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A fractal-fractional gingerbread-man map generalized by $p-$fractal-fractional difference operator

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By using the generalization of the gamma function ($p-$gamma function: $\Gamma_p(.)$), we introduce a generalization of the fractal-fractional calculus which is called $p-$fractal-fractional calculus. Examples are illustrated including the basic power functions. As applications, we formulate the $p-$fractal-fractional difference operators. A class of maps, called gingerbread-man maps, is investigated. We present a new idea of a stability for continuous system, based on three parameters. Sufficient conditions are illustrated to obtain the stability of the system.

Keywords: fractional calculus, fractal calculus, fractional difference operator, fractal-fractional differential operator, fractal-fractional calculus, fractal-fractional discrete operator.

2020 Mathematics Subject Classification: 26A33.

Introduction

Combining the concepts of fractal and fractional operators in the notion of fractal-fractional operators [1], which are operators applied to functions defined on fractal sets and utilizing fractional calculus. Fractal-fractional operators are an interdisciplinary field spanning science, mathematics, and electronics [2, 3]. They have been applied in several domains, including as data processing, image evaluation, and the mathematical modeling of intricate systems including non-local and self-similar behavior [4, 5].

The normal gamma function is extended in the generalized gamma function (see [6]), which can be expressed for positive real numbers. It has several uses in the fields of mathematics, physics, engineering, and statistics (see [7–10]), when p is a positive integer. The fact that both the scale parameter and the form parameter are included makes its properties more complicated than those of the standard one. It is a key tool in many mathematical and scientific contexts, especially when dealing with circumstances with intricate and diverse data distributions [11, 12].

We offer a generalization of the fractal-fractional calculus utilizing the generalization of the gamma function called $p-$gamma function. The discrete $p-$fractal-fractional operators are also developed. We demonstrate that well-known examples are included in the generalized operators. The paper is organized as follows: Section 2 deals with the methods and observations (our main results). Section 3 provides the conclusion of this analysis with suggestions, as future works.

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1 Presentation and Notices

This section deals with all concepts that will be used in the sequel.

Definition 1. The $p-$gamma function can be generalized as follows [6] (see Table 1 for some values of $p-$gamma function)

$$\Gamma_p(\zeta) = \lim_{m\to\infty} \frac{m!p^m(m\,p)^{\frac{\zeta}{p}-1}}{(\zeta)_{m,p}}, \quad p>0,$$

where $(\zeta)_{m,p} := \zeta(\zeta+p)(\zeta+2p)\dots(\zeta+(m-1)p)$ and $(\zeta)_{m,p} = \dfrac{\Gamma_p(\zeta+m\,p)}{\Gamma_p(\zeta)}$. Moreover, $\Gamma(\zeta) = \lim\limits_{p\to 1}\Gamma_p(\zeta)$, $\Gamma_p(\zeta) = p^{\zeta/p-1}\Gamma(\zeta/p)$, $\Gamma_p(\zeta+p) = \zeta\Gamma_p(\zeta)$, and $\Gamma_p(p) = 1$.

Table 1

Exact and numerical solution of example 1

ζ	$\Gamma_1(\zeta)$	$\Gamma_2(\zeta)$	$\Gamma_3(\zeta)$	$\Gamma_4(\zeta)$	$\Gamma_5(\zeta)$
1	1.000	1.253	1.288	1.282	1.267
2	1.000	1.000	0.939	0.886	0.845
3	2.000	1.253	1.000	0.867	0.782
4	6.000	2.000	1.288	1.000	0.844
5	24.000	3.760	1.878	1.282	1.000

Definition 2. [1] Suppose that $\varphi(\chi)$ is a differentiable over $(0,b)$, then the Caputo fractal-fractional derivative is given as follows ($\mu,\nu \in (n-1,n]$):

$$^{\mathcal{C}}\Delta_\chi^{\mu,\nu}\varphi(\chi) := \frac{1}{\Gamma(n-\mu)}\int_0^\chi \frac{d\varphi(t)}{dt^\nu}(\chi-t)^{n-\mu-1}dt.$$

And for a continuous function $\varphi(\chi)$ and fractal differentiable over $(0,b)$, the Riemann-Liouville fractal-fractional differential operator is given by the formula ($\mu,\nu \in (n-1,n]$):

$$^{\mathcal{RL}}\Delta_\chi^{\mu,\nu}\varphi(\chi) := \frac{1}{\Gamma(n-\mu)}\frac{d}{d\chi^\nu}\int_0^\chi \varphi(t)(\chi-t)^{n-\mu-1}dt,$$

where

$$\frac{d\varphi(\chi)}{d\chi^\nu} = \lim_{\chi\to t}\frac{\varphi(\chi)-\varphi(t)}{\chi^\nu - t^\nu}.$$

More generalization is formulated for the above operators, as follows ($\mu,\nu,\gamma \in (n-1,n]$):

$$^{\mathcal{C}}\Delta_\chi^{\mu,\nu,\gamma}\varphi(\chi) := \frac{1}{\Gamma(n-\mu)}\int_0^\chi \frac{d^\gamma\varphi(t)}{dt^\nu}(\chi-t)^{n-\mu-1}dt.$$

And

$$^{\mathcal{RL}}\Delta_\chi^{\mu,\nu,\gamma}\varphi(\chi) := \frac{\frac{d^\gamma}{d\chi^\nu}}{\Gamma(n-\mu)}\int_0^\chi \varphi(t)(\chi-t)^{n-\mu-1}dt,$$

where

$$\frac{d^\gamma\varphi(\chi)}{d\chi^\nu} = \lim_{\chi\to t}\frac{\varphi^\gamma(\chi)-\varphi^\gamma(t)}{\chi^\nu - t^\nu}.$$

Correspondingly, the fractal-fractional integral operator of order $\mu,\nu>0$ is formulated by the structure:

$$Y_\chi^{\mu,\nu}\varphi(\chi) := \frac{\nu}{\Gamma(\mu)}\int_0^\chi t^{\mu-1}\varphi(t)(\chi-\tau)^{\mu-1}\mathrm{dt}, \quad \mu,\nu>0.$$

Combining the definition of $p-$gamma function and the fractal-fractional operators to get the generalized fractal-fractional operators, as follows:

Definition 3. Suppose that $\varphi(\chi)$ is a differentiable over the open interval $(0, b)$. Then the Caputo $p-$fractal-fractional derivative is given as follows $(\mu, \nu \in (n-1, n])$:

$$ {}_{p}^{\mathcal{C}}\Delta_{\chi}^{\mu,\nu}\varphi(\chi) := \frac{1}{p\Gamma_p(n-\mu)}\int_0^{\chi} \frac{d\varphi(t)}{dt^{\nu/p}}(\chi-t)^{n-\mu/p-1}dt. $$

And for a continuous function $\varphi(t)$ and fractal differentiable over $(0, b)$, the Riemann-Liouville $p-$fractal-fractional differential operator is given by the formula $(\mu, \nu \in (n-1, n])$:

$$ {}_{p}^{\mathcal{RL}}\Delta_{\chi}^{\mu,\nu}\varphi(\chi) := \frac{\frac{d}{d\chi^{\nu/p}}}{p\Gamma_p(n-\mu)}\int_0^{\chi} \varphi(t)(\chi-t)^{n-\mu/p-1}dt, $$

where

$$ \frac{d\varphi(\chi)}{d\chi^{\nu/p}} = \lim_{\chi\to t}\frac{\varphi(\chi)-\varphi(t)}{\chi^{\nu/p}-t^{\nu/p}}. $$

More generalization is considered for the above operators, as follows $(\mu, \nu \in (n-1, n])$:

$$ {}_{p}^{\mathcal{C}}\Delta_{\chi}^{\mu,\nu,\gamma}\varphi(\chi) := \frac{1}{p\Gamma_p(n-\mu)}\int_0^{\chi} \frac{d^{\gamma/p}\varphi(t)}{dt^{\nu/p}}(\chi-t)^{n-\mu/p-1}dt. $$

And

$$ {}_{p}^{\mathcal{RL}}\Delta_{\chi}^{\mu,\nu,\gamma}\varphi(\chi) := \frac{\frac{d^{\gamma/p}}{d\chi^{\nu/p}}}{p\Gamma_p(n-\mu)}\int_0^{\chi} \varphi(t)(\chi-t)^{n-\mu/p-1}dt, $$

where

$$ \frac{d^{\gamma/p}\varphi(\chi)}{d\chi^{\nu/p}} = \lim_{\chi\to t}\frac{\varphi^{\gamma/p}(\chi)-\varphi^{\gamma/p}(t)}{\chi^{\nu/p}-t^{\nu/p}}. $$

Correspondingly, the $p-$fractal-fractional integral operator of order $\mu, \nu > 0$ is formulated by the structure:

$$ {}_{p}Y_{\chi}^{\mu,\nu}\varphi(\chi) := \frac{\nu}{p\Gamma_p(\mu)}\int_0^{\chi} t^{\mu/p-1}\varphi(t)(\chi-\tau)^{\mu/p-1}dt. $$

Example 1. Now for the generalized $p-$fractal-fractional operators, we have

$$ \frac{d\varphi(\chi)}{d\chi^{\nu}} = \lim_{\chi\to t}\frac{\chi^m - t^m}{\chi^{\nu/p}-t^{\nu/p}} = \frac{mpt^{m-\nu/p}}{\nu} $$

then $(\mu, \nu \in (0, 1])$

$$ \begin{aligned} {}_{p}^{\mathcal{C}}\Delta_{\chi}^{\mu,\nu}\chi^m &= \frac{mp}{\nu p\Gamma_p(1-\mu)}\int_0^{\chi}\left(t^{m-\nu/p}\right)(\chi-t)^{-\mu/p}dt \\ &= \frac{m}{\nu\Gamma_p(1-\mu)}\frac{\Gamma\left(1-\frac{\mu}{p}\right)\Gamma\left(m-\frac{\nu}{p}+1\right)\chi^{\frac{mp+p-\mu-\nu}{p}}}{\Gamma\left(m-\frac{-2p+\mu+\nu}{p}\right)}, \end{aligned} $$

$$ \begin{aligned} {}_{p}^{\mathcal{RL}}\Delta_{\chi}^{\mu,\nu}(\chi^m) &= \frac{1}{p\Gamma_p(1-\mu)}\frac{d}{d\chi^{\nu/p}}\int_0^{\chi}(t^m)(\chi-t)^{-\mu/p}dt \\ &= \frac{1}{p\Gamma_p(1-\mu)}\frac{d}{d\chi^{\nu/p}}\left(\frac{\Gamma(m+1)\Gamma\left(1-\frac{\mu}{p}\right)\chi^{m-\frac{\mu}{p}+1}}{\Gamma\left(m-\frac{\mu}{p}+2\right)}\right) \\ &= \frac{1}{p\Gamma_p(1-\mu)}\left(\frac{\Gamma(m+1)\Gamma(1-\mu/p)}{\Gamma(m-\mu/p+2)}\right)\left(\frac{d}{d\chi^{\nu/p}}\chi^{m-\mu/p+1}\right). \end{aligned} $$

But

$$\frac{d}{d\chi^{\nu/p}}\chi^{m-\mu/p+1} = \frac{(-\mu+mp+p)}{\nu}\chi^{(-\mu+mp+p-\nu)/p}$$

thus, we have

$${}_{p}^{\mathcal{RL}}\Delta_{\chi}^{\mu,\nu}(\chi^m) = \frac{1}{p\Gamma_p(1-\mu)}\left(\frac{\Gamma(m+1)\Gamma(1-\mu/p)}{\Gamma(m-\mu/p+2)}\right)\frac{(-\mu+mp+p)\chi^{(-\mu+mp+p-\nu)/p}}{\nu}.$$

The $p-$fractal-fractional integral implies that

$$\begin{aligned}
{}_{p}Y_{\chi}^{\mu,\nu}\chi^m &= \frac{\nu\int_0^{\chi} t^{\mu/p-1}t^m(\chi-t)^{\mu/p-1}dt}{p\Gamma_p(\mu)} \\
&= \frac{\nu}{p\Gamma_p(\mu)}\frac{\Gamma\left(\frac{\mu}{p}\right)\chi^{m+\frac{2\mu}{p}-1}\Gamma\left(m+\frac{\mu}{p}\right)}{\Gamma\left(m+\frac{2\mu}{p}\right)} \\
&= \frac{\nu\chi^{m+\frac{2\mu}{p}-1}\Gamma\left(m+\frac{\mu}{p}\right)}{p^{\frac{\mu}{p}-1}\Gamma\left(m+\frac{2\mu}{p}\right)}.
\end{aligned}$$

$$\frac{d^{\gamma/p}\varphi(\chi)}{d\chi^{\nu/p}} = \lim_{\chi\to t}\frac{\chi^{(\gamma/p)m}-t^{(\gamma/p)m}}{\chi^{\nu/p}-t^{\nu/p}} = \frac{\gamma m t^{(\gamma m-\nu)/p}}{\nu},$$

then we have

$$\begin{aligned}
{}_{p}^{\mathcal{C}}\Delta_{\chi}^{\mu,\nu,\gamma}(\chi^m) &= \frac{\gamma m\int_0^{\chi}\left(t^{(\gamma m-\nu)/p}\right)(\chi-t)^{-\mu/p}dt}{\nu p^{(1-\mu)/p}\Gamma\left(\frac{1-\mu}{p}\right)} \\
&= \frac{\gamma m\chi^{\frac{\gamma m-\mu+p-\nu}{p}}}{\nu p^{(1-\mu)/p}}\left(\frac{\Gamma(1-\frac{\mu}{p})\Gamma\left(\frac{\gamma m+p-\nu}{p}\right)}{\Gamma\left(\frac{1-\mu}{p}\right)\Gamma\left(-\frac{-\gamma m-2p+\nu+\mu}{p}\right)}\right).
\end{aligned}$$

$$\begin{aligned}
{}_{p}^{\mathcal{RL}}\Delta_{\chi}^{\mu,\nu,\gamma}(\chi^m) &= \frac{1}{p^{(1-\mu)/p}\Gamma\left(\frac{1-\mu}{p}\right)}\frac{d^{\gamma/p}}{d\chi^{\nu/p}}\int_0^{\chi}(t^m)(\chi-t)^{-\mu/p}dt \\
&= \frac{\gamma\left(p+mp-\mu\right)\chi^{-(p(\nu-(1+m)\gamma)+\gamma\mu)/p^2}}{\nu p^{(1-\mu)/p+1}}\left(\frac{\Gamma(m+1)\Gamma(1-\mu/p)}{\Gamma\left(\frac{1-\mu}{p}\right)\Gamma(m-\mu/p+2)}\right).
\end{aligned}$$

1.1 p−fractal-fractional differences operators

In light of the fact that the forward and backward difference operators are characterized as follows:

$$\curlywedge g(\chi) = g(\chi+1)-g(\chi), \quad \curlyvee g(\chi) = g(\chi)-g(\chi-1)$$

satisfying the iteration $\curlywedge^k = \curlywedge(\curlywedge^{k-1})$ and $\curlyvee^k = \curlyvee(\curlyvee^{k-1})$. And for the fractional order $\curlywedge^\mu = \curlywedge^n(\curlywedge^{-n+\mu})$ and $\curlyvee^\mu = (-1)^n \curlyvee^n (\curlyvee^{-n+\mu})$, where $n = [\mu] + 1$. In [13], the Caputo fractional difference is defined as follows:

$$\begin{aligned} {}^{C} \curlywedge^\mu g(\chi) &= \curlywedge^{-(n-\mu)} \curlywedge^n g(\chi) \\ &= \frac{1}{\Gamma(n-\mu)} \sum_{k=0}^{\chi-(n-\mu)} (\chi - k - 1)^{n-\mu-1} \curlywedge_k^n g(\chi), \end{aligned}$$

where the factor χ^μ is defined by

$$\chi^\mu = \frac{\Gamma(\chi+1)}{\Gamma(\chi+1-\mu)}, \quad \mu > 0;$$

and

$$\begin{aligned} {}^{C} \curlyvee^\mu g(\chi) &= \curlyvee^{-(n-\mu)} \curlyvee^n g(\chi) \\ &= \frac{1}{\Gamma(n-\mu)} \sum_{k=\chi+(n-\mu)}^{b} (k - 1 - \chi)^{n-\mu-1} \curlyvee_k^n g(\chi), \end{aligned}$$

correspondingly, the fractional integral difference operators are as follows:

$$\curlywedge^{-\mu} g(\chi) = \frac{1}{\Gamma(\mu)} \sum_{k=0}^{\chi-\mu} (\chi - k - 1)^{\mu-1} \curlywedge_k^n g(\chi),$$

$$\curlyvee^{-\mu} g(\chi) = \frac{1}{\Gamma(\mu)} \sum_{k=\chi+\mu}^{b} (k - 1 - \chi)^{\mu-1} \curlyvee_k^n g(\chi).$$

We have the next generalized process of fractal-fractional difference formula.

Definition 4. The Caputo fractal-fractional difference operators are defined as follows:

$${}^{C} \curlywedge^{\mu,\nu} g(\chi) = \frac{1}{\Gamma(n-\mu)} \sum_{k=0}^{\chi-(n-\mu)} (\chi^\nu - (k+1)^\nu)^{n-\mu-1} \curlywedge_k^n g(\chi),$$

where the factor $(\chi^\nu)^\mu$ is defined by

$$(\chi^\nu)^\mu = \frac{\Gamma(\chi^\nu+1)}{\Gamma(\chi^\nu+1-\mu)}, \quad \mu, \nu > 0;$$

and

$${}^{C} \curlyvee^{\mu,\nu} g(\chi) = \frac{1}{\Gamma(n-\mu)} \sum_{k=\chi+(n-\mu)}^{b} ((k-1)^\nu - \chi^\nu)^{n-\mu-1} \curlyvee_k^n g(\chi),$$

correspondingly, the fractal-fractional integral difference operators are given by the formulas

$$\curlywedge^{-\mu,\nu} g(\chi) = \frac{1}{\Gamma(\mu)} \sum_{k=0}^{\chi-\mu} (\chi^\nu - (k+1)^\nu)^{\mu-1} \curlyvee_k^n g(\chi),$$

$$\curlyvee^{-\mu,\nu} g(\chi) = \frac{1}{\Gamma(\mu)} \sum_{k=\chi+\mu}^{b} ((k-1)^\nu - \chi^\nu)^{\mu-1} \curlyvee_k^n g(\chi).$$

More generalization is given by using $p-$fractal-fractional formula.

Definition 5. The Caputo $p-$fractal-fractional difference operators are defined as follows:

$$ {}_{p}^{C}\text{⅄}^{\mu,\nu}\, g(\chi) = \frac{1}{p\Gamma_p(n-\mu)} \sum_{k=0}^{\chi-(n-\mu)} (\chi^{\nu}-(k+1)^{\nu})^{n-\mu/p-1}\, \text{⅄}_k^n\, g(\chi), $$

where the factor $\chi^{\mu/p}$ is defined by

$$ (\chi^{\nu})^{\mu/p} = \frac{\Gamma_p(\chi^{\nu}+1)}{\Gamma_p(\chi^{\nu}+1-\mu/p)}, \quad \mu,\nu,p>0; $$

$$ {}_{p}^{C}\Upsilon^{\mu,\nu}\, g(\chi) = \frac{1}{p\Gamma_p(n-\mu)} \sum_{k=\chi+(n-\mu)}^{b} ((k-1)^{\nu}-\chi^{\nu})^{n-\mu/p-1}\, \Upsilon_k^n\, g(\chi), $$

correspondingly, the $p-$fractal-fractional integral difference operators are given by the formulas

$$ {}_{p}\text{⅄}^{-\mu,\nu}\, g(\chi) = \frac{1}{p\Gamma_p(\mu)} \sum_{k=0}^{\chi-\mu} (\chi^{\nu}-(k+1)^{\nu})^{\mu/p-1}\, \text{⅄}_k^n\, g(\chi), $$

$$ {}_{p}\Upsilon^{-\mu,\nu}\, g(\chi) = \frac{1}{p\Gamma_p(\mu)} \sum_{k=\chi+\mu}^{b} ((k-1)^{\nu}-\chi^{\nu})^{\mu/p-1}\, \Upsilon_k^n\, g(\chi). $$

Definition 6. The Caputo $p-$fractal-fractional difference operators are defined as follows:

$$ {}_{p}^{C}\text{⅄}^{\mu,\nu}\, g(\chi) = \frac{1}{p^{(n-\mu)/p)}\Gamma(\frac{(n-\mu)}{p})} \sum_{k=0}^{\chi-(n-\mu)} (\chi^{\nu}-(k+1)^{\nu})^{n-\mu/p-1}\, \text{⅄}_k^n\, g(\chi), $$

where the factor $(\chi^{\nu})^{\mu}$ is defined by

$$ (\chi^{\nu})^{\mu/p} = \frac{p^{\frac{(\mu)}{p^2}}\Gamma\left(\frac{(\chi^{\nu}+1)}{p}\right)}{\Gamma\left(\frac{(\chi^{\nu}+1-\mu/p)}{p}\right)}, $$

$$ {}_{p}^{C}\Upsilon^{\mu,\nu}\, g(\chi) = \frac{1}{p^{(n-\mu)/p)}\Gamma(\frac{(n-\mu)}{p})} \sum_{k=\chi+(n-\mu)}^{b} ((k-1)^{\nu}-\chi^{\nu})^{n-\mu/p-1}\, \Upsilon_k^n\, g(\chi), $$

correspondingly, the fractal-fractional integral difference operators are given by the formulas

$$ {}_{p}\text{⅄}^{-\mu,\nu}\, g(\chi) = \frac{1}{p^{\mu/p}\Gamma(\frac{\mu}{p})} \sum_{k=0}^{\chi-\mu} (\chi^{\nu}-(k+1)^{\nu})^{\mu/p-1}\, \text{⅄}_k^n\, g(\chi), $$

$$ {}_{p}\Upsilon^{-\mu,\nu}\, g(\chi) = \frac{1}{p^{\mu/p}\Gamma(\frac{\mu}{p})} \sum_{k=\chi+\mu}^{b} ((k-1)^{\nu}-\chi^{\nu})^{\mu/p-1}\, \Upsilon_k^n\, g(\chi). $$

Note that for a positive integer q and $\mu > 0$, we have

$$_p \curlywedge^{-\mu,\nu} \left(\curlywedge^q g(\chi)\right) = \curlywedge^q \left({_p} \curlywedge^{-\mu,\nu} g(\chi)\right) - \sum_{k=0}^{q-1} \frac{(\chi^\nu - a^\nu)^{k+\mu/p-q}}{p^{\mu/p}\Gamma(\mu/p + k - q + 1)} \curlywedge^k g(a),$$

$$_p \curlyvee^{-\mu,\nu} \left(\curlyvee^q g(\chi)\right) = \curlyvee^q \left({_p} \curlyvee^{-\mu,\nu} g(\chi)\right) - \sum_{k=0}^{q-1} \frac{(b^\nu - \chi^\nu)^{k+\mu/p-q}}{p^{\mu/p}\Gamma(\mu/p + k - q + 1)} \curlyvee^k g(b).$$

As a consequence, when change $n - \mu/p$ instead of μ/p and n instead of q a calculation yields the following result:

Proposition 1. If $\nu, \mu, p > 0$ then

$$^C_p \curlywedge^{\mu,\nu} g(\chi) = {^{RL}_p} \curlywedge^{\mu,\nu} g(\chi) - \sum_{k=0}^{n-1} \frac{(\chi^\nu - a^\nu)^{k-\mu/p}}{p^{\mu/p}\Gamma(k + 1 - \frac{\mu}{p})} \curlywedge^k g(a),$$

and $(\chi \in [a,b], n = [\mu/p] + 1)$

$$^C_p \curlyvee^{\mu,\nu} g(\chi) = {^{RL}_p} \curlyvee^{\mu,\nu} g(\chi) - \sum_{k=0}^{n-1} \frac{(b^\nu - \chi^\nu)^{k-\mu/p}}{p^{\mu/p}\Gamma(k + 1 - \frac{\mu}{p})} \curlyvee^k g(b).$$

When $p = 1$ and $\nu = 1$, we obtain the result in [13] – Theorem 14.

1.2 Generalized gingerbread-man map (GGMM)

A gingerbread-man map is a two-dimensional chaotic map agreeing to the theory of dynamical systems. When specific initial circumstances and initial parameters are used, the map is chaotic. This map looks like a gingerbread man when the set of chaotic solutions is designed [14]

$$\begin{aligned} x_{k+1} &= 1 - y_k + |x_k|, \\ y_{k+1} &= x_k, \quad k \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{1}$$

By using the generalized $p-$fractal-fractional operator $^C_p \curlywedge^{\mu,\nu}$, the system turns into the following equations:

$$\begin{aligned} {^C_p} \curlywedge^{\mu,\nu} x(k) &= 1 - y(k + \frac{\mu}{p} - 1) + \left| x(k + \frac{\mu}{p} - 1) \right| \\ &- x(k + \frac{\mu}{p} - 1), \\ {^C_p} \curlywedge^{\mu,\nu} y(k) &= x(k + \frac{\mu}{p} - 1) - y(k + \frac{\mu}{p} - 1), \end{aligned}$$

where $\mu, \nu \in (0, 1],\ p \geq 1,\ k \in \mathbb{N}_{1+\frac{\mu}{p}}$. By using the generalized $p-$fractal-fraction integral form $^C_p \curlywedge^{-\mu,\nu}$ with some preparations, we have

$$x(n) = x_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} (n^\nu - (k+1)^\nu)^{\frac{\mu}{p}-1} \left(1 - y(k + \frac{\mu}{p} - 1) + |x(k + \frac{\mu}{p} - 1)| - x(k + \frac{\mu}{p} - 1)\right),$$

$$y(n) = y_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} (n^\nu - (k+1)^\nu)^{\frac{\mu}{p}-1} \left(x(k + \frac{\mu}{p} - 1) - y(k + \frac{\mu}{p} - 1)\right).$$

Now by using the factor

$$(\chi^\nu - \tau^\nu)^{\mu/p} = \frac{p^{\frac{(\mu)}{p^2}}\Gamma\left(\frac{(\chi^\nu-\tau^\nu+1)}{p}\right)}{\Gamma\left(\frac{(\chi^\nu-\tau^\nu+1-\mu/p)}{p}\right)}, \quad \mu,\nu,p>0,$$

we get the system (see Figure 1):

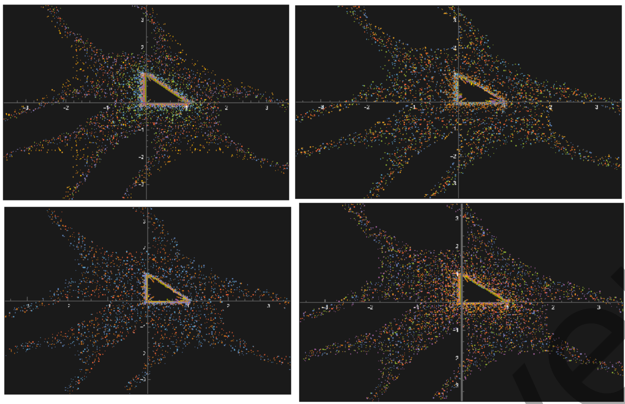

Figure 1. The plot of system (1) and system (2), when for different $p-$fractal-fractional value satisfying $\nu = \dfrac{\log\left(-\frac{9997p^{p/\mu}\Gamma(p/\mu)+10000}{9997p^{p/\mu}\Gamma(p/\mu)-10000}\right)}{\log(\mu)}$, $\nu = \dfrac{\log\left(-\frac{9998p^{p/\mu}\Gamma(p/\mu)+10000}{9998p^{p/\mu}\Gamma(p/\mu)-10000}\right)}{\log(\mu)}$, and $\nu = \dfrac{\log\left(-\frac{9999p^{p/\mu}\Gamma(p/\mu)+10000}{9999p^{p/\mu}\Gamma(p/\mu)-10000}\right)}{\log(\mu)}$ respectively. The iteration is selected for $n = 1$ to 1000.

$$\begin{aligned} x(n) &= x_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} \frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\frac{\mu}{p}}{p}\right)} (n^\nu-(k+1)^\nu)^{\frac{\mu}{p}-1} \\ &\times \left(1 - y(k+\frac{\mu}{p}-1) + |x(k+\frac{\mu}{p}-1)| - x(k+\frac{\mu}{p}-1)\right), \\ y(n) &= y_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} \frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\frac{\mu}{p}}{p}\right)} \left(x(k+\frac{\mu}{p}-1) - y(k+\frac{\mu}{p}-1)\right). \end{aligned} \tag{2}$$

System (2) can be recognized as follows:

$$\begin{aligned} x(n) &= x_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} \frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\mu/p}{p}\right)} \\ &\times (n^\nu-(k+1)^\nu)^{\frac{\mu}{p}-1} \left(1 - y(k+\frac{\mu}{p}-1) - 2x(k+\frac{\mu}{p}-1)\right), \\ y(n) &= y_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} \frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\frac{\mu}{p}}{p}\right)} \left(x(k+\frac{\mu}{p}-1) - y(k+\frac{\mu}{p}-1)\right). \end{aligned}$$

Thus, the characteristic polynomial is $P_1(\lambda) = \lambda^2 + 3\lambda + 3$, where the eigenvalues are $\lambda_{1,2} = 1/2(-3 \pm i\sqrt{3})$ corresponding to the eigenvectors $v_{1,2} = (1/2(-1 \pm i\sqrt{3}), 1)$. Hence, the system is in the steady behavior. Moreover, the equilibrium point is $(1/3, 1/3)$, while the fixed point is $2/7, 1/7$. In addition, we have

$$x(n) = x_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} \frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\mu/p}{p}\right)} \left(1 - y(k+\frac{\mu}{p}-1)\right),$$

$$y(n) = y_0 + \frac{1}{p^{\frac{\mu}{p}}\Gamma(\frac{\mu}{p})} \sum_{k=1}^{n-1-\mu} \frac{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)+1}{p}\right)}{\Gamma\left(\frac{(n^\nu-(k+1)^\nu)-\mu/p}{p}\right)} \left(x(k+\frac{\mu}{p}-1) - y(k+\frac{\mu}{p}-1)\right).$$

Obviously, the characteristic polynomial is $P_1(\lambda) = \lambda^2 + \lambda + 1$, where the eigenvalues are $\lambda_{1,2} = 1/2(-1 \pm i\sqrt{3})$ corresponding to the eigenvectors $v_{1,2} = (1/2(-1 \pm i\sqrt{3}), 1)$. Hence, the system is in the steady behavior. Moreover, the equilibrium point is $(1, 1)$, while the fixed point is $2/3, 1/3$. The stability can be realized by the following generalized result.

Application 1. In this part, we introduce an application on control theory using the operator ${}^{\mathcal{C}}_{p}\Delta^{\mu,\nu,\gamma}_{\chi}\varphi(\chi)$. A p-fractal-fractional PID controller can be presented by

$$u(\chi) = K_a e(\chi) + K_i\, {}_{p}Y^{\mu,\nu}_{\chi} e(\chi) + K_d\, {}^{\mathcal{C}}_{p}\Delta^{\mu,\nu,\gamma}_{\chi} e(\chi),$$

where

- $u(\chi)$ is the Control signal (output of the controller);
- K_a: Proportional gain (adjusts the control response to the current error);
- K_i: Integral gain (adjusts the control response based on the accumulated past error);
- K_d: Derivative gain (adjusts the control response based on the predicted future error);
- $e(\chi)$: Error signal;
- ${}_{p}Y^{\mu,\nu}_{\chi} e(\chi)$: $p-$fractal-fractional integral operator, which is accounting for memory effects and fractal properties;
- ${}^{\mathcal{C}}_{p}\Delta^{\mu,\nu,\gamma}_{\chi} e(\chi)$: $p-$fractal-fractional differential operator, which is capturing anomalous diffusion and self-similar properties in the system.

Note that when $\mu = \nu = p = 1$, we obtain the integer case. By adding fractal and fractional dynamics, this controller goes beyond conventional PID control, improving performance in complicated, memory-dependent, and nonlinear systems. The stabilization of chaotic systems by the use of control mechanisms that lessen unpredictability and guarantee a desired steady-state or periodic behavior is known as chaos suppression. Feedback control, adaptive control, and sliding mode control are examples of traditional chaotic control techniques. However, by combining memory effects and multi-scale dynamics, fractal-fractional controllers provide special benefits.

By using the fractal-fractional weight w_k as follows

$$w_k := (n^\nu - k^\nu)^{\mu/p} = \frac{p^{\frac{(\mu)}{p^2}}\Gamma\left(\frac{(n^\nu-k^\nu+1)}{p}\right)}{\Gamma\left(\frac{(n^\nu-k^\nu+1-\mu/p)}{p}\right)}, \quad \mu, \nu, p > 0,$$

the controller of system (2) can be defined as follows:

$$u_n = -K_a e_n - K_i \sum_{k=0}^{n} w_k e_k - K_d(e_n - e_{n-1}),$$

where $e_n = x_n - x_d$ is the error with a desired fixed point x_d. The system exhibits a wide variety of x-values and stays chaotic for low K_a values. The system changes into a more steady, periodic behavior as K_a rises. The system completely stabilizes and displays a single fixed point after a particular threshold is reached. This demonstrates how more control gains reduce chaos and make a system more predictable.

The stability will be studied in the next section.

2 Stability

This section deals witht the sufficient conditions of the stability of the suggested system. If every zero of a polynomial with real coefficients has a negative real part, the polynomial is stable and/or a Hurwitz polynomial. A stable polynomial's coefficients share identical sign, as is widely recognized [15]. On the other hand, all of a matrix's eigenvalues, having negative real portions, indicates that the matrix is stable. In the domain of matrices, stability is frequently crucial to control theory and dynamic systems. We analyze the characteristic polynomial for λ in order to get the eigenvalues. The eigenvalues of matrix M are represented by the solutions (λ). The system is stable if the real components of all the eigenvalues are negative. Since stability describes how a system behaves in time, it is an essential topic in many domains, such as differential equations, control theory, and signal processing. We start with the linear system.

2.1 Linear system

We start with the next system.

Definition 7. Let $f(\chi) = f(\chi; \chi_0, f_0)$ be the solution of

$$ {}_{p}^{C}\curlywedge^{\mu,\nu} f(\chi) = F(\chi, f)$$

with the following details:

- $f(\chi)$ has a structure over $[\chi_0, \infty)$;
- the point $(\chi, f(\chi)) \in \mathbb{E}$, where

$$\mathbb{E} := \{(\chi, \chi) : \chi \in (\chi_1, \infty),\ \|f\| < \chi_0, \chi > \chi_0\}.$$

Then f is called stable whenever a positive real number $\eta > 0$ exists for all solutions $f(\chi) = f(\chi; \chi_0, f_0) \in \mathbb{E}$ achieving the relation

$$\|f_1 - f_0\| < \eta,$$

and for arbitrary numbers $\varepsilon > 0$ and $0 < \zeta \leq \eta$, the inequality

$$\|f_1 - f_0\| < \zeta \Rightarrow \|f(\chi; \chi_0, f_0) - f(\chi; \chi_0, f_1)\| < \varepsilon, \quad \chi \in [\chi_0, \infty).$$

Additionally,

$$\lim_{\chi\to\infty} \|f(\chi; \chi_0, f_0) - f(\chi; \chi_0, f_1)\| = 0$$

then the solution f is asymptotically stable.

Theorem 2. Assume the linear system

$$ {}_{p}^{C}\curlywedge^{\mu,\nu} \begin{pmatrix} f(\chi) \\ g(\chi) \end{pmatrix} = \Upsilon_{2\times 2} \begin{pmatrix} f(\chi) \\ g(\chi) \end{pmatrix}. \quad (3)$$

Then the system is stable if and only if the solutions are bounded.

Moreover, if the characteristic polynomial of Υ is stable, the system is asymptotically stable.

Proof. Via creating a matrix-valued function with two variables, Υ, as follows:

$$\Upsilon(x,y)=\begin{bmatrix} \upsilon_{11}(x,y) & \upsilon_{12}(x,y) & \dots & \upsilon_{1n}(x,y) \\ \upsilon_{21}(x,y) & \upsilon_{22}(x,y) & \dots & \upsilon_{2n}(x,y) \\ \vdots & \vdots & \ddots & \vdots \\ \upsilon_{m1}(x,y) & \upsilon_{m2}(x,y) & \dots & \upsilon_{mn}(x,y) \end{bmatrix}.$$

Each element $\upsilon_{ij}(x,y)$ of the matrix is a scalar-valued function of x and y. Note that when $x=y$, then we obtain the matrix. In this case, $m=2$ and $n=2$, each element of the matrix is a linear combination of x and y.

Now, the boundedness of solutions of system (3) implies that there exists a fixed number $\varrho>0$ achieving the inequality $\|\Upsilon\|<\varrho$, where $\|.\|$ represents the max norm of the matrix $(\|\Upsilon\|=\max\limits_{1\le j\le n}\sum\limits_{i=1}^{m}|\upsilon_{ij}|)$. This leads to

$$\|f(\chi)-f_0(\chi)\|<\frac{\varepsilon}{2\varrho},\quad \|g(\chi)-g_0(\chi)\|<\frac{\varepsilon}{2\varrho},\quad \varepsilon>0.$$

As a result, we get

$$\|f(\chi;\chi_0,f_0)-f(\chi;\chi_0,\chi_1)\|=\|\Upsilon(\chi,\chi_0)(f_0-f_1)\|<\frac{\varrho\varepsilon}{2\varrho}=\frac{\varepsilon}{2}.$$

In a similar vein, there is

$$\|g(\chi;\chi_0,g_0)-g(\chi;\chi_0,g_1)\|=\|\Upsilon(\chi,\chi_0)(g_0-g_1)\|<\frac{\varrho\varepsilon}{2\varrho}=\frac{\varepsilon}{2}.$$

Let $R=(f,g)^t$, then

$$\begin{aligned}\|R(\chi)-R_0(\chi)\| &\le \|\Upsilon(\chi,\chi_0)\,(R(\chi)-R_0(\chi))\,\| \\ &\le \varrho\|R(\chi)-R_0(\chi)\| \\ &< \varrho(\frac{\varepsilon}{2\varrho}+\frac{\varepsilon}{2\varrho}) \\ &= \varepsilon.\end{aligned}$$

Based on the definition of stability, system (3) is stable.

On the other hand, the stability of the results, involving the zero-value solution, means that the inequality is satisfied by a constant with a positive value ω for a positive number $\varepsilon>0$, as follows:

$$\|R(\chi)\|<\omega\Rightarrow\|\Upsilon(\chi)R(\chi)\|<\varepsilon.$$

In particular,

$$\|f(\chi)\|=\|f(\chi;\chi_0,f_0)\|<\varepsilon/2$$

and

$$\|g(\chi)\|=\|g(\chi;\chi_0,g_0)\|<\varepsilon/2.$$

Hence, all stable solutions are bounded.

Now, the findings are asymptotically stable, provided the characteristic polynomial corresponding to Υ is stable (all its roots are negative).

$$\begin{aligned}\|f(\chi;\chi_0,f_0)-f(\chi;\chi_0,f_1)\| &\le \varrho\exp\left(\frac{\Upsilon(\chi^\nu-\chi_0^\nu)^{\mu/p}}{\mu/p}\right)\|f_1-f_0\|\\ &\le \varrho\exp(-\varepsilon_1\frac{\chi^{\mu\nu/p}}{\mu\nu/p}),\quad 0<\varepsilon_1<\varepsilon=0,\\ &\chi\to\infty,\quad \mu,\nu\in(0,1],\quad p\ge 1.\end{aligned}$$

In the same manner, we get

$$\begin{aligned}\|g(\chi;\chi_0,g_0)-g(\chi;\chi_0,g_1)\| &\le c\exp\left(\frac{\Upsilon(\chi^\nu-\chi_0^\nu)^{\mu/p}}{\mu/p}\right)\|g_1-g_0\|\\ &\le c\exp\left(-\varepsilon_1\frac{\chi^{\nu\mu/p}}{\mu\nu/p}\right),\quad 0<\varepsilon_1<\varepsilon=0,\\ &\chi\to\infty,\quad \mu,\nu\in(0,1],\quad p\ge 1,\end{aligned}$$

which implies the asymptotically stable outcomes.

Corollary 1. Suppose that all of the eigenvalues of the sup norm $\|\Upsilon\|<1$ fall inside the interval [0,1]. Then the system is stable, while if each solution of system (3) is bounded, then the system is asymptotically stable.

Proof. Assume the characteristic polynomial Υ such that $\|\Upsilon\|<1$ and all its eigenvalues are in the interval [0, 1]. Then it is an invertible positive contraction [15]. Then $\Upsilon_{2\times2}^{-1}-\mathbb{I}_d$ is positive semi-definite, with positive determinant $|\Upsilon_{2\times2}|>0$ (one can find the details in the proof of Proposition 3.5 in [16]). This leads to Υ characteristic polynomial is real stable. Hence, asymptotically stable is valid property, in light of Theorem 2.

2.2 Non-homogeneous system

We have the following result:

Theorem 3. Every solution for a non-homogeneous system that fits system (3)

$$^{C}_{p}\lambda^{\mu,\nu}\begin{pmatrix}f(\chi)\\ g(\chi)\end{pmatrix}=\Upsilon_{2\times2}\begin{pmatrix}f(\chi)\\ g(\chi)\end{pmatrix}+\begin{pmatrix}h_1(\chi)\\ h_2(\chi)\end{pmatrix}$$

is stable if and only if they are bounded and

$$\|H\|<\flat,\quad \flat\in(0,\infty),\quad H=(h_1,h_2)^t.$$

The system is asymptotically stable if the characteristic polynomial Υ is stable satisfying $\|\Upsilon\|<\varrho$ and

$$\varrho<\frac{e}{\flat},\quad \flat>0,\quad \|\Upsilon\|\le\varrho.$$

Proof. Let $\|H(\chi)\|<\flat,\ \flat>0$. The condition of the theorem yields that

$$\begin{aligned}\|f(\chi)\| &\le \varrho\exp\left(\varrho\flat\frac{(\chi^\nu-\chi_0^\nu)^{\mu/p}}{\mu/p}\right)\|f_0\|\\ &\le \varrho\exp\left((\varrho\flat-e)\frac{\chi^{\nu\mu/p}}{\mu/p}\right)\|f_0\|\\ &=0,\quad \varrho\flat-e<0,\quad \nu,\mu\in(0,1],\quad \chi\to\infty.\end{aligned}$$

This proves the result.

Example 2. Consider the system

$$
\begin{aligned}
&{}^{C}_{p}\curlywedge^{\mu,\nu} x(\chi) = 1 - y(\chi) + |x(\chi)| - x(\chi),\\
&{}^{C}_{p}\curlywedge^{\mu,\nu} y(\chi) = x(\chi) - y(\chi).
\end{aligned}
$$

Then it can be divided into two cases, as follows:

$$
\begin{aligned}
&{}^{C}_{p}\curlywedge^{\mu,\nu} x(\chi) = 1 - y(\chi),\\
&{}^{C}_{p}\curlywedge^{\mu,\nu} y(\chi) = x(\chi) - y(\chi).
\end{aligned}
\tag{4}
$$

And

$$
\begin{aligned}
&{}^{C}_{p}\curlywedge^{\mu,\nu} x(\chi) = 1 - y(\chi) - 2x(\chi),\\
&{}^{C}_{p}\curlywedge^{\mu,\nu} y(\chi) = x(\chi) - y(\chi).
\end{aligned}
\tag{5}
$$

For system (4), the characteristic polynomial is $\Upsilon(\lambda) = \lambda^2 + \lambda + 1$, with two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$ and $\|\Upsilon\| = \max(0+1, 1+1) = 2 < \varrho,\ \varrho > 2$. Moreover, $\|H\| = 1 < \flat,\ \flat > 1$. Thus, we have $2 < \varrho < e$ and $1 < \flat < e/\varrho$ yields $\varrho < e/\flat$. Thus, in view of Theorem 3, system (4) is asymptotically stable.

Now for system (5), we have the following data: $\Upsilon(\lambda) = \lambda^2 + 3\lambda + 3$, with two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(-3 + i\sqrt{3}\right)$ and $\|\Upsilon\| = \max(2+1, 1+1) = 3 < \varrho,\ \varrho > 3$. Moreover, $\|H\| = 1 < \flat,\ \flat > 1$. As a consequence, the inequality $\varrho < e/\flat$ has no solution. Therefore, Theorem 3 is not applicable.

A mathematical method called perturbation analysis can be employed to examine how a system behaves in reactions to minor perturbations or adjustments. It is especially helpful in comprehending a system's sensitivity and stability. Experts can learn more about a system's overall functioning and make predictions about its eventual configurations by examining how it reacts to disturbances. When examining how dynamic systems behave when subjected to tiny perturbations, perturbation analysis is an invaluable resource. Its capacity to shed light on the system's general effectiveness, stability, and reactivate has made it a popular approach in a variety of scientific and technical fields.

The next example is a perturbation sample of the above system (see Figures 2, 3 and 4 for different values of ϵ_1 and ϵ_2).

Example 3. Consider the following system,

$$
\begin{aligned}
&{}^{C}_{p}\curlywedge^{\mu,\nu} x(\chi) = 1 - y(\chi) + |x(\chi)| - x(\chi) + \epsilon_1,\\
&{}^{C}_{p}\curlywedge^{\mu,\nu} y(\chi) = x(\chi) - y(\chi) + \epsilon_2.
\end{aligned}
$$

Then it can be divided into two cases, as follows:

$$
\begin{aligned}
&{}^{C}_{p}\curlywedge^{\mu,\nu} x(\chi) = 1 - y(\chi) + \epsilon_1,\\
&{}^{C}_{p}\curlywedge^{\mu,\nu} y(\chi) = x(\chi) - y(\chi) + \epsilon_2.
\end{aligned}
\tag{6}
$$

And

$$
\begin{aligned}
&{}^{C}_{p}\curlywedge^{\mu,\nu} x(\chi) = 1 - y(\chi) - 2x(\chi) + \epsilon_1,\\
&{}^{C}_{p}\curlywedge^{\mu,\nu} y(\chi) = x(\chi) - y(\chi) + \epsilon_2.
\end{aligned}
\tag{7}
$$

For system (6), the characteristic polynomial is $\Upsilon(\lambda) = \lambda^2 + \lambda + 1$, with two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$ and $\|\Upsilon\| = \max\left(0+1, 1+1\right) = 2 < \varrho,\ \varrho > 2$. Moreover, $\|H\| = \max(1+\epsilon_1, \epsilon_2) = 1 + \epsilon_1 < \flat,\ \flat > 1 + \epsilon_1,\ \epsilon_2 \leq 1 + \epsilon_1$. Thus, we have $2 < \varrho < e$ and $0 < \epsilon_1 < \frac{(e-\varrho)}{\varrho}$ yields $\varrho < \frac{e}{\flat} < \frac{e}{1+\epsilon_1}$. Thus, in view of Theorem 3, system (6) (similarly, for system (7)) is asymptotically stable.

Now for system (7), we have the following data: $\Upsilon(\lambda) = \lambda^2 + 3\lambda + 3$, with two complex eigenvalues $\lambda_{1,2} = \frac{1}{2}\left(-3 + i\sqrt{3}\right)$ and $\|\Upsilon\| = \max\left(2+1, 1+1\right) = 3 < \varrho,\ \varrho > 3$. Moreover, $\|H\| = \max(1+\epsilon_1, \epsilon_2) = 1 + \epsilon_1 < \flat,\ \flat > 1 + \epsilon_1,\ \epsilon_2 \leq \epsilon_1$. Then it follows that the inequality $\varrho < e/\flat$ has a solution, whenever $\varrho > 3$ and $-1 < \epsilon_1 < \frac{(e-\varrho)}{\varrho}$. Therefore, Theorem 3 indicates asymptotically solutions. A comparison of the system is shown in Figure 5 whenever $\mu = \nu = 0.999$ and $p = 1$, and vice versa when $\mu = \nu = 1$ and $p = 1$ for various values of ϵ_1 and ϵ_2.

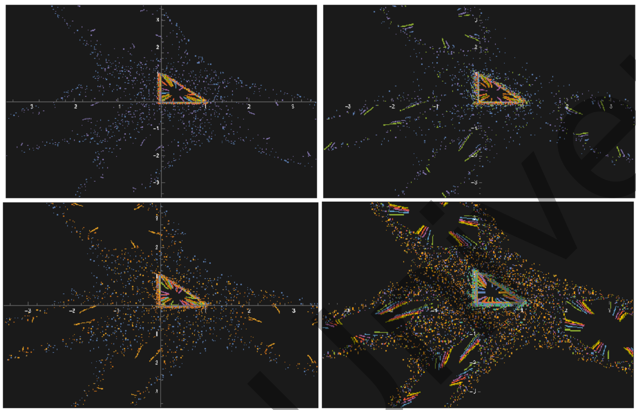

Figure 2. The plot of perterbation System when $\mu = \nu = 0.9992$ and $p = 1$ for $\epsilon_1 = \epsilon_2 = 0.0001, 0.001, 0.01$ and 0.1 respectively. The iteration is selected for $n = 1$ to 1000.

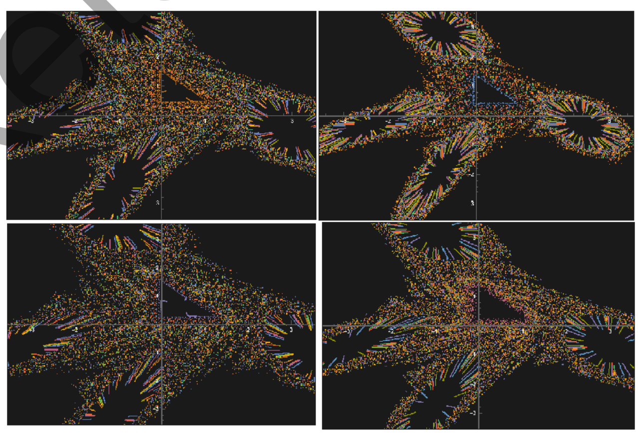

Figure 3. The plot of perterbation System when $\mu = \nu = 0.9992$ and $p = 1$ for $\epsilon_1 = \epsilon_2 = 0.5$, $\epsilon_1 = 0.3, \epsilon_2 = 0.5$, $\epsilon_1 = 0.5, \epsilon_2 = 0.3$ and $\epsilon_1 = 0.3, \epsilon_2 = 0.3$ respectively. The iteration is selected for $n = 1$ to 1000.

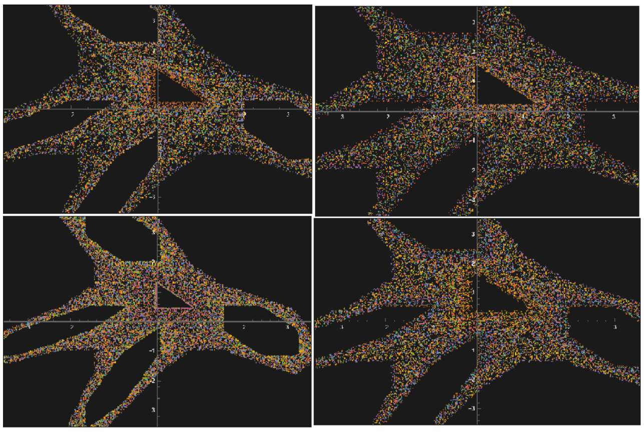

Figure 4. The plot of perterbation System when $\mu = \nu = 0.9999$ and $p = 1$ for $\epsilon_1 = \epsilon_2 = 0.3$, $\epsilon_1 = 0.5, \epsilon_2 = 0.3$, $\epsilon_1 = 0.3, \epsilon_2 = 0.5$ and $\epsilon_1 = 0.5, \epsilon_2 = 0.5$ respectively. The iteration is selected for $n = 1$ to 1000.

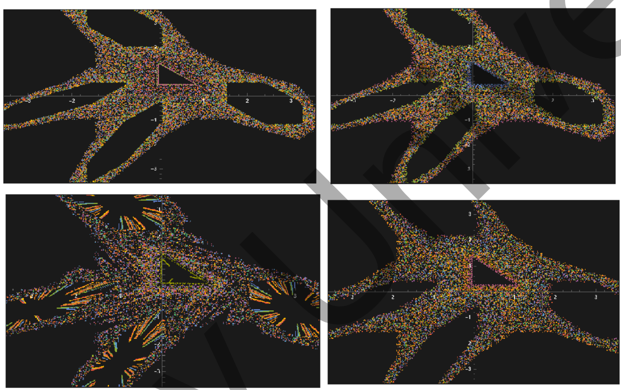

Figure 5. A comparison of the system, when $\mu = \nu = 0.999$ and $p = 1$ vice versa $\mu = \nu = 1$ and $p = 1$ for $\epsilon_1 = \epsilon_2 = 0.5$, (first line) and $\epsilon_1 = 0.3, \epsilon_2 = 0.3$ (second line) respectively. The iteration is selected for $n = 1$ to 1000.

Conclusions and suggestions

By utilizing the generalized gamma function (Γ_p), the fractal-fractional operators are modified. Moreover, the difference operators corresponding to the suggested $p-$fractal-fractional operators are introduced. Examples for the continuous types are illustrated. As an application, we suggested to study the generalized gingerbread-man map (GGMM). Some special cases are indicated for such a system. Stability of the linear and nonlinear systems are examined. We presented a set of conditions to obtain the asymptotic stability behavior of the proposed systems. A perturbation system is formulated with different graphics, based on the values of the perturbation factors. For future works, one can consider different types of stability of the generalized map.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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