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Volterra's equation of the second kind with «incompressible» kernel

In the article the singular Volterra's integral equation of the second kind is considered, which because of the «incompressible» of the kernel classical methods of solutions are not applicable. It is shown that the corresponding homogeneous equation at $|\lambda|>1$ has a continuous spectrum, and the multiplicity of the characteristic numbers grows with increasing $|\lambda|$. By the Carleman-Vekua method the equation is reduced to Abelequation. The eigenfunctions of the equation are found in an explicit form.

Key words: the singular Volterra's integral equation of the second kind, «incompressible» kernel, eigenfunction, Abel's equation.

When solving model problems for parabolic equations in domains with moving boundary tne singular integral equations of the following form arise

$$\varphi(t)-\lambda\int_0^t K(t,\tau)\,\varphi(\tau)\,d\tau = f(t), \quad t>0, \tag{1}$$

where

$$K(t,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right)+\frac{1}{(t-\tau)^{1/2}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\}.$$

The kernel $K(t,\tau)$ has the following properties:

1) $K(t,\tau)\ge 0$ and continuously at $0<\tau\le t<\infty$;

2) $\lim\limits_{t\to t_0}\int_{t_0}^{t} K(t,\tau)\,d\tau=0,\ t_0\ge\varepsilon>0$;

3) $\lim\limits_{t\to 0}\int_0^t K(t,\tau)\,d\tau=1,\ \lim\limits_{t\to+\infty}\int_0^t K(t,\tau)\,d\tau=1$ [1].

Such equations are called by us Volterra integral equations with «incompressible» kernel. The feature the equation in question consists in property 3) of the kernel $K(t,\tau)$ and is expressed in the fact that the corresponding inhomogeneous equation can not be solved by the method of successive approximations for $|\lambda|>1$. Obviously, if $|\lambda|<1$ then the equation (1) has a unique solution, that can be found by the method of successive approximations. Case $|\lambda|=1$ was considered in [1]. Therefore, further in this paper, weas sume that $|\lambda|>1$.

Equations of the form (1) was first considered in Refs of S.N.Kharin in which the asymptotics of integrals of the double layer potentials was studied, and approximate solutions of some applied problems were constructed [2, 3]. And in the further such integral equations were the object of research by many authors.

It should be noted that to this kind of singular integral equations also boundary value problems for spectrally loaded parabolic equations are reduced when the load line moves by law $x=\alpha(t)$ [4–7].

By means of relations:

$$t+\tau=2t-(t-\tau),\quad \frac{(t+\tau)^2}{4a^2(t-\tau)}=\frac{t\tau}{a^2(t-\tau)}+\frac{t-\tau}{4a^2},$$

we reduce the equation (1) to the form

$$\varphi(t)-\int_0^t \frac{1}{2a\sqrt{\pi}}\left\{\frac{2t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}-\frac{1}{(t-\tau)^{1/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}+\frac{1}{(t-\tau)^{1/2}}\right\}\times$$
$$\times\exp\left\{-\frac{t-\tau}{4a^2}\right\}\varphi(\tau)\,d\tau=f(t). \tag{2}$$

From [8;183] it follows, that it suffices to find a solution «simplified» equation

$$\varphi(t)-\lambda\int_0^t k(t,\tau)\,\varphi(\tau)\,d\tau=f(t). \tag{3}$$

where

$$k(t,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{2t}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}+\frac{1}{(t-\tau)^{1/2}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)\right\}.$$

To investigate the full equation (3) we will allocate part of its characteristic, namely

$$\varphi(t)-\lambda\int_0^t k_o(t,\tau)\,\varphi(\tau)\,d\tau=f_1(t), \tag{4}$$

where

$$k_o(t,\tau)=\frac{t}{a\sqrt{\pi}(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\};$$

$$f_1(t)=f(t)+\lambda\int_0^t k_h(t,\tau)\,\varphi(\tau)\,d\tau, \tag{5}$$

where

$$k_h(t,\tau)=\frac{1}{2a\sqrt{\pi}(t-\tau)^{1/2}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right).$$

Equation (4) is characteristic for equation (2), since

$$\lim_{t\to 0}\int_0^t k_o(t,\tau)\,d\tau=1;\ \lim_{t\to 0}\int_0^t k_h(t,\tau)\,d\tau=0.$$

Considering the right side of equation (4) known, we find its solution, ie solution of the characteristic equation (4).

Similarly, [9; 174], integral equation (4) is reduced to an equation with a difference kernel. To do this, we make in it replacements:

$$t=\frac{1}{y},\quad \tau=\frac{1}{x};\ \psi(y)=\frac{1}{\sqrt{y}}\varphi\left(\frac{1}{y}\right);\quad f_2(y)=\frac{1}{\sqrt{y}}f_1\left(\frac{1}{y}\right). \tag{6}$$

Then we obtain the equation of the form

$$\psi(y)-\lambda\int_y^\infty \frac{1}{a\sqrt{\pi}(x-y)^{3/2}}\exp\left\{-\frac{1}{a^2(x-y)}\right\}\psi(x)\,dx=f_2(y).\quad (y>0). \tag{7}$$

Solution of equation (7) can be found either by the operational method, or by its reduction to Riemann boundary value problem [9].

If we denote $L[\psi(y)]=\overline{\psi}(p)$ as the Laplace transform, we can prove the following theorem of the convolution

$$L\left[\int_y^\infty K(y-x)\,\psi(x)\,dx\right]=\overline{K}(-p)\,\overline{\psi}(p), \tag{8}$$

where

$$\overline{K}(-p)=\int_0^\infty K(-t)\,e^{pt}\,dt.$$

$L\left[\frac{k}{2\sqrt{\pi}\,t^{3/2}}\exp\left\{-\frac{k^2}{4t}\right\}\right]=e^{-k\sqrt{p}}$, then by virtue of (8), equation (7) is transformed to

$$\overline{\psi}(p)\cdot\left(1-\lambda e^{-\frac{2}{a}\sqrt{-p}}\right)=\overline{f_2}(p).$$

The corresponding homogeneous equation has the form

$$\overline{\psi}(p)\cdot\left(1-\lambda e^{-\frac{2}{a}\sqrt{-p}}\right)=0. \tag{9}$$

Nonzero solutions of equation (9) hold in the case when

$$1-\lambda e^{-\frac{2}{a}\sqrt{-p}}=0. \tag{10}$$

If $p=p_k$ are the roots of equation (10), the eigenfunctions of equation (7) will have the form [9]

$$\psi_k(y)=C_k e^{p_k y},\quad C_k=const. \tag{11}$$

We shall find the roots of equation (10). When $|\lambda|\geq 1$ we have $e^{\frac{2}{a}\sqrt{-p}}=\lambda$ [9]. Taking the logarithm, we obtain

$$\frac{2}{a}\sqrt{-p}=\ln|\lambda|+i(\arg\lambda+2k\pi);\quad k=0,1,2,...$$

$$-p_k=\frac{a^2}{4}\left(\ln^2|\lambda|-(\arg\lambda+2k\pi)^2\right)+i\frac{a^2}{4}\ln|\lambda|^2(\arg\lambda+2k\pi);\quad k=0,1,2,... \tag{12}$$

For boundedness of functions (11) at infinity it is necessary that $\mathrm{Re}(-p_k)\geq 0$, ie $\ln^2|\lambda|\geq(\arg\lambda+2k\pi)^2$ or

$$-\ln|\lambda|\leq\arg\lambda+2k\pi\leq\ln|\lambda|.$$

Hence, $-N_1\leq k\leq N_2$, where $N_1=\left[\frac{\ln|\lambda|+\arg\lambda}{2\pi}\right]$, $N_2=\left[\frac{\ln|\lambda|-\arg\lambda}{2\pi}\right]$, k is number of eigenfunctions of (11). Obviously, the larger $|\lambda|$, the greater the eigenfunctions.

Thereby, $\forall\lambda$, $|\lambda|\geq 1$ we have

$$\psi_{об.одн}(y)=\sum_{k=-N_1}^{N_2}C_k e^{p_k y}.$$

Passing to the variables (6), we obtain the solution of the homogeneous equation (4)

$$\varphi_{об.одн}(t)=\sum_{k=-N_1}^{N_2}C_k\cdot\frac{1}{\sqrt{t}}e^{p_k/t}.$$

Where $\mathrm{Re}\,p_k\leq 0$ by (12).

We note that if $\lambda=1$, then $p_0=0$. This case is considered in detail in [1] and [10].

Thus, the solution of inhomogeneous equation (7) has the form

$$\psi(y)=f_2(y)+\lambda\int_y^\infty r_-(y-x)\,f_2(x)\,dx+\sum_{k=-N_1}^{N_2}C_k e^{p_k y},\quad (C_k-const), \tag{13}$$

where

$$r_-(y)=\frac{1}{a\sqrt{\pi}(-y)^{3/2}}\sum_{n=1}^{\infty}\frac{n}{\lambda^n}\cdot\exp\left\{-\frac{n^2}{a^2(-y)}\right\}.$$

Performing reverse replacements (6) to (13), we obtain the solution of the inhomogeneous equation (4)

$$\varphi(t)=f_1(t)+\lambda\int_0^t r(t,\tau)\,f_1(\tau)\,d\tau+\sum_{k=-N_1}^{N_2}C_k\cdot\frac{1}{\sqrt{t}}e^{p_k/t}. \tag{14}$$

Where

$$r(t,\tau)=\frac{t}{a\sqrt{\pi}\left(t-\tau\right)^{3/2}}\sum_{n=1}^{\infty}\frac{n}{\lambda^{n}}\cdot\exp\left\{-n^{2}\frac{t\tau}{a^{2}(t-\tau)}\right\}. \tag{15}$$

We shall now begin to solving equation (3), i.e. the simplified variant of initial equation (1).

Using the formula for the solution of characteristic equation (14), taking into account relation (5) for the function $f_1(t)$, we obtain

$$\varphi(t)=f(t)+\lambda\int_{0}^{t}\frac{1}{2a\sqrt{\pi\left(t-\tau\right)}}\left(1-\exp\left\{-\frac{t\tau}{a^{2}(t-\tau)}\right\}\right)\varphi(\tau)\,d\tau+$$

$$+\lambda\int_{0}^{t}r\left(t,\tau\right)\left(f(\tau)+\lambda\int_{0}^{\tau}\frac{1}{2a\sqrt{\pi\left(\tau-\tau_{1}\right)}}\left(1-\exp\left\{-\frac{\tau\tau_{1}}{a^{2}(\tau-\tau_{1})}\right\}\right)\varphi(\tau_{1})\,d\tau_{1}\right)d\tau+\sum_{k=-N_{1}}^{N_{2}}C_{k}\cdot\frac{1}{\sqrt{t}}e^{p_{k}/t}$$

Changing the order of integration in the right-hand side of this equation and interchanging the roles of τ and τ_1 we have

$$\varphi(t)=\lambda\int_{0}^{t}\left\{\frac{1}{2a\sqrt{\pi\left(t-\tau\right)}}\left(1-\exp\left\{-\frac{t\tau}{a^{2}(t-\tau)}\right\}\right)+\lambda\int_{\tau}^{t}r\left(t,\tau_{1}\right)\frac{1}{2a\sqrt{\pi\left(\tau_{1}-\tau\right)}}\times\right.$$

$$\left.\times\left(1-\exp\left\{-\frac{\tau_{1}\tau}{a^{2}(\tau_{1}-\tau)}\right\}\right)d\tau_{1}\right\}\varphi(\tau)\,d\tau+f(t)+\lambda\int_{0}^{t}r\left(t,\tau\right)f(\tau)\,d\tau+\sum_{k=-N_{1}}^{N_{2}}C_{k}\cdot\frac{1}{\sqrt{t}}e^{p_{k}/t} \tag{16}$$

We compute the inner integral in (16)

$$J\left(t,\tau;\lambda\right)=\int_{\tau}^{t}r\left(t,\tau_{1}\right)\frac{1}{2a\sqrt{\pi\left(\tau_{1}-\tau\right)}}\left(1-\exp\left\{-\frac{\tau_{1}\tau}{a^{2}(\tau_{1}-\tau)}\right\}\right)d\tau_{1}=$$

$$=\frac{t}{2a^{2}\pi}\sum_{n=1}^{\infty}\int_{\tau}^{t}\frac{n}{\lambda^{n}\left(t-\tau_{1}\right)^{3/2}\sqrt{\left(\tau_{1}-\tau\right)}}\exp\left\{-n^{2}\frac{t\tau_{1}}{a^{2}(t-\tau_{1})}\right\}\left(1-\exp\left\{-\frac{\tau_{1}\tau}{a^{2}(\tau_{1}-\tau)}\right\}\right)d\tau_{1}$$

or

$$J\left(t,\tau;\lambda\right)=\frac{t}{2a^{2}\pi}\sum_{n=1}^{\infty}\frac{n}{\lambda^{n}}\cdot\int_{\tau}^{t}\frac{1}{\left(t-\tau_{1}\right)^{3/2}\sqrt{\left(\tau_{1}-\tau\right)}}\exp\left\{-n^{2}\frac{t\tau_{1}}{a^{2}(t-\tau_{1})}\right\}\left(1-\exp\left\{-\frac{\tau_{1}\tau}{a^{2}(\tau_{1}-\tau)}\right\}\right)d\tau_{1}=$$

$$=\frac{t}{2a^{2}\pi}\sum_{n=1}^{\infty}\frac{n}{\lambda^{n}}\cdot I_{n}(t,\tau). \tag{17}$$

As a result of computing the integral $I_n(t,\tau)$ takes the form

$$I_{n}(t;\tau)=\frac{a\sqrt{\pi}}{nt\sqrt{t-\tau}}\left(\exp\left\{-\frac{n^{2}t\tau}{a^{2}(t-\tau)}\right\}-\exp\left\{-\frac{(n+1)^{2}t\tau}{a^{2}(t-\tau)}\right\}\right).$$

Substituting in (17), we obtain

$$J\left(t,\tau;\lambda\right)=\frac{1}{2a\sqrt{\pi(t-\tau)}}\sum_{n=1}^{\infty}\frac{1}{\lambda^{n}}\left(\exp\left\{-\frac{n^{2}t\tau}{a^{2}(t-\tau)}\right\}-\exp\left\{-\frac{(n+1)^{2}t\tau}{a^{2}(t-\tau)}\right\}\right)=$$

$$=\frac{1}{2a\lambda\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{t\tau}{a^{2}(t-\tau)}\right\}.$$

Then equation (16) can be rewritten as

$$\varphi(t)=\lambda\int_{0}^{t}\left\{\frac{1}{2a\sqrt{\pi\left(t-\tau\right)}}\left(1-\exp\left\{-\frac{t\tau}{a^{2}(t-\tau)}\right\}\right)+\frac{1}{2a\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{t\tau}{a^{2}(t-\tau)}\right\}\right\}\varphi(\tau)\,d\tau+$$

$$+f(t)+\lambda\int_{0}^{t}r\left(t,\tau\right)f(\tau)\,d\tau+\sum_{k=-N_{1}}^{N_{2}}C_{k}\cdot\frac{1}{\sqrt{t}}e^{p_{k}/t}$$

Finally, after introducing the notation

$$f_2(t) = f(t) + \lambda \int_0^t r(t,\tau) f(\tau)\, d\tau,$$

where $r(t,\tau)$ is defined by (15), we obtain

$$\varphi(t) - \frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} d\tau = f_2(t) + \sum_{k=-N_1}^{N_2} C_k \cdot \frac{1}{\sqrt{t}} e^{p_k/t}. \tag{18}$$

Thus, initial «simplified» integral equation (3) is reduced to equation (18) that is Abel integral equation of the second kind.

According to [8; 117]

Solution of the Abel equation of the second kind

$$y(x) + \mu \int_a^x \frac{y(t)}{\sqrt{x-t}} dt = f(x)$$

has the form

$$y(x) = F(x) + \pi\mu^2 \int_a^x \exp\left[\pi\mu^2 (x-t)\right] F(t)\, dt, \tag{19}$$

where

$$F(x) = f(x) - \mu \int_a^x \frac{f(t)}{\sqrt{x-t}} dt.$$

We find the solution of Abel equation (18) for $f_2(t) = 0,$ that is, we will find a solution of corresponding homogeneous equation (3) for each $k;\ \ -N_1 \le k \le N_2$ (eigenfunctions). Under this condition, equation (18) for each $k;\ \ -N_1 \le k \le N_2,$ has the form

$$\varphi_k(t) - \frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{\varphi_k(\tau)}{\sqrt{t-\tau}} d\tau = \frac{1}{\sqrt{t}} e^{p_k/t}.$$

The solution of this equation can be written as (see (19))

$$\varphi_k(t) = F_k(t) + \frac{\lambda^2}{4a^2} \int_0^t \exp\left(\frac{\lambda^2 (t-\tau)}{4a^2} \right) F_k(\tau) d\tau,$$

where

$$F_k(t) = \frac{1}{\sqrt{t}} e^{p_k/t} + \frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{e^{p_k/\tau}}{\sqrt{\tau(t-\tau)}} d\tau = \frac{1}{\sqrt{t}} e^{p_k/t} + \frac{\lambda\sqrt{\pi}}{2a} erfc\left(\frac{\sqrt{-p_k}}{\sqrt{t}} \right).$$

In calculating the last integral we use the formula [11; formula 3.471 (2)].

Function $F_k(t)$ is bounded for $\forall t \in [0; +\infty)$ at $t \to +\infty$ and $F_k(0) = 0.$

Thus, the eigenfunctions of equation (3) have the form

$$\varphi_k(t) = \frac{1}{\sqrt{t}} e^{p_k/t} + \frac{\lambda\sqrt{\pi}}{2a} erfc\left(\frac{\sqrt{-p_k}}{\sqrt{t}} \right) +$$

$$+ \frac{\lambda^2}{4a^2} \int_0^t \exp\left(\frac{\lambda^2 (t-\tau)}{4a^2} \right) \cdot \left\{ \frac{1}{\sqrt{\tau}} e^{p_k/\tau} + \frac{\lambda\sqrt{\pi}}{2a} erfc\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}} \right) \right\} d\tau.$$

We rewrite the last function in the form

$$\varphi_k(t) = \frac{1}{\sqrt{t}} e^{p_k/t} + \frac{\lambda\sqrt{\pi}}{2a} erfc\left(\frac{\sqrt{-p_k}}{\sqrt{t}} \right) +$$

$$+ \frac{\lambda^2}{4a^2} \exp\left(\frac{\lambda^2 t}{4a^2} \right) \left\{ \int_0^t \exp\left(-\frac{\lambda^2}{4a^2}\tau + \frac{p_k}{\tau} \right) \frac{1}{\sqrt{\tau}} d\tau + \frac{\lambda\sqrt{\pi}}{2a} \int_0^t erfc\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}} \right) \cdot \exp\left(-\frac{\lambda^2}{4a^2}\tau \right) d\tau \right\}$$

or

$$\varphi_k(t)=\frac{1}{\sqrt{t}}e^{p_k/t}+\frac{\lambda\sqrt{\pi}}{2a}erfc\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right)+\frac{\lambda^2}{4a^2}\exp\left(\frac{\lambda^2 t}{4a^2}\right)\left\{I_1(t;\lambda)+\frac{\lambda\sqrt{\pi}}{2a}I_2(t;\lambda)\right\}, \quad (20)$$

where

$$I_1(t;\lambda)=\int_0^t \exp\left(-\frac{\lambda^2}{4a^2}\tau+\frac{p_k}{\tau}\right)\frac{1}{\sqrt{\tau}}d\tau;$$

$$I_2(t;\lambda)=\int_0^t erfc\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}}\right)\cdot\exp\left(-\frac{\lambda^2}{4a^2}\tau\right)d\tau.$$

After replacing $z=\sqrt{\tau}$ the integral $I_1(t;\lambda)$ can be written as

$$I_1(t;\lambda)=2\int_0^{\sqrt{t}}\exp\left(-\frac{\lambda^2}{4a^2}z^2+\frac{p_k}{z^2}\right)dz.$$

We compute the integral $I_2(t;\lambda)$ by parts:

$$\left\| \begin{matrix} u=erfc\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}}\right); & dv=\exp\left(-\frac{\lambda^2}{4a^2}\tau\right)d\tau; \\ du=\frac{\sqrt{-p_k}}{\sqrt{\pi}\,\tau^{3/2}}e^{p_k/\tau}d\tau; & v=-\frac{4a^2}{\lambda^2}\exp\left(-\frac{\lambda^2}{4a^2}\tau\right) \end{matrix} \right\|.$$

Then

$$I_2(t;\lambda)=-\frac{4a^2}{\lambda^2}\exp\left(-\frac{\lambda^2}{4a^2}t\right)\cdot erfc\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right)+\frac{4a^2\sqrt{-p_k}}{\lambda^2\sqrt{\pi}}\int_0^t\exp\left(-\frac{\lambda^2}{4a^2}\tau+\frac{p_k}{\tau}\right)\frac{1}{\tau^{3/2}}d\tau.$$

After replacing $z=\sqrt{\tau}$ we obtain

$$I_2(t;\lambda)=-\frac{4a^2}{\lambda^2}\exp\left(-\frac{\lambda^2}{4a^2}t\right)\cdot erfc\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right)+\frac{8a^2\sqrt{-p_k}}{\lambda^2\sqrt{\pi}}\int_0^{\sqrt{t}}\exp\left(-\frac{\lambda^2}{4a^2}z^2+\frac{p_k}{z^2}\right)\frac{1}{z^2}dz.$$

After substituting the expressions obtained for $I_1(t;\lambda)$ and $I_2(t;\lambda)$ into (20) we have

$$\varphi_k(t)=\frac{1}{\sqrt{t}}e^{p_k/t}+\frac{\lambda\sqrt{\pi}}{2a}erfc\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right)+\frac{\lambda^2}{4a^2}\exp\left(\frac{\lambda^2 t}{4a^2}\right)\left\{2\int_0^{\sqrt{t}}\exp\left(-\frac{\lambda^2}{4a^2}z^2+\frac{p_k}{z^2}\right)dz+\right.$$

$$\left.+\frac{\lambda\sqrt{\pi}}{2a}\left[-\frac{4a^2}{\lambda^2}\exp\left(-\frac{\lambda^2}{4a^2}t\right)\cdot erfc\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right)+\frac{8a^2\sqrt{-p_k}}{\lambda^2\sqrt{\pi}}\int_0^{\sqrt{t}}\exp\left(-\frac{\lambda^2}{4a^2}z^2+\frac{p_k}{z^2}\right)\frac{1}{z^2}dz\right]\right\}.$$

After some simple transformations we obtain

$$\varphi_k(t)=\frac{1}{\sqrt{t}}e^{p_k/t}+\frac{\lambda^2}{2a^2}\exp\left(\frac{\lambda^2 t}{4a^2}\right)\int_0^{\sqrt{t}}\exp\left(-\frac{\lambda^2}{4a^2}z^2+\frac{p_k}{z^2}\right)\left(1+\frac{2a\sqrt{-p_k}}{\lambda z^2}\right)dz=$$

$$=\frac{1}{\sqrt{t}}e^{p_k/t}+\frac{\lambda}{a}\exp\left(\frac{\lambda^2 t}{4a^2}\right)\int_0^{\sqrt{t}}\exp\left(-\frac{\lambda^2}{4a^2}z^2+\frac{p_k}{z^2}\right)\left(\frac{\lambda}{2a}+\frac{\sqrt{-p_k}}{z^2}\right)dz=$$

$$=\frac{1}{\sqrt{t}}e^{p_k/t}-\frac{\lambda}{a}\exp\left(\frac{\lambda^2 t}{4a^2}\right)\int_0^{\sqrt{t}}\exp\left(\frac{p_k}{z^2}-\frac{\lambda^2}{4a^2}z^2\right)d\left(\frac{\sqrt{-p_k}}{z}-\frac{\lambda}{2a}z\right).$$

After the introduction of replacement $\xi = \frac{\sqrt{-p_k}}{z} - \frac{\lambda}{2a} z$ we obtain

$$\varphi_k(t) = \frac{1}{\sqrt{t}} e^{p_k/t} + \frac{\lambda}{a} \exp\left(\frac{\lambda^2 t}{4a^2} - \frac{\sqrt{-p_k}}{a} \right) \int\limits_{\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}}^{+\infty} \exp\left(-\xi^2\right) d\xi =$$

$$= \frac{1}{\sqrt{t}} e^{p_k/t} + \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\lambda^2 t}{4a^2} - \frac{\sqrt{-p_k}}{a} \right) \cdot erfc\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}} \right).$$

Thus, the function

$$\varphi_k(t) = \frac{1}{\sqrt{t}} e^{p_k/t} + \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\lambda^2 t}{4a^2} - \frac{\sqrt{-p_k}}{a} \right) \cdot erfc\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}} \right) \qquad (21)$$

is an eigenfunction of «simplified» equation (3) for each $k; \quad -N_1 \le k \le N_2$,
where

$$N_1 = \left[\frac{\ln|\lambda| + \arg\lambda}{2\pi} \right], \quad N_2 = \left[\frac{\ln|\lambda| - \arg\lambda}{2\pi} \right].$$

Then, the function

$$\varphi(t) = \sum_{k=-N_1}^{N_2} C_k \varphi_k(t) \qquad (22)$$

is a solution of Abel equation (18) for $f_2(t) = 0$, that isa solution of «simplified» homogeneous equation (3), and the functions $\varphi_k(t)$ and values p_k are determined by formulas (21) and (12) respectively.

We note that after multiplying equality (22) by $\exp\left(-\frac{t}{4a^2} \right)$, we obtain the solution of the homogeneous equation corresponding to original equation (1)

$$\varphi(t) = \sum_{k=-N_1}^{N_2} C_k \cdot \left\{ \frac{1}{\sqrt{t}} \exp\left(-\frac{p_k}{t} - \frac{t}{4a^2} \right) + \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\left(\lambda^2 - 1\right)t}{4a^2} - \frac{\sqrt{-p_k}}{a} \right) \cdot erfc\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}} \right) \right\}.$$

Thus, the following theorem holds

Theorem. Inhomogeneous equation (1) is solvable for any function $f(t)$: $\sqrt{t} f(t) \in L(0;\infty) \cap C(0;\infty)$. The corresponding homo geneouse quation has $\left(N_1 + N_2 + 1\right)$ eigen functions

$$\varphi_k(t) = \frac{1}{\sqrt{t}} \exp\left(-\frac{p_k}{t} - \frac{t}{4a^2} \right) + \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\left(\lambda^2 - 1\right)t}{4a^2} - \frac{\sqrt{-p_k}}{a} \right) \cdot erfc\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}} \right)$$

and the general solution of equation (1) can be written as

$$\varphi(t) = F(t) + \frac{\lambda^2}{4a^2} \int_0^t \exp\left(\frac{\lambda^2 (t-\tau)}{4a^2} \right) F(\tau) d\tau + \sum_{k=-N_1}^{N_2} C_k \varphi_k(t),$$

where

$$F(t) = f_2(t) - \frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{f_2(\tau)}{\sqrt{t-\tau}} d\tau,$$

and the function $f_2(t) = f(t) + \lambda \int_0^t r\left(t,\tau\right) f(\tau) d\tau$, where $r(t,\tau)$ is defined by formula (15).

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«Қысылмайтын» ядросы бар екінші текті Вольтерраның бір теңдеуі жайында

Мақалада ядросының «сығылмауына» байланысты шешімдердің дәстүрлі әдістерін қолдану мүмкін емес екінші текті Вольтерра ерекше интегралдық теңдеуі қарастырылған. Сәйкес біртекті теңдеуінде $|\lambda|>1$ болғанда тұтас спектр болатыны, сонымен қатар $|\lambda|$ жоғарлағанда сипаттамалық сандарының еселенуі де өсетіні көрсетілді. Теңдеу Карлеман-Векуа әдісінің көмегімен Абель теңдеуіне әкелінді. Теңдеудің меншікті функциялары айқын түрде табылды.

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Об одном уравнении Вольтерра второго рода с «несжимаемым» ядром

В статье рассмотрено особое интегральное уравнение Вольтерра второго рода, к которому в силу «несжимаемости» ядра не применимы классические методы решения. Показано, что соответствующее однородное уравнение при $|\lambda|>1$ имеет сплошной спектр, причем кратность характеристических чисел растет с возрастанием $|\lambda|$. Уравнение методом Карлемана-Векуа сведено к уравнению Абеля. Собственные функции уравнения найдены в явном виде.

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Integral transforms and boundary value problems

This article is devoted to determine the solution of the none stationary heat conduction equation for unlimited space and to investigate the two-dimensional Helmholtz equation. The solutions of the considered boundary value problems are obtained with the use of the mixed Fourier transform and of the double Fourier transform. From these line items in work it is illustrated how the integral transforms method can be used to obtain the solution of boundary value problems for partial differential equations of different kinds. In addition, the Green's function is built for the two-dimensional Poisson equation in this article.

Key words: a heat conduction equation, the two-dimensional Helmholtz equation, the two-dimensional Poisson equation, the mixed Fourier transform, the double Fourier transform, a Green's function.

Many boundary value problems in applied mathematics, mathematical physics, and engineering science can be effectively solved by the use of the Fourier transform, the Fourier cosine transform, or the Fourier sine transform. These transforms are very useful for solving partial differential or integral equations for the following reasons. First, these equations are replaced by ordinary differential equations, which enable us to find the solution of the transform function. The solution of the given equation is then obtained in the original variables by inverting the transform solution. Second, the Fourier transform of the elementary source term is used for determination of the fundamental solution that illustrates the basic ideas behind the construction and implementation of Green's functions. Third, the transform solution combined with the convolution theorem provides an elegant representation of the solution for the boundary value and initial value problems [1].

The boundary value problem for $u(x,y,z,t)$ satisfies the following heat conduction equation and boundary conditions

$$u_t = a^2\Delta_3 u, \quad -\infty < x, y < +\infty, \quad 0 < z, t < +\infty; \tag{1}$$

$$u\big|_{z=0} = f(x,y,t), \quad u\big|_{t=0} = g(x,y,z). \tag{2}$$

We use the mixed Fourier transform [2] defined by (3)

$$\tilde{u}(\lambda,\mu,\nu,t) = \frac{1}{\sqrt{2\pi^3}} \int\int_{-\infty}^{+\infty} e^{i[\lambda x+\mu y]} dxdy \int_0^{+\infty} u(x,y,z,t)\sin\nu z dz, \tag{3}$$

to the problem (1), (2) which reduces to

$$\begin{cases} \tilde{u}_t + a^2(\lambda^2+\mu^2+\nu^2)\tilde{u} = \sqrt{\dfrac{2}{\pi}} a^2 \nu F(\lambda,\mu,t); \\ \tilde{u}(\lambda,\mu,\nu,0) = G(\lambda,\mu,\nu). \end{cases} \tag{4}$$

Thus, this transformed problem (4) is solved to obtain

$$\tilde{u}(\lambda,\mu,\nu,t) = G(\lambda,\mu,\nu)e^{-a^2(\lambda^2+\mu^2+\nu^2)t} + \sqrt{\frac{2}{\pi}} a^2\nu \int_0^t F(\lambda,\mu,\tau) e^{-a^2(\lambda^2+\mu^2+\nu^2)(t-\tau)} d\tau = \tilde{u}_1 + \tilde{u}_2. \tag{5}$$