

References

1. Soldatov A.P., Ramazanov M.I., Shaldykova B.A. About the boundary value problems for the spectrally-loaded parabolic operator. I // Vestnik of KarSU. Mathematics series. — 2011. — № 2 (61). — P. 10–18.
2. Dzhenaliev M.T., Ramazanov M.I. On the boundary value problem for the spectrally-loaded heat conduction operator // Siberian Mathematical Journal. — 2006. — Vol. 47. — № 3. — P. 527–547.
3. Dzhenaliev M.T., Ramazanov M.I. On a boundary value problem for a spectrally-loaded heat operator. I // Siberian Mathematical Journal. — 2007. — Vol. 43. — № 4. — P. 498–508.
4. Amangaliyeva M.M., Ahmanova D.M., Dzhenaliev M.T., Ramazanov M.I. The boundary value problems for the spectrally-loaded heat conduction operator with approaching loaded line in zero or infinity // Different. equations. — 2011. — Vol. 47. — № 2. — P. 231–243.
5. Dzhenaliev M.T., Shaldykova B.A., Kusainova B.S. Sliding boundary problem for spectrally-loaded heat conduction operator. I // Vestnik of KarSU. Mathematics series. — 2010. — № 1 (57). — P. 12–19.
6. Shaldykova B.A. Sliding boundary problem for spectrally-loaded heat conduction operator. II // Vestnik of KarSU. Mathematics series. — 2010. — № 2 (58). — P. 75–82.

УДК 517.62

Approximated solution of differential equations with vertical asymptotes**Тік асимптотасы бар дифференциалдық теңдеулердің жуықталған шешімі**

Uxikbayev K.B.

Suleyman Demirel University, Almaty (E-mail: kanamath@gmail.com)

Мақалада тік асимптотасы болатын дифференциалды теңдеулердің жуықталған шешімін табу әдісі қаралады. Бірінші дәрежелі дифференциалдық теңдеулер үшін Коши есебінің шешімінің максималды болу интервалын бағалау және ескеру үшін бір қадамдық сандық әдістердің өзгертілген түрі ұсынылған. Сонымен қатар ол әдістің жинақтылығы және ауытқу шамасы қарастырылған.

В статье рассматривается метод нахождения приближенного решения дифференциальных уравнений, имеющих вертикальные асимптоты. Предлагается модификация одношаговых численных методов, позволяющая оценивать и учитывать максимальный интервал существования решения задачи Коши для обыкновенных дифференциальных уравнений с тем, чтобы не строить лишние смысла «приближенные решения» за границами этого промежутка, если он конечен. Рассмотрены сходимость предлагаемого метода и погрешность.

Application of step-by-step methods of the solution of a problem of Cauchy for the ordinary differential equation

$$y' = f(x, y), x \geq 0; \quad (1)$$

$$y(0) = \alpha \quad (2)$$

meets serious difficulties if the solution at $y(x)$ is not continuous on all numerical axis.

Really, customary definition of the solution as x , value forces to choose argument functions as a step. Calculations with such step do not allow to "notice", for example, a vertical asymptote of the solution.

In this study I offer modified version of the single-step methods which allows to estimate and consider the maximum interval of existence of the solution of a problem of Cauchy for the ordinary differential equations [1–6].

This modification is based on geometrical idea of consideration of the solution as a curve on a plane Oxy. At such point of view as a step it is obvious to choose the distance between points $(x_i, y_i), (x_{i+1}, y_{i+1})$, which approximates the solution.

Let's apply this idea to Euler's method described by formulas $x_{i+1} = x_i + h, y_{i+1} = y_i + h_i f(x_i, y_i)$. As here the integrated curve is replaced with a straight line as constant step H we will choose distance between points $(x_i, y_i), (x_{i+1}, y_{i+1})$.

$$H = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} = h_i \sqrt{1 + f^2(x_i, y_i)}.$$

From here, $h_i = \frac{H}{\sqrt{1 + f^2(x_i, y_i)}}$. Thus, Euler's method can be written in such type:

$$x_{i+1} = x_i + \frac{H}{\sqrt{1 + f^2(x_i, y_i)}}, \quad y_{i+1} = y_i + \frac{Hf(x_i, y_i)}{\sqrt{1 + f^2(x_i, y_i)}}. \quad (3)$$

Let's result conditions of an extremity of the maximum interval of existence of the solution of a problem (1), (2) and we will find out behavior under these conditions of the approached solution constructed under formulas (3). The interval $[0, b)$ is considered as the maximum interval of existence of the solution $y(x)$, if $b = \infty$ or if there is no final limit $\lim_{x \rightarrow b^-} y(x)$. The corresponding solution $y(x)$ defined on $[0, b)$, is called as full. Let's note that continuity of f enough for existence of the full solution and continuity of any solution on the maximum interval.

THEOREM 1. Let $\delta > 0$ and there exist such positive numbers A, δ , which $\forall x \geq 0, y \geq \alpha$ fulfill the inequality:

$$f(x, y) \geq Ay^{1+\delta}. \quad (4)$$

Then: If there exist full solution $y(x), x \in [0, b)$, of problem (1), (2),

$$b \leq \frac{1}{A\delta\alpha^\delta} < \infty, \quad \lim_{x \rightarrow b^-} y(x) = \infty;$$

For the approximated solution constructed under formulas (3), limiting parities take place $\lim_{i \rightarrow \infty} x_i < \infty$, $\lim_{i \rightarrow \infty} y_i = \infty$.

Theorem 1 indicates the "high-quality affinity" between the approximate and exact solutions of (1), (2). To study the convergence of the proposed method is convenient to replace the problem (1), (2), the Cauchy problem for a system of two equations:

$$\begin{cases} y'_i = \frac{f(x, y)}{\sqrt{1 + f^2(x, y)}}; \\ x'_i = \frac{1}{\sqrt{1 + f^2(x, y)}}; \end{cases} \quad (5)$$

$$x(0) = 0, y(0) = \alpha. \quad (6)$$

Solution of this system are the functions $x = x(t), y = y(t)$, defining the parametric equations of the integral curve of (1), (2). Formula (3) are the formulas of Euler's method solution to problem (5), (6) with the step H of the parameter t . Require the following statement about the continuous dependence of solutions $(x(t), y(t))$ to (5) of the initial conditions. Choose an arbitrary number $T > 0, \beta, \gamma$. Denote by $(\bar{x}(t), \bar{y}(t))$ is the solution to (5), for which $\bar{x}(T) = x(T) + \beta, \bar{y}(T) = y(T) + \beta$.

THEOREM 2. Let for $x \geq 0, y \geq \alpha > 0$, function $f(x, y)$ satisfies the inequality (4) and has continuous partial derivatives, and $\exists B > 0$, such that

$$|f'_x(x, y)| \leq Bf(x, y), \quad |f'_y(x, y)| \leq Bf(x, y). \quad (7)$$

Then there exist such decreasing function $N(E)$, known for $E \in [\alpha, \infty)$, which for any $x(T), y(T), \beta, \gamma$ from inequalities $x(T) \geq 0, x(T) + \beta \geq 0, y(T) \geq E, y(T) + \beta \geq E$ implies inequality

$$\sup | \bar{x}(t) - x(t), \bar{y}(t) - y(t) | \leq N(E) |\beta, \gamma|. \quad (8)$$

Theorem 3 estimates the accuracy of solution taken from formulas (3).

THEOREM 3. Let function $f(x, y)$ satisfies the conditions of theorem 2. Then $\exists M > 0$, such for the solution $(x(t), y(t))$ of problem (5), (6) and it's approximation constructed by formulas (3), the inequality $\sup_{i=1,2,\dots} |(x_i - x(iH))|_{R^2} \leq MH^2$ is true.

Example: Let's consider a problem

$$\begin{cases} y' = y^2 + 1; \\ y(0) = 0. \end{cases} \quad (9)$$

Here, $f(x, y) = y^2 + 1$, let $H = 0.1$; $x_0 = 0, y_0 = 0, f_0 = 0$. Let's use formulas (3), so, we have step 1:

$$x_1 = 0 + \frac{0.1}{\sqrt{1+1^2}} = \frac{0.1}{\sqrt{2}} \approx 0.07,$$

$$y_1 = 0 + \frac{0.1 \cdot 1}{\sqrt{2}} \approx 0.07;$$

$$f_1 = 1 + 0.07^2 \approx 1.0049;$$

step 2:

$$x_2 = 0.07 + \frac{0.1}{\sqrt{1+1.0049^2}} \approx 0.141;$$

$$y_2 = 0.07 + \frac{0.1 \cdot 1.0049}{\sqrt{1+1.0049^2}} \approx 0.142;$$

$$f_2 = 1 + 0.142^2 \approx 1.019;$$

step 3:

$$x_3 = 0.141 + \frac{0.1}{\sqrt{1+1.019^2}} \approx 0.210;$$

$$y_3 = 0.142 + \frac{0.1 \cdot 1.019}{\sqrt{1+1.019^2}} \approx 0.212;$$

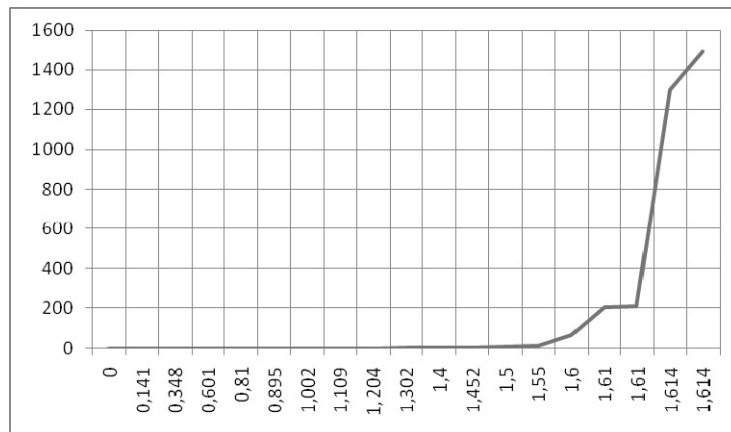
$$f_3 = 1 + 0.212^2 \approx 1.044.$$

The further calculations are similar. All value x, y are shown in the table

Table

Steps	Approximated x	Approximated y	$y(x_i)$ -exact solution
0	0	0	0
2	0,141	0,142	0,142
5	0,348	0,359	0,363
9	0,601	0,669	0,686
13	0,810	1,009	1,051
15	0,895	1,190	1,247
19	1,002	1,471	1,564
23	1,109	1,856	2,009
28	1,204	2,346	2,603
36	1,302	3,140	3,630
51	1,400	4,637	5,798
66	1,452	6,136	8,378
92	1,500	8,736	14,101
160	1,550	15,435	49,078
660	1,600	65,535	-
2095	1,610	209,035	-
2101	1,610	209,635	-
12965	1,614	1296,035	-
14905	1,614	1489,935	-

This is the graph of the table shown above.



The exact solution of a problem (9) is function $y = \tan x$. Its vertical asymptote is $x = \frac{\pi}{2}$. As we see, we also have found an asymptote by the approximated calculation.

References

1. *Hartman F.* Ordinary differential equations. — M., 1970.
2. *Kamke E.* The guide on the ordinary differential equations. — M.: Science, 1976.
3. *Bahvalov N.S.* Numerical methods (the analysis, algebra, the ordinary differential equations): The main edition of the physical and mathematical literature of publishing house. — M.: Science, 1975.
4. *Gavrilov N.I.* Methods, theories of the ordinary differential equations. The state publishing house. — M.: Science, Higher school, 1962.
5. *Pak V.V., Nosenko Y.L.* Calculus: Textbook. — D.: Stalker, 1997.
6. *Zaguskin V.L.* The guide on numerical methods of the solution of the equations // M.: Fizmatgiz, 1960. — 216 p.