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On One Solution of a Periodic Boundary-Value Problem for a Third-Order Pseudoparabolic Equation

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Abstract—This article is devoted to the study of the solvability of a periodic boundary-value problem for a third-order pseudoparabolic equation with a mixed derivative. Nonlocal problems for pseudoparabolic equations have been investigated by many authors. Of particular interest in the study of these problems is caused in connection with their applied values. Such problems include highly porous media with a complex topology, and first of all, soil and ground. To solve this problem, new functions are introduced and the boundary-value problem for a third-order pseudoparabolic equation is reduced to a periodic boundary-value problem for a system of hyperbolic equations with a second-order mixed derivative. Based on the equivalence of the boundary-value problem for a system of hyperbolic equations and the periodic boundary-value problem for a family of systems of ordinary differential equations, two-parameter families of algorithms for finding an approximate solution are constructed and the conditions for unambiguous solvability of the problem under study are established.

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1. INTRODUCTION

On $\Omega = [0,\omega] \times [0,T]$ we consider the semi-periodic boundary-value problem

$$\frac{\partial^3 u}{\partial x^2 \partial t} = A(x,t)\frac{\partial^2 u}{\partial x^2} + B(x,t)\frac{\partial u}{\partial x} + C(x,t)\frac{\partial u}{\partial t} + D(x,t)u + f(x,t), \quad (x,t) \in \Omega,. \tag{1}$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega], \tag{2}$$

$$u(0,t) = \varphi(t), \quad t \in [0,T], \tag{3}$$

$$\frac{\partial u(0,t)}{\partial x} = \psi(t), \quad t \in [0,T], \tag{4}$$

where $(n \times n)$ are the matrices $A(x,t), B(x,t), C(x,t), D(x,t)$, n-vector functions $f(x,t)$ are continuous on Ω, n-vector functions $\varphi(t)$, $\psi(t)$ are continuously differentiable on $[0,T]$, here

$$||u(x,t)|| = \max_{i=\overline{1,n}} |u_i(x,t)|, \quad ||A(x,t)|| = \max_{i=\overline{1,n}} \sum_{j=1}^{n} |a_{ij}(x,t)|.$$

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Let $C(\Omega, R^n)$ be the function space $u : \Omega \to R^n$ continuous on Ω, with the norm $||u||_0 = \max\limits_{(x,t)\in\overline{\Omega}} ||u(x,t)||$. A function $u(x,t) \in C(\Omega, R^n)$, having partial derivatives

$$\frac{\partial u(x,t)}{\partial x} \in C(\Omega, R^n), \quad \frac{\partial u^2(x,t)}{\partial x^2} \in C(\Omega, R^n),$$
$$\frac{\partial u(x,t)}{\partial t} \in C(\Omega, R^n), \quad \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \in C(\Omega, R^n)$$

is called a solution to problem (1)–(3) if it satisfies system (1) for all $(x,t) \in \Omega$, and conditions (2)–(4).

Boundary-value problems for evolution equations of the pseudoparabolic type in certain areas are well studied [1–4]. In this paper, a periodic boundary-value problem for a third-order pseudoparabolic equation is investigated using the method of a parameterization [5]. Previously, this method was used in the study of boundary-value problems for a system of hyperbolic equations with a second-order mixed derivative [6–18].In this paper, we propose an algorithm for finding an approximate solution, and set the coefficient attributes for the unique solvability of the periodic boundary-value problem (1)–(4).

To find a solution, we introduce the functions $z(x,t) = \frac{\partial u(x,t)}{\partial x}$, $w(x,t) = \frac{\partial u(x,t)}{\partial t}$ and write problem (1)–(4) in the form

$$\frac{\partial^2 z}{\partial x \partial t} = A(x,t)\frac{\partial z}{\partial x} + B(x,t)z + C(x,t)w + D(x,t)u + f(x,t), \quad (x,t) \in \Omega, \tag{5}$$

$$z(x,0) = z(x,T), \quad x \in [0,\omega], \tag{6}$$

$$z(0,t) = \psi(t), \quad t \in [0,T], \tag{7}$$

$$w(x,t) = \dot{\varphi}(t) + \int\limits_0^x \frac{\partial z(\xi,t)}{\partial t} d\xi, \tag{8}$$

$$u(x,t) = \varphi(t) + \int\limits_0^x z(\xi,t) d\xi. \tag{9}$$

For fixed $u(x,t)$ and $w(x,t)$ problem (5)–(7) is a semi-periodic boundary-value problem for a system of third-order hyperbolic equations.

We re-introduce the notation $v(x,t) = \frac{\partial z(x,t)}{\partial x}$, and also reduce problem (5)–(9) to a family of periodic boundary-value problems for a system of ordinary differential equations of the form

$$\frac{\partial v}{\partial t} = A(x,t)v + B(x,t)z + C(x,t)w + D(x,t)u + f(x,t), \quad (x,t) \in \Omega, \tag{10}$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega], \tag{11}$$

$$z(x,t) = \psi(t) + \int\limits_0^x v(\xi,t) d\xi, \quad (x,t) \in \Omega. \tag{12}$$

2. MAIN RESULT

To solve problem (8)–(12), we apply the method of a parametrization.

By the step $h > 0 : Nh = T$ we make fragmentation $[0,T) = \bigcup\limits_{r=1}^{N} [(r-1)h, rh)$, $N = 1, 2, \ldots$. Moreover, the area Ω is divided into N parts. By $v_r(x,t)$, $z_r(x,t)$, $u_r(x,t)$ we denote, respectively, the restriction of the function $v(x,t)$, $z(x,t), u(x,t)$ in $\Omega_r = [0,\omega] \times [(r-1)h, rh)$, $\quad r = \overline{1,N}$. By $\lambda_r(x)$ we denote the value of the function $v_r(x,t)$ at $t = (r-1)h$, i.e. $\lambda_r(x) = v_r(x,(r-1)h)$ and make the replacement $\widetilde{v}_r(x,t) = v_r(x,t) - \lambda_r(x), r = \overline{1,N}$. We obtain an equivalent boundary-value problem with unknown functions $\lambda_r(x)$:

$$\frac{\partial \widetilde{v}_r}{\partial t} = A(x,t)\widetilde{v}_r + A(x,t)\lambda_r(x) + B(x,t)z_r(x,t) + C(x,t)w_r + D(x,t)u_r + f(x,t), \qquad (13)$$

$$\widetilde{v}_r(x,(r-1)h) = 0, \quad x \in [0,\omega], \quad r = \overline{1,N}, \qquad (14)$$

$$\lambda_1(x) - \lambda_N(x) - \lim_{t \to T-0} \widetilde{v}_N(x,t) = 0, \quad x \in [0,\omega], \qquad (15)$$

$$\lambda_s(x) + \lim_{t \to sh-0} \widetilde{v}_s(x,t) - \lambda_{s+1}(x) = 0, \quad x \in [0,\omega], \quad s = \overline{1,N-1}. \qquad (16)$$

$$z_r(x,t) = \psi(t) + \int\limits_0^x \widetilde{v}_r(\xi,t)d\xi + \int\limits_0^x \lambda_r(\xi)d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}, \qquad (17)$$

$$w_r(x,t) = \dot{\varphi}(t) + \int\limits_0^x \frac{\partial z_r(\xi,t)}{\partial t}d\xi, \qquad (18)$$

$$u_r(x,t) = \varphi(t) + \int\limits_0^x z_r(\xi,t)d\xi, \quad (x,t) \in \Omega_r, \quad r = \overline{1,N}, \qquad (19)$$

where (16) is the condition for combining functions in the internal lines of the partition.

Problem (13), (14) for fixed $\lambda_r(x), z_r(x,t), w_r(x,t), u_r(x,t)$ is a one-parameter family of Cauchy problems for systems of ordinary differential equations, where $x \in [0,\omega]$, and is equivalent to the integral equation

$$\widetilde{v}_r(x,t) = \int\limits_{(r-1)h}^t A(x,\tau)\widetilde{v}_r(x,\tau)d\tau + \int\limits_{(r-1)h}^t A(x,\tau)d\tau \cdot \lambda_r(x) + \int\limits_{(r-1)h}^t F(x,\tau,z_r,w_r,u_r)d\tau, \qquad (20)$$

where

$$\int\limits_{(r-1)h}^t F(x,\tau,z_r,w_r,u_r)d\tau = \int\limits_{(r-1)h}^t B(x,\tau)z_r(x,\tau)d\tau + \int\limits_{(r-1)h}^t C(x,\tau)w_r(x,\tau)d\tau$$
$$+ \int\limits_{(r-1)h}^t D(x,\tau)u_r(x,\tau)d\tau + \int\limits_{(r-1)h}^t f(x,\tau)d\tau.$$

Instead of $\widetilde{v}_r(x,\tau)$ we substitute the corresponding right-handed part of (20) and by repeating this process ν ($\nu = 1, 2, ...$) times we obtain

$$\widetilde{v}_r(x,t) = D_{\nu r}(x,t)\lambda_r(x) + F_{\nu r}(x,t,z_r,w_r,u_r) + G_{\nu r}(x,t,\widetilde{v}_r), \quad r = \overline{1,N}, \qquad (21)$$

where

$$D_{\nu r}(x,t) = \sum_{j=0}^{\nu-1} \int\limits_{(r-1)h}^t A(x,\tau_1)d\tau_1 \ldots \int\limits_{(r-1)h}^{\tau_j} A(x,\tau_{j+1})d\tau_{j+1} \ldots d\tau_1,$$

$$F_{\nu r}(x,t,z_r,w_r,u_r) = \int\limits_{(r-1)h}^t \big[B(x,\tau_1)z_r(x,\tau_1) + C(x,\tau_1)w_r(x,\tau_1) + D(x,\tau_1)u_r(x,\tau_1)$$

$$+ f(x,\tau_1)\big]d\tau_1 + \sum_{j=1}^{\nu-1}\int\limits_{(r-1)h}^{t} A(x,\tau_1)\dots\int\limits_{(r-1)h}^{\tau_{j-1}} A(x,\tau_j)\int\limits_{(r-1)h}^{\tau_j}\big[B(x,\tau_{j+1})z_r(x,\tau_{j+1})$$

$$+ C(x,\tau_{j+1})w_r(x,\tau_{j+1}) + D(x,\tau_{j+1})u_r(x,\tau_{j+1}) + f(x,\tau_{j+1})\big]d\tau_{j+1}d\tau_j\dots d\tau_1,$$

$$G_{\nu r}(x,t,\widetilde{v}_r) = \int\limits_{(r-1)h}^{t} A(x,\tau_1)\dots\int\limits_{(r-1)h}^{\tau_{\nu-2}} A(x,\tau_{\nu-1})\int\limits_{(r-1)h}^{\tau_{\nu-1}} A(x,\tau_\nu)\widetilde{v}_r(x,\tau_\nu)d\tau_\nu d\tau_{\nu-1}\dots d\tau_1,$$

$\tau_0 = t, r = \overline{1,N}$. Passing to the limit as $t \to rh-0$ in (21) we have

$$\lim_{t\to rh-0}\widetilde{v}_r(x,t) = D_{\nu r}(x,rh)\lambda_r(x) + F_{\nu r}(x,rh,z_r,w_r,u_r) + G_{\nu r}(x,rh,\widetilde{v}_r),$$

$x \in [0,\omega], r = \overline{1,N}$. Substituting in (15), (16) instead of $\lim\limits_{t\to rh-0}\widetilde{v}_r(x,t), r = \overline{1,N}$, the corresponding right-handed parts for unknown functions $\lambda_r(x), r = \overline{1,N}$, we obtain the system of functional equations:

$$Q_\nu(x,h)\lambda(x) = -F_\nu(x,h,z,w,u) - G_\nu(x,h,\widetilde{v}), \tag{22}$$

where

$$Q_\nu(x,h) = \begin{vmatrix} I & 0 & \dots & 0 & -[I + D_{\nu N}(x,Nh)] \\ I + D_{\nu 1}(x,h) & -I & \dots & 0 & 0 \\ 0 & I + D_{\nu 2}(x,2h) & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I + D_{\nu,N-1}(x,(N-1)h) & -I \end{vmatrix},$$

$$F_\nu(x,h,z,w,u)$$
$$= (-F_{\nu N}(x,Nh,z_N,w_N,u_N), F_{\nu 1}(x,h,z_1,w_1,u_1),\dots,F_{\nu,N-1}(x,(N-1)h,z_{N-1},w_{N-1},u_{N-1})),$$

$$G_\nu(x,h,\widetilde{v}) = (-G_{\nu N}(x,Nh,\widetilde{v}_N), G_{\nu 1}(x,h,\widetilde{v}_1),\dots,G_{\nu,N-1}(x,(N-1)h,\widetilde{v}_{N-1})),$$

I is the unit matrix of dimension n. To find a system of five functions $\{\lambda_r(x), \widetilde{v}_r(x,t), z_r(x,t), w_r(x,t), u_r(x,t)\}$, $r = \overline{1,N}$, we have a closed system consisting of equations (22), (21), (17), (18) and (19). Assuming the invertibility of the matrix $Q_\nu(x,h)$ for all $x \in [0,\omega]$, from equation (21), where

$$\widetilde{v}_r(x,t) = 0, z_r(x,t) = \psi(t), w_r(x,t) = \dot{\varphi}(t), u_r(x,t) = \varphi(t),$$

we find $\lambda^{(0)}(x) = (\lambda_1^{(0)}(x), \lambda_2^{(0)}(x),\dots,\lambda_N^{(0)}(x))'$: $\lambda^{(0)}(x) = -[Q_\nu(x,h)]^{-1}\{F_\nu(x,h,\psi,\dot{\varphi},\varphi) + G_\nu(x,h,0)\}$. Using equation (20), for $\lambda_r(x) = \lambda_r^{(0)}(x)$ we find the functions $\{\widetilde{v}_r^{(0)}(x,t)\}$, $r = \overline{1,N}$, i.e.

$$\widetilde{v}_r^{(0)}(x,t) = D_{\nu r}(x,t)\lambda_r^{(0)}(x) + F_{\nu r}(x,t,\psi,\dot{\varphi},\varphi) + G_{\nu r}(x,t,0).$$

The functions $z_r^{(0)}(x,t), w_r^{(0)}(x,t), u_r^{(0)}(x,t), r = \overline{1,N}$, are determined from the relations

$$z_r^{(0)}(x,t) = \psi(t) + \int\limits_0^x \widetilde{v}_r^{(0)}(\xi,t)d\xi + \int\limits_0^x \lambda_r^{(0)}(\xi)d\xi, \quad (x,t)\in\Omega_r, \quad r = \overline{1,N},$$

$$w_r^{(0)}(x,t) = \dot{\varphi}(t) + \int\limits_0^x \frac{\partial z_r^{(0)}(\xi,t)}{\partial t}d\xi, \quad (x,t)\in\Omega_r, \quad r = \overline{1,N},$$

$$u_r^{(0)}(x,t)=\varphi(t)+\int\limits_0^x z_r^{(0)}(\xi,t)d\xi,\quad (x,t)\in\Omega_r,\quad r=\overline{1,N}.$$

For the initial approximation of the problem (13)–(18) we take the system $(\lambda_r^{(0)}(x),\ \widetilde{v}_r^{(0)}(x,t),$ $z_r^{(0)}(x,t),u_r^{(0)}(x,t)),r=\overline{1,N}$ and successive approximations are constructed according to the following algorithm:

Step 1. A) Assuming that

$$z_r(x,t)=z_r^{(0)}(x,t),\quad w_r(x,t)=w_r^{(0)}(x,t),\quad u_r(x,t)=u_r^{(0)}(x,t),\quad r=\overline{1,N},$$

are first approximations in $\lambda_r(x),\widetilde{v}_r(x,t),r=\overline{1,N}$, we find by solving problem (13)–(16). By taking

$$\lambda_r^{(1,0)}(x)=\lambda_r^{(0)}(x),\quad \widetilde{v}_r^{(1,0)}(x,t)=\widetilde{v}_r^{(0)}(x,t),$$

the system couple $\{\lambda_r^{(1)}(x),\widetilde{v}_r^{(1)}(x,t)\},r=\overline{1,N}$, we find as the limit of the sequence $\lambda_r^{(1,m)}(x)$, $\widetilde{v}_r^{(1,m)}(x,t)$, are defined the next way:

Step 1.1. Assuming the invertibility of the matrix $Q_\nu(x,h)$, $x\in[0,\omega]$, from the equation (22), where $\widetilde{v}_r(x,t)=\widetilde{v}_r^{(1,0)}(x,t)$, we find $\lambda^{(1,1)}(x)=(\lambda_1^{(1,1)}(x),\lambda_2^{(1,1)}(x),\ldots,\lambda_N^{(1,1)}(x))'$:

$$\lambda^{(1,1)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,z^{(0)},w^{(0)},u^{(0)})+G_\nu(x,h,\widetilde{v}^{(1,0)})\Big\}.$$

Substituting the found $\lambda_r^{(1,1)}(x)$, $r=\overline{1,N}$ into (21), we find

$$\widetilde{v}_r^{(1,1)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(1,1)}(x)+F_{\nu r}(x,t,z^{(0)},w^{(0)},u^{(0)})+G_{\nu r}(x,t,\widetilde{v}^{(1,0)}).$$

Step 1.2. From equation (22), where $\widetilde{v}_r(x,t)=\widetilde{v}_r^{(1,1)}(x,t)$, we define

$$\lambda^{(1,2)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,z^{(0)},w^{(0)},u^{(0)})+G_\nu(x,h,\widetilde{v}^{(1,1)})\Big\}.$$

By using expression (20) again, we find the functions $\{\widetilde{v}_r^{(1,2)}(x,t)\}$, $r=\overline{1,N}$,

$$\widetilde{v}_r^{(1,2)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(1,2)}(x)+F_{\nu r}(x,t,z^{(0)},w^{(0)},u^{(0)})+G_{\nu r}(x,t,\widetilde{v}^{(1,1)}).$$

At the $(1,m)$ step, we obtain the system of couple $\{\lambda_r^{(1,m)}(x),\widetilde{v}_r^{(1,m)}(x,t)\},r=\overline{1,N}$.

Let's suppose that the solution of problem (13)–(16) is a sequence of systems of couples $\{\lambda_r^{(1,m)}(x),$ $\widetilde{v}_r^{(1,m)}(x,t)\}$ is defined and for $m\to\infty$ converges to continuous, respectively, on $x\in[0,\omega]$, $(x,t)\in\Omega_r$ functions $\lambda_r^{(1)}(x),\widetilde{v}_r^{(1)}(x,t),r=\overline{1,N}$.

B) The functions $z_r^{(1)}(x,t),w_r^{(1)}(x,t),u_r^{(1)}(x,t),r=\overline{1,N}$, are determined from the relations

$$z_r^{(1)}(x,t)=\psi(t)+\int\limits_0^x\widetilde{v}_r^{(1)}(\xi,t)d\xi+\int\limits_0^x\lambda_r^{(1)}(\xi)d\xi,\quad (x,t)\in\Omega_r,\quad r=\overline{1,N},$$

$$w_r^{(1)}(x,t)=\dot{\varphi}(t)+\int\limits_0^x\frac{\partial z_r^{(1)}(\xi,t)}{\partial t}d\xi,\quad (x,t)\in\Omega_r,\quad r=\overline{1,N},$$

$$u_r^{(1)}(x,t)=\varphi(t)+\int\limits_0^x z_r^{(1)}(\xi,t)d\xi,\quad (x,t)\in\Omega_r,\quad r=\overline{1,N}.$$

Step 2. A) Assuming that

$$z_r(x,t)=z_r^{(1)}(x,t),\quad w_r(x,t)=w_r^{(1)}(x,t),\quad u_r(x,t)=u_r^{(1)}(x,t),\quad r=\overline{1,N},$$

are the second approximations in $\lambda_r(x), \widetilde{v}_r(x,t), r=\overline{1,N}$, we find solving problem (13)–(16). Taking $\lambda_r^{(2,0)}(x)=\lambda_r^{(1)}(x), \quad \widetilde{v}_r^{(2,0)}(x,t)=\widetilde{v}_r^{(1)}(x,t)$, the system of couples $\{\lambda_r^{(2)}(x), \widetilde{v}_r^{(2)}(x,t)\}, r=\overline{1,N}$, we find as the limit of the sequence $\lambda_r^{(2,m)}(x), \widetilde{v}_r^{(2,m)}(x,t)$, that defines in the following way:

Step 2.1. Assuming the matrix $Q_\nu(x,h), x\in[0,\omega]$ is invertible, from equation (22), where $\widetilde{v}_r(x,t)=\widetilde{v}_r^{(2,0)}(x,t)$, we find $\lambda^{(2,1)}(x)=(\lambda_1^{(2,1)}(x), \lambda_2^{(2,1)}(x), ..., \lambda_N^{(2,1)}(x))'$:

$$\lambda^{(2,1)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,z^{(1)},w^{(1)},u^{(1)})+G_\nu(x,h,\widetilde{v}^{(2,0)})\Big\}.$$

By substituting the found $\lambda_r^{(2,1)}(x)$, $r=\overline{1,N}$, in (21) we find

$$\widetilde{v}_r^{(2,1)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(2,1)}(x)+F_{\nu r}(x,t,z^{(1)},w^{(1)},u^{(1)})+G_{\nu r}(x,t,\widetilde{v}^{(2,0)}).$$

Step 2.2. From equation (22), where $\widetilde{v}_r(x,t)=\widetilde{v}_r^{(2,1)}(x,t)$, we define

$$\lambda^{(2,2)}(x)=-[Q_\nu(x,h)]^{-1}\Big\{F_\nu(x,h,z^{(1)},w^{(1)},u^{(1)})+G_\nu(x,h,\widetilde{v}^{(2,1)})\Big\}.$$

Using expression (21) again, we find the functions $\{\widetilde{v}_r^{(2,2)}(x,t)\}$, $r=\overline{1,N}$:

$$\widetilde{v}_r^{(2,2)}(x,t)=D_{\nu r}(x,t)\lambda_r^{(2,2)}(x)+F_{\nu r}(x,t,z^{(1)},w^{(1)},u^{(1)})+G_{\nu r}(x,t,\widetilde{v}^{(2,1)}).$$

At the $(2,m)$ step, we obtain the system of couples $\{\lambda_r^{(2,m)}(x), \widetilde{v}_r^{(2,m)}(x,t)\}$, $r=\overline{1,N}$.

Let's suppose that the solution to problem (13)–(16) is a sequence of systems of couples $\{\lambda_r^{(2,m)}(x)$, $\widetilde{v}_r^{(2,m)}(x,t)\}$ are defined and at $m\to\infty$ converges to $\{\lambda_r^{(2)}(x), \widetilde{v}_r^{(2)}(x,t)\}$, $r=\overline{1,N}$.

B) The functions $z_r^{(2)}(x,t)$, $w_r^{(2)}(x,t)$, $u_r^{(2)}(x,t)$, $r=\overline{1,N}$, are determined from the ratios

$$z_r^{(2)}(x,t)=\psi(t)+\int\limits_0^x \widetilde{v}_r^{(2)}(\xi,t)d\xi+\int\limits_0^x \lambda_r^{(2)}(\xi)d\xi, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},$$

$$w_r^{(2)}(x,t)=\dot{\varphi}(t)+\int\limits_0^x \frac{\partial z_r^{(2)}(\xi,t)}{\partial t}d\xi, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N},$$

$$u_r^{(2)}(x,t)=\varphi(t)+\int\limits_0^x z_r^{(2)}(\xi,t)d\xi, \quad (x,t)\in\Omega_r, \quad r=\overline{1,N}.$$

By continuing the process, at the step k we obtain the system $\{\lambda_r^{(k)}(x)$, $\widetilde{v}_r^{(k)}(x,t)$, $z_r^{(k)}(x,t)$, $w_r^{(k)}(x,t)$, $u_r^{(k)}(x,t)\}$, $r=\overline{1,N}$. The conditions of the following statement provide feasibility and convergence of the proposed algorithm, as well as unique solvability problems (13)–(19).

Theorem 1. *Let's suppose that for some* $h>0: Nh=T, N=1,2,...,$ *and* $\nu, \nu\in\mathbb{N}$, $(nN\times nN)$ *the matrix* $Q_\nu(x,h)$ *is invertible for all* $x\in[0,\omega]$ *and the inequalities are carried out*

1) $||[Q_\nu(x,h)]^{-1}||\le\gamma_\nu(x,h)$;

2) $q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}\le\mu<1$, *where* $q_\nu(x,h)=1+\gamma_\nu(x,h)\sum\limits_{j=1}^{\nu}\frac{(\alpha(x)h)^j}{j!}$.

Then there is a unique solution of problem (13)–(19) and the estimates are valid

$$a) \quad \max\Big\{\max_{r=\overline{1,N}}||\lambda_r^*(x)-\lambda_r^{(k)}(x)||+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}||\widetilde{v}_r^*(x,t)-\widetilde{v}_r^{(k)}(x,t)||,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big|\Big|\frac{\partial\widetilde{v}_r^*(x,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{(k)}(x,t)}{\partial t}\Big|\Big|,$$

$$\int\limits_0^x\Big(\max_{r=\overline{1,N}}||\lambda_r^*(\xi)-\lambda_r^{(k)}(\xi)||+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}||\widetilde{v}_r^*(\xi,t)-\widetilde{v}_r^{(k)}(\xi,t)||\Big)d\xi\Big\}$$

$$\leq d_0(x)\sum_{j=k-1}^{\infty}\frac{1}{j!}\Big(\int\limits_0^x d_0(\xi)d\xi\Big)^j\int\limits_0^x\max\Big\{d_1(\xi),d_2(\xi),\int\limits_0^{\xi}d_1(\widetilde{\xi})d\widetilde{\xi}\Big\}d\xi$$

$$\times\max\Big\{\max_{t\in[0,T]}||\psi(t)||,\max_{t\in[0,T]}||\dot{\varphi}(t)||,\max_{t\in[0,T]}||\varphi(t)||,||f||_0\Big\},$$

b) $$\max\Big\{\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}||z_r^*(x,t)-z_r^{(k)}(x,t)||,\ \max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}||w_r^*(x,t)-w_r^{(k)}(x,t)||,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}||u_r^*(x,t)-u_r^{(k)}(x,t)||\Big\}$$

$$\leq\int\limits_0^x\max\Big\{\max_{r=\overline{1,N}}||\lambda_r^*(\xi)-\lambda_r^{(k)}(\xi)||+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}||\widetilde{v}_r^*(\xi,t)-\widetilde{v}_r^{(k)}(\xi,t)||,$$

$$\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}\Big|\Big|\frac{\partial\widetilde{v}_r^*(\xi,t)}{\partial t}-\frac{\partial\widetilde{v}_r^{(k)}(\xi,t)}{\partial t}\Big|\Big|,$$

$$\int\limits_0^{\xi}\Big(\max_{r=\overline{1,N}}||\lambda_r^*(\widetilde{\xi})-\lambda_r^{(k)}(\widetilde{\xi})||+\max_{r=\overline{1,N}}\sup_{t\in[(r-1)h,rh)}||\widetilde{v}_r^*(\widetilde{\xi},t)-\widetilde{v}_r^{(k)}(\widetilde{\xi},t)||\Big)d\widetilde{\xi}\Big\}d\xi,\quad k=1,2,...,$$

where

$$\alpha(x)=\max_{t\in[0,T]}||A(x,t)||,\quad\beta(x)=\max_{t\in[0,T]}||B(x,t)||,\quad\sigma(x)=\max_{t\in[0,T]}||C(x,t)||,$$

$$\delta(x)=\max_{t\in[0,T]}||D(x,t)||,\quad\rho_1(x)=\beta(x)+\sigma(x)+\delta(x)+1,$$

$$\rho_2(x)=\alpha(\xi)\Big(1+q_\nu(\xi,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(\xi)h)^j}{j!}\Big)+1,$$

$$\rho_3(x)=\frac{(q_\nu(x,h)+\gamma_\nu(x,h))h\sum\limits_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}},$$

$$d_0(x)=\max\Big\{\rho_3(x)(\rho_1(x)-1),\int\limits_0^x[\rho_2(\xi)(\rho_1(\xi)-1)]d\xi,\int\limits_0^x\rho_3(\xi)(\rho_1(\xi)-1)d\xi\Big\},$$

$$d_1(x)=\Big[\frac{1+\gamma_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}{1-q_\nu(x,h)\frac{(\alpha(x)h)^\nu}{\nu!}}q_\nu(x,h)+\gamma_\nu(x,h)\Big]\Big[h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\beta(x)\int\limits_0^x\rho(\xi)\theta_\nu(\xi,h)d\xi$$

$$+h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\sigma(x)\int\limits_0^x\rho_1(\xi)\rho_2(\xi)d\xi+h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\delta(x)\int\limits_0^x\int\limits_0^{\xi}\rho(\xi_1)\theta_\nu(\xi_1,h)d\xi_1d\xi$$

$$+\frac{(\alpha(x)h)^\nu}{\nu!}\rho_1(x)q_\nu(x,h)h\sum_{j=0}^{\nu-1}\frac{(\alpha(x)h)^j}{j!}\Big],$$

$$d_2(x) = \int\limits_0^x \rho_2(\xi)\Big[\beta(\xi)\int\limits_0^\xi \rho_1(\xi_1)\theta_\nu(\xi_1,h)d\xi_1 + \sigma(\xi)\int\limits_0^\xi \rho_2(\xi_1)\rho_1(\xi_1)d\xi_1$$
$$+ \delta(\xi)\int\limits_0^\xi\int\limits_0^{\xi_1} \rho(\xi_2)\theta_\nu(\xi_2,h)d\xi_2 d\xi_1\Big]d\xi.$$

The proof of the theorem is given according to the scheme of the algorithm.

Because of the equivalence of problems (1)–(4) and (13)–(19), Theorem 1 implies

Theorem 2. *Let's suppose that the conditions of Theorem 1 are satisfied. Then problem (1)–(4) has a unique solution* $u^*(x,t)$ *and the estimate are valid*

$$\max\left\{\left\|\frac{\partial u^*(x,t)}{\partial x} - \frac{\partial u^{(k)}(x,t)}{\partial x}\right\|_0, \left\|\frac{\partial u^*(x,t)}{\partial t} - \frac{\partial u^{(k)}(x,t)}{\partial t}\right\|_0, \left\|u_r^*(x,t) - u_r^{(k)}(x,t)\right\|_0\right\}$$
$$\leq \int\limits_0^x d_0(\xi)\sum_{j=k-1}^{\infty}\frac{1}{j!}\Big(\int\limits_0^\xi d_0(\xi_1)d\xi\Big)^j \int\limits_0^\xi \max\Big\{d_1(\xi_1), d_2(\xi_1), \int\limits_0^{\xi_1} d_1(\xi_2)d\xi_2\Big\}d\xi_1 d\xi$$
$$\times \max\Big\{\max_{t\in[0,T]}||\psi(t)||, \max_{t\in[0,T]}||\dot{\varphi}(t)||, \max_{t\in[0,T]}||\varphi(t)||, ||f||_0\Big\}.$$

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