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Boundary value problem for fractional diffusion equation in a curvilinear angle domain

We consider a boundary value problem for the fractional diffusion equation in an angle domain with a curvilinear boundary. Existence and uniqueness theorems for solutions are proved. It is shown that Hölder continuity of the curvilinear boundary ensures the existence of solutions. The uniqueness is proved in the class of functions that vanish at infinity with a power weight. The solution to the problem is constructed explicitly in terms of the solution of the Volterra integral equation.

Keywords: noncylindrical domain, curvilinear angle domain, boundary value problem, fractional diffusion equation.

Introduction and problem statement

Consider the equation

$$\left(\frac{\partial^\alpha}{\partial y^\alpha} - \frac{\partial^2}{\partial x^2} \right) u(x, y) = f(x, y), \quad (0 < \alpha \leq 1) \quad (1)$$

where $\frac{\partial^\alpha}{\partial y^\alpha}$ denotes a fractional derivative with respect to y of order α with the origin at the point $x = 0$.

The fractional diffusion equations and their generalizations have attracted great attention in recent years. Research of (1) began in works [1–4].

To give an idea of the variety of problems considered for this equation and the multiplicity of approaches to studying them, we mention [5–33]. An overview is provided in [31]. A more detailed survey can be found in the article [34]. We also point out the monographs [35–37], which reflect many of these approaches and contain vast bibliographies concerning the issue.

Interest in the study of fractional differential equations is also fueled by applications in physics and simulation (see e.g. [38–41]).

Fractional differentiation is given in the Riemann-Liouville sense [38], i.e.

$$\frac{\partial^\alpha}{\partial y^\alpha} u(x, y) = D_{0y}^\alpha u(x, y) = \frac{\partial}{\partial y} D_{0y}^{\alpha-1} u(x, y)$$

and

$$D_{0y}^{\alpha-1} u(x, y) = \frac{1}{\Gamma(1-\alpha)} \int_0^y u(t)(y-t)^{-\alpha} dt.$$

We will consider the equation (1) in a curvilinear angle domain Ω that is defined by

$$\Omega = \{(x, y) : y > 0, x > z(y)\},$$

where $z(y)$ is a non-decreasing continuous function such that $z(0) = 0$.

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A function $u(x, y)$ is called a regular solution of equation (1) in the domain Ω if $y^{1-\alpha}u(x, y) \in C(\bar{\Omega})$, and, moreover, $u(x, y)$ has continuous derivatives in Ω with respect to x up to the second order, and the function $D_{0y}^{\alpha-1}u(x, y)$ is continuously differentiable as function of y for a fixed x at interior points of Ω , and $u(x, y)$ satisfies equation (1) at all points of Ω .

Our purpose is to solve the following problem: *find a regular solution of the equation (1) in the domain Ω satisfying initial and boundary value conditions*

$$\lim_{y \rightarrow 0} D_{0y}^{\alpha-1}u(x, y) = \tau(x) \quad (x > 0) \tag{2}$$

and

$$u(z(y), y) = \varphi(y) \quad (y > 0) \tag{3}$$

where $\tau(x)$ and $\varphi(y)$ are given continuous functions.

The problems in domains with curvilinear boundary were considered in [42], [26]. For the problems for parabolic equations in angles domains, we refer to [43–46].

1 Notations and preliminaries

In what follows, we use the denotations

$$w_\mu(x, y) = x^{\mu-1} \phi(-\beta, \mu; -|x|y^{-\beta}), \tag{4}$$

$$w(x, y) = w_0(x, y), \quad \text{and} \quad \beta = \frac{\alpha}{2}.$$

In (4), ϕ denotes the Wright function [47], [48],

$$\phi(a, b; z) = \sum_{k=0}^{\infty} \frac{z^k}{k! \Gamma(ak + b)} \quad (a > -1).$$

The asymptotics of the Wright function can be given in the form [48]

$$\phi(-\beta, \mu; -t) = \exp(-\rho t^\sigma) \left[A t^\delta + O(t^{\delta-\sigma}) \right] \quad (t \rightarrow \infty), \tag{5}$$

where $\beta \in (0, 1)$, $\sigma = \frac{1}{1-\beta}$, $\rho = (1-\beta)\beta^{\frac{\beta}{1-\beta}}$, and $\delta = \frac{1-2\mu}{2(1-\beta)}$.

In particular, formula (5) implies

$$|w_\mu(x, y)| \leq C |x|^{-\theta} y^{\beta\theta + \mu - 1}, \tag{6}$$

where

$$\theta \geq \begin{cases} 0, & (-\mu) \notin \mathbb{N} \cup \{0\}, \\ -1, & (-\mu) \in \mathbb{N} \cup \{0\}, \end{cases} \quad \text{and} \quad C = C(\beta, \mu, \theta).$$

Here and subsequently, by C we denote positive constants, which may be different in different cases, indicating in parentheses the parameters on which they depend, if necessary, $C = C(\alpha, \beta, \dots)$.

The differentiation formulas for the Wright function [48]

$$\frac{d}{dx} \phi(-\beta, \mu; x) = \phi(-\beta, \mu - \beta; x)$$

and [49]

$$D_{0y}^\nu \left[y^{\mu-1} \phi\left(-\beta, \mu; -\frac{c}{y^\beta}\right) \right] = y^{\mu-\nu-1} \phi\left(-\beta, \mu - \nu; -\frac{c}{y^\beta}\right) \quad (c > 0)$$

give

$$\frac{\partial}{\partial x} w_\mu(x, y) = -\text{sign}(x)w_{\mu-\beta}(x, y), \quad D_{0y}^\nu w_\mu(x, y) = w_{\mu-\nu}(x, y), \quad (7)$$

and

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2} \right) w_\mu(x, y) = 0 \quad (\mu \in \mathbb{R}, \quad |x| > 0, \quad y > 0).$$

We need later the following two statements.

Lemma 1. Let $\psi(t) \in L(0, y)$, let the function $\psi(t)$ be left continuous at the point $t = y$, and let

$$0 \leq z(y) - z(t) \leq C(y - t)^\delta \quad (\delta > \beta, \quad 0 < t < y).$$

Then

$$\lim_{\substack{x \rightarrow z(y) \\ x > z(y)}} \int_0^y \psi(t)w(x - z(t), y - t) dt = \psi(y) + \int_0^y \psi(t)w(z(y) - z(t), y - t) dt.$$

Lemma 2. Let $t^{1-\alpha}q(t) \in C[0, y]$, let $q(t)$ be Hölder continuous in a left neighborhood of y with an exponent $\delta > \alpha$,

$$|q(t) - q(y)| < C(y - t)^\delta \quad (y - \varepsilon < t < y, \quad \varepsilon > 0),$$

and let

$$t^{1-\alpha}q(t) \leq y^{1-\alpha}q(y) \quad \text{for all } t \in (0, y).$$

Then

$$(D_{0t}^\alpha q)_{t=y} \geq 0,$$

and $(D_{0t}^\alpha q)_{t=y} = 0$ if and only if $q(t) = Ct^{\alpha-1}$.

The proof of Lemma 1 is given in [26]. Lemma 2 is proved in [42] (see also [50]).

2 Existence theorem

Theorem 1. Let

$$0 \leq z(y_2) - z(y_1) \leq C(y_2 - y_1)^\delta \quad (0 \leq y_1 < y_2, \quad \delta > \beta), \quad (8)$$

$$\tau(x) \in C[0, \infty), \quad y^{1-\alpha}\varphi(y) \in C[0, \infty), \quad \tau(0) = \Gamma(\alpha) [y^{1-\alpha}\varphi(y)]_{y=0}, \quad (9)$$

$$y^{1-\alpha}f(x, y) \in C(\bar{\Omega}),$$

$$\lim_{x \rightarrow \infty} \tau(x) \exp\left(-\omega x^{\frac{2}{2-\alpha}}\right) = 0, \quad (10)$$

$$\lim_{x \rightarrow \infty} y^{1-\alpha}f(x, y) \exp\left(-\omega x^{\frac{2}{2-\alpha}}\right) = 0, \quad (11)$$

for every $\omega > 0$, and let the function $f(x, y)$ be representable in the form

$$f(x, y) = D_{0y}^{-\delta}g(x, y) \quad (12)$$

for some $\delta > \beta$ and $y^{1-\alpha+\delta}g(x, y) \in C(\bar{\Omega})$.

Then there exists a solution to problem (1), (2), and (3); it can be represented in the form:

$$u(x, y) = \int_0^y \psi(t)w(x - z(t), y - t) dt + T(x, y) + F(x, y), \quad (13)$$

where

$$T(x, y) = \frac{1}{2} \int_0^\infty \tau(s) w_\beta(x - s, y) ds, \quad F(x, y) = \frac{1}{2} \int_0^y \int_{z(t)}^\infty f(s, t) w_\beta(x - s, y - t) ds dt,$$

and the function $\psi(x)$ is a solution of the integral equation

$$\psi(y) + \int_0^y \psi(t) w(z(y) - z(t), y - t) dt = \varphi(y) - T(z(y), y) - F(z(y), y). \tag{14}$$

Proof. Consider separately each of three summands on the right side of (13), namely $F(x, y)$, $T(x, y)$, and $\Psi(x, y)$, where

$$\Psi(x, y) = \int_0^y \psi(t) w(x - z(t), y - t) dt.$$

Let us start with $F(x, y)$. By (5), (6), and (7), with (11) and (12), we get

$$|F(x, y)| \leq C y^{3\beta-1},$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x^2} F(x, y) &= \frac{1}{2} \frac{\partial}{\partial x} \int_0^y \int_{z(t)}^\infty \text{sign}(x - s) g(s, t) w_\delta(x - s, y - t) ds dt = \\ &= - \int_0^y \frac{(y - t)^{\delta-1}}{\Gamma(\delta)} g(x, t) ds dt + \frac{1}{2} \int_0^y \int_{z(t)}^\infty g(s, t) w_{\delta-\beta}(x - s, y - t) ds dt = \\ &= -f(x, y) + \frac{1}{2} D_{0y}^\alpha \int_0^y \int_{z(t)}^\infty f(s, t) w_\beta(x - s, y - t) ds dt. \end{aligned}$$

This means that

$$y^{1-\alpha} F(x, y) \in C(\bar{\Omega}), \quad \lim_{y \rightarrow 0} D_{0y}^{\alpha-1} F(x, y) = 0, \tag{15}$$

and

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2} \right) F(x, y) = f(x, y).$$

Consider now $T(x, y)$. By (5), (7), and (10), one can check that

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2} \right) T(x, y) = 0.$$

Rewrite $T(x, y)$ in the form

$$T(x, y) = \frac{1}{2} \int_0^\infty [\tau(s) - \tau(x)] w_\beta(x - s, y) ds + \frac{\tau(x)}{2} \int_0^\infty w_\beta(x - s, y) ds.$$

Using (5), (6), and (7), leads to

$$\int_0^\infty w_\beta(x - s, y) ds = \left(\int_0^x + \int_x^\infty \right) w_\beta(x - s, y) ds = \frac{2y^{\alpha-1}}{\Gamma(\alpha)} - w_\alpha(x, y)$$

and

$$\left| \int_0^\infty [\tau(s) - \tau(x)] w_\beta(x - s, y) ds \right| \leq \left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^\infty \right) |\tau(s) - \tau(x)| w_\beta(x - s, y) ds +$$

$$+ C \sup_{t \in (x-\varepsilon, x+\varepsilon)} |\tau(t) - \tau(y)| \int_{x-\varepsilon}^{x+\varepsilon} w_\beta(x-s, y) ds \leq C y^{\alpha-1} [y^\theta + \omega(\varepsilon)],$$

where $\theta > 0$ and $\omega(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. This gives that

$$y^{1-\alpha} T(x, y) = C (\bar{\Omega} \setminus \{0, 0\}) \quad \text{and} \quad D_{0y}^{\alpha-1} T(x, y) = \tau(x).$$

Let us examine the behavior of $T(x, y)$ in a neighborhood of $(0, 0)$. Assume that $\sigma = \sigma(y)$ is a continuous function defined in a right neighborhood of the point $y = 0$ such that $(\sigma(y), y) \in \Omega$ (i.e. $\sigma(y) > z(y)$) and $\sigma(y) \rightarrow 0$ as $y \rightarrow 0$. It is easy to see that

$$\begin{aligned} T(\sigma(y), y) &= \frac{y^{\alpha-1}}{2} \int_0^{\sigma/y^\beta} \tau(sy^\beta) \phi\left(-\beta, \beta; s - \frac{\sigma}{y^\beta}\right) ds + \\ &+ \frac{y^{\alpha-1}}{2} \int_0^\infty \tau(sy^\beta + \sigma) \phi(-\beta, \beta; -s) ds \end{aligned}$$

and

$$\lim_{y \rightarrow 0} y^{1-\alpha} T(\sigma(y), y) = \frac{\tau(0)}{\Gamma(\alpha)} - \frac{\tau(0)}{2} \lim_{y \rightarrow 0} \phi\left(-\beta, \alpha; -\frac{\sigma(y)}{y^\beta}\right). \quad (16)$$

Finally, consider $\Psi(x, y)$. By (6) and (8), we have

$$|w(z(y) - z(t), y - t)| \leq C |z(y) - z(t)| (y - t)^{-\beta-1} \leq C (y - t)^{\delta-\beta-1}.$$

This means that integral equation (14) has a unique solution $\psi(y)$. A simple computation gives

$$\left(D_{0y}^\alpha - \frac{\partial^2}{\partial x^2}\right) \Psi(x, y) = 0,$$

and

$$\lim_{y \rightarrow 0} D_{0y}^{\alpha-1} \Psi(x, y) = 0 \quad \text{and} \quad y^{1-\alpha} \Psi(x, y) = C (\bar{\Omega} \setminus \{0, 0\}).$$

Now we investigate the behavior of $\Psi(x, y)$ in a neighborhood of $(0, 0)$. It follows from formulas (9), (14), (15), (16) that

$$y^{1-\alpha} \psi(y) \in C[0, \infty) \quad \text{and} \quad \lim_{y \rightarrow 0} y^{1-\alpha} \psi(y) = \frac{\tau(0)}{2\Gamma(\alpha)}. \quad (17)$$

Set $\psi_0(y) = y^{1-\alpha} \psi(y)$, $(\sigma(y), y) \in \Omega$ ($\sigma(y) > z(y)$), and $\sigma(y) \rightarrow 0$ as $y \rightarrow 0$. It is easy to see that

$$\begin{aligned} \Psi(\sigma(y), y) &= \int_0^y \psi(t) [w(\sigma(y) - z(t), y - t) - w(\sigma(y) - z(y), y - t)] dt + \\ &+ \int_0^y t^{1-\alpha} [\psi_0(t) - \psi_0(y)] w(\sigma(y) - z(y), y - t) dt + \\ &+ \psi_0(y) \int_0^y t^{1-\alpha} w(\sigma(y) - z(y), y - t) dt. \end{aligned}$$

This yields that

$$\lim_{y \rightarrow 0} y^{1-\alpha} \Psi(\sigma(y), y) = \psi_0(0) \lim_{y \rightarrow 0} \phi\left(-\beta, \alpha; -\frac{\sigma(y) - z(y)}{y^\beta}\right).$$

Taking into account (16), (17), and

$$\lim_{y \rightarrow 0} \phi\left(-\beta, \alpha; -\frac{\sigma(y) - z(y)}{y^\beta}\right) = \lim_{y \rightarrow 0} \phi\left(-\beta, \alpha; -\frac{\sigma(y)}{y^\beta}\right),$$

we get

$$y^{1-\alpha} [\Psi(x, y) + T(x, y)] \in C(\bar{\Omega})$$

and, consequently,

$$y^{1-\alpha} u(x, y) \in C(\bar{\Omega}).$$

To complete the proof, it remains to show that the function (13) satisfies (3). Indeed, by Lemma 1 we have

$$\begin{aligned} u(z(y), y) &= \lim_{x \rightarrow z(y)} \Psi(x, y) + T(z(y), y) + F(z(y), y) = \\ &= \psi(y) + \Psi(z(y), y) + T(z(y), y) + F(z(y), y). \end{aligned}$$

Due to (14) we get

$$\psi(y) + \Psi(z(y), y) = \varphi(y) - T(z(y), y) - T(z(y), y).$$

Combining the last two equations leads to (3).

3 Solution uniqueness

Theorem 2. Let $z(y) \in C[0, \infty)$. There exists at most one regular solution to the problem (1), (2), and (3), satisfying

$$\lim_{x \rightarrow \infty} \sup_{0 < y < T} |y^{1-\alpha} u(x, y)| = 0 \tag{18}$$

for every $T > 0$.

Proof. Let $u(x, y)$ be a solution of the homogeneous problem

$$\left(\frac{\partial^\alpha}{\partial y^\alpha} - \frac{\partial^2}{\partial x^2} \right) u(x, y) = 0, \quad \left(D_{0y}^{\alpha-1} u \right)_{y=0} = 0, \quad u(z(y), y) = 0. \tag{19}$$

Set

$$\Omega_T = \Omega \cap \{y \leq T\} \quad \text{for some } T > 0, \quad \text{and } v(x, y) = y^{1-\alpha} u(x, y).$$

By (18), (19), and the equality $\left(D_{0y}^{\alpha-1} u \right)_{y=0} = \Gamma(\alpha)v(x, 0)$, we can conclude that there is a point $(\xi, \eta) \in \Omega_T$ such that

$$v(\xi, \eta) \geq v(x, y) \quad \text{for all } (x, y) \in \Omega_T. \tag{20}$$

Lemma 2, with (19) and (20), yields that

$$\left[D_{0y}^\alpha u(\xi, y) \right]_{y=\eta} \geq 0 \quad \text{and} \quad u_{xx}(\xi, \eta) \leq 0.$$

This means that $\left[D_{0y}^\alpha u(\xi, y) \right]_{y=\eta} = 0$, and consequently $u(\xi, y) = Cy^{\alpha-1}$. Taking into account that $v(x, 0) = 0$, this implies that $C = 0$. Thus, we get

$$u(x, y) \leq 0 \quad \text{for all } (x, y) \in \Omega_T.$$

Similarly, considering the function $v(x, y) = -y^{1-\alpha} u(x, y)$ gives

$$u(x, y) \geq 0 \quad \text{for all } (x, y) \in \Omega_T.$$

Thus, we can conclude that $u(x, y) = 0$ for all $(x, y) \in \Omega_T$. The arbitrary choice of the number T implies that $u(x, y) = 0$ in Ω .

Acknowledgments

The study was supported by the Ministry of Science and Education of the Republic of Kazakhstan (Grant Projects AP09259780 (2021–2023) and AP09258892 (2021–2023)).

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ҚИСЫҚ СЫЗЫҚТЫ БҰРЫШТЫҚ ОБЛЫСТАҒЫ БӨЛШЕК РЕТТІ ДИФфуЗИЯЛЫҚ ТЕҢДЕУ ҮШІН ШЕТТІК ЕСЕП

Мақалада қисық сызықты бұрыштық облыстағы бөлшек ретті диффузиялық теңдеудің шеттік есебі зерттелген. Қарастырылып отырған облыстағы есептің шешімінің бар болуы мен жалғыздығы туралы теоремалар дәлелденді. Гельдер бойынша қисық сызықты шекараның үздіксіздігі шешімдердің

бар болуын қамтамасыз ететіндігін көрсеткен. Шешімнің жалғыздығы шексіздікте дәрежелік салмақпен нөлге айналатын функциялар класында дәлелденді. Есептің шешуі Вольтерраның интегралдық теңдеуінің шешуі арқылы айқын түрде құрылады.

Кілт сөздер: цилиндрлік емес облыс, қисық сызықты бұрыштық облыс, шеттік есеп, бөлшек ретті диффузиялық теңдеу.

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Краевая задача для дробного диффузионного уравнения в криволинейной угловой области

В статье рассмотрена и доказана теорема существования и единственности рассматриваемой краевой задачи. Показано, что непрерывность криволинейной границы по Гельдеру обеспечивает существование решений. Единственность решения задачи доказана в классе функций, обращающихся в нуль на бесконечности со степенным весом. Вычисление ответа задачи построено в явном виде через решение интегрального уравнения Вольтерра.

Ключевые слова: нецилиндрическая область, криволинейная угловая область, краевая задача, дробное диффузионное уравнение.

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