T if T_1 and T are mutually model consistent (i.e. models of theory T_1 are embedded in models of theory T , and models of theory T_1 are embedded in models of theory T). The theory T_1 is model complete.

A model companion to a theory does not always exist, but if it does, it is unique.

Theorem 11. If the class \mathcal{K} of existentially closed models of a theory T is a formula-definable class, then \mathcal{K} is a model companion of the theory T .

Proof. Follows from Corollary 2.4 and Theorem 2. (P. Eklof, G. Sabbagh [4]).

Some important types of companions of incomplete theories and their model-theoretic properties have been studied in the works [5–7].

In the work of J. Ax [8], the concept of pseudofiniteness was first defined. The groundworks obtained to date for pseudofinite structures directly depend on the results of J. Ax. The basic definitions of pseudofiniteness are as follows.

Definition 5. [8] An infinite structure \mathcal{M} of a fixed language L is *pseudofinite* if for all L -sentences φ , $\mathcal{M} \models \varphi$ implies that there is a finite L -structure \mathcal{M}_0 such that $\mathcal{M}_0 \models \varphi$. The theory $T = Th(\mathcal{M})$ of the pseudofinite structure \mathcal{M} is called *pseudofinite*.

Many beautiful theorems in model theory of the 1950s-60s were proved using ultraproducts. Set theorists love ultraproducts because they give rise to elementary embeddings, a staple of large cardinal theory. J. Ax in [8] connect the notion of pseudofiniteness and the construction of ultraproducts.

Proposition 3.1. [8] Fix a language L and an L -structure \mathcal{M} . Then the following are equivalent:

- 1) an L -structure \mathcal{M} is pseudofinite;
- 2) $\mathcal{M} \models T_f$, where T_f is the common theory of all finite L -structures;
- 3) \mathcal{M} is elementarily equivalent to an ultraproduct of finite L -structures.

In classical logic, the following property is a straightforward consequence of pseudofiniteness.

Proposition 3.2. Let \mathcal{M} be a pseudofinite structure and $f : M^k \rightarrow M^k$ be a definable function. Then f is injective if and only if f is surjective.

The study of countably infinite and countably categorical smoothly approximable structures is relevant in many areas of mathematics, including topology, analysis, and algebra.

Definition 6. [10] Let Σ be a countable signature and let \mathcal{M} be a countable and ω -categorical Σ -structure. Σ -structure \mathcal{M} (or $Th(\mathcal{M})$) is said to be *smoothly approximable* if there is an ascending chain of finite substructures $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \dots \subseteq \mathcal{M}$ such that $\bigcup_{i \in \omega} \mathcal{M}_i = \mathcal{M}$ and for every i , and for every $\bar{a}, \bar{b} \in \mathcal{M}_i$ if $tp_{\mathcal{M}}(\bar{a}) = tp_{\mathcal{M}}(\bar{b})$, then there is an automorphism σ of M such that $\sigma(\bar{a}) = \bar{b}$ and $\sigma(\mathcal{M}_i) = \mathcal{M}_i$, or equivalently, if it is the union of an ω -chain of finite homogeneous substructures; or equivalently, if any sentence in $Th(\mathcal{M})$ is true of some finite homogeneous substructure of \mathcal{M} .

It is noted that the concept of a “finitely homogeneous substructure” does not mean that the substructure is homogeneous.

Smoothly approximated structures were first examined in generality in [10], subsequently in [11]. The model theory of smoothly approximable structures has been developed much further by G. Cherlin and E. Hrushovski [12].

A. Lachlan introduced the concept of *smoothly approximable* structures to change the direction of analysis from finite to infinite, that is, to classify large finite structures that appear to be *smooth approximations* to an infinite limit.

When proving the above properties for examples, in order to avoid textual routine, the following known results are used.

Corollary 3.1. [10] Every ω -categorical, ω -stable structure over a language with just finitely many function symbols is smoothly approximated.

Corollary 3.2. [10] If \mathcal{M} is smoothly approximated, then $Th(\mathcal{M})$ is not finitely axiomatisable.

Remark. Any smoothly approximable structures are pseudofinite, but the converse is not always true.

Example 1. Theory T of the class of all Boolean algebras, T_1 theory of atomless Boolean algebras. It is known that T_1 is a model companion for T . It is clear that $T_1 \cdot T_1$ will be the theory of atomless

Boolean algebra, and all countable atomless Boolean algebras are isomorphic. If some Boolean algebra A has an atom, then its theory $Th(A)$ will satisfy $Th(A) \cdot T_1 \neq T_1$. This means that the class of models T_1 is a formula-definable class. Since the class of models of a theory T is a variety, then by Theorem 11, this class is a formula-definable class. Thus, we have obtained an example of a formula-definable class of models of theory T in which theory T_1 is a model companion and the class of all models of theory T_1 is a formula-definable class. A Boolean algebra is known to be pseudofinite if and only if each element has an atom [20]. It is clear that the theory of this model companion is not pseudofinite. Since the T_1 theory is finitely axiomatizable, the countable model of the model companion is not smoothly approximable by Corollary 3.2.

Example 2. Theory of T abelian groups of exponent of a prime number p . The complete theory T_1 of the infinite model of a theory T is a model companion of a theory T since the infinite model of a theory T is an existentially closed model and categorical. It is clear that the class of models of the theory T is formula-definable, the determinant of this class is the infinite model of the theory T_1 . However, the model companion of T_1 is not a formula-definable class. The theory of this model companion is, of course, pseudofinite. The infinite countable model of the model companion is ω -categorical, ω -stable, and by Corollary 3.1. is smoothly approximable.

Example 3. Theory T of one equivalence relation. The class of models of theory T is a formula-definable class; its determinant is a model with an infinite number of classes, and each class contains an infinite number of elements. The theory of the T_1 model, in which the infinite countable model contains for each $1 \leq n \leq \omega$ an infinite number of n -element classes, is a model companion of the theory of T . The class of models of the theory of T_1 is not formula-definable since for some non-existentially closed models B in the theory of T , $Th(B) \cdot T_1 = T_1$ holds. In the work [21], it is proved that any theory with one equivalence relation is pseudofinite. It is clear that theory T_1 is pseudofinite. Also, this work proves that any countably categorical model of this theory is smoothly approximable. Therefore, an infinite countable model of T_1 theory is smoothly approximable by [21].

Example 4. Theory T of linear order. The model companion of theory T is the theory T_1 of dense linear order without endpoints. The classes of models of theory T and the class of models of theory T_1 are not formula-definable classes of models. If it is a formula-definable class of models, it must be closed under the product of models, but this is not the case. Theory T_1 is not pseudofinite (see [22]). The infinite countable model of theory T is not smoothly approximable since no automorphism permutes elements.

Conclusion

The paper shows that the formula-definable class of algebraic systems is idempotently formula-definable and is an axiomatizable class of algebraic systems. Any variety of algebraic systems is an idempotently formula-definite class. If the class \mathcal{K} of all existentially closed algebraic systems of a theory T is formula-definable, then a theory of the class \mathcal{K} is a model companion of the theory T . Also, the paper discusses examples of some theories on the properties of formula-definability, pseudofiniteness and smoothly approximability of their model companion.

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Author Contributions

All authors participated in the revision of the manuscript and approved the final submission. All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Кейбір теориялардың модельдік компаньондарының қасиеттері

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σ сигнатурасының алгебралық жүйелерінің \mathcal{K} класы формуламен анықталатын класс деп аталады, егер σ сигнатурасының кез келген B алгебралық жүйесі бар болса, онда тек $B \in \mathcal{K}$ үшін, яғни $Th(B) \cdot Th(A) = Th(A)$ орындалатындай σ сигнатурасының A алгебралық жүйесі табылса. Мақалада алгебралық жүйелердің формуламен анықталатын класы идемпотентті түрде формуламен айқындалатын класс және алгебралық жүйелердің аксиоматизацияланатын класы екендігі көрсетілген. Алгебралық жүйелердің кез келген түрі идемпотентті түрде формуламен анықталатын класс болып саналады. T теориясының барлық экзистенциалды тұйық алгебралық жүйелерінің \mathcal{K} класы формуламен анықталатын болса, онда \mathcal{K} класының теориясы T теориясының модельдік компаньоны болып табылады. Сондай-ақ, мақалада формуламен анықталатын, псевдоақырлы және олардың модельдік компаньонының тегіс аппроксимациялану қасиеттері туралы кейбір теориялардың мысалдары талқыланған.

Кілт сөздер: модельдік компаньон, псевдоақырлы теория, формула бойынша анықталатын класс, тегіс аппроксимацияланатын құрылым.

Свойства модельного компаньона некоторых теорий

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Класс \mathcal{K} алгебраических систем сигнатуры σ называется формульно-определимым, если существует алгебраическая система сигнатуры σ , такая что для любой алгебраической системы сигнатуры σ выполняется $B \in \mathcal{K}$ тогда и только тогда, когда $Th(B) \cdot Th(A) = Th(A)$. В статье показано, что формульно-определимый класс алгебраических систем является идемпотентно формульно-определимым и аксиоматизируемым классом алгебраических систем. Любое многообразие алгебраических систем является идемпотентно формульно-определимым классом. Если класс \mathcal{K} всех экзистенциально замкнутых алгебраических систем теории формульно-определим, то теория класса \mathcal{K} является модельным компаньоном теории T . Также в статье рассмотрены примеры некоторых теорий на свойства формульно-определимости, псевдоконечности и гладкой аппроксимируемости моделей их модельного компаньона.

Ключевые слова: модельный компаньон, псевдоконечная теория, формульно-определимый класс, гладко аппроксимируемая структура.

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