

## TRIGONOMETRIC WIDTHS OF THE NIKOL'SKII–BESOV CLASSES IN THE LEBESGUE SPACE WITH MIXED NORM

G. Akishev

UDC 517.51

We establish exact-order estimates for the trigonometric widths of the Nikol'skii–Besov classes of periodic functions of many variables in the Lebesgue space with mixed norm.

### Introduction

Let  $\bar{x} = (x_1, \dots, x_m) \in \mathbb{T}^m = [0, 2\pi)^m$ . By  $L_{\bar{p}}(\mathbb{T}^m)$ , we denote a space of Lebesgue-measurable functions  $f(\bar{x})$  defined on  $\mathbb{R}^m$ ,  $2\pi$ -periodic in each variable, and such that

$$\|f\|_{\bar{p}} = \left[ \int_0^{2\pi} \left[ \dots \left[ \int_0^{2\pi} |f(\bar{x})|^{p_1} dx_1 \right]^{p_2/p_1} \dots \left[ \int_0^{2\pi} |f(\bar{x})|^{p_m/(p_{m-1})} dx_m \right]^{1/p_m} \right] < \infty,$$

where  $\bar{p} = (p_1, \dots, p_m)$ ,  $1 \leq p_j < \infty$ ,  $j = 1, \dots, m$  (see [1, p. 128]). In the case where  $p_1 = p, \dots, p_m = p$ , instead of  $\|\cdot\|_{\bar{p}}$ ,  $L_{\bar{p}}(\mathbb{T}^m)$  and  $B_{\bar{p},\theta}^r$ , we use the notation  $\|\cdot\|_p$ ,  $L_p(\mathbb{T}^m)$  and  $B_{p,\theta}^r$ .

The function  $f \in L_1(\mathbb{T}^m) = L(\mathbb{T}^m)$  is expanded in the Fourier series

$$\sum_{\bar{n} \in \mathbb{Z}^m} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle},$$

where  $\langle \bar{n}, \bar{x} \rangle = \sum_{j=1}^m n_j x_j$ ,  $a_{\bar{n}}(f)$  are the Fourier coefficients of the function  $f \in L_1(\mathbb{T}^m)$  in the multiple trigonometric system  $\{e^{i\langle \bar{n}, \bar{x} \rangle}\}_{\bar{n} \in \mathbb{Z}^m}$ , and  $\mathbb{Z}^m$  is a space of points from  $\mathbb{R}^m$  with integer-valued coordinates.

For the function  $f \in L(\mathbb{T}^m)$  and number  $s \in \mathbb{Z}_+$ , we set

$$\sigma_s(f, \bar{x}) = \sum_{\bar{n} \in \rho(s)} a_{\bar{n}}(f) e^{i\langle \bar{n}, \bar{x} \rangle},$$

$$\rho(s) = \left\{ \bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : [2^{s-1}] \leq \max_{j=1, \dots, m} |k_j| < 2^s \right\},$$

where  $[a]$  is the integer part of the number  $a$ .

We define the Nikol'skii–Besov classes [1, 2] as follows: Let  $1 < p_j < \infty$ ,  $j = 1, \dots, m$ ,  $1 \leq \theta \leq \infty$ , and  $r > 0$ . Then,

Karaganda State University, Karaganda, Kazakhstan.

Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 66, No. 6, pp. 723–732, June, 2014. Original article submitted October 27, 2012; revision submitted April 1, 2014.

$$B_{\bar{p},\theta}^r = \left\{ f \in L_{\bar{p}}(\mathbb{T}^m) : \left( \sum_{s \in \mathbb{Z}_+} 2^{sr\theta} \|\sigma_s(f)\|_{\bar{p}}^\theta \right)^{1/\theta} \leq 1 \right\},$$

$$H_{\bar{p}}^r = \left\{ f \in L_{\bar{p}}(\mathbb{T}^m) : \sup_{s \in \mathbb{Z}_+} 2^{sr} \|\sigma_s(f)\|_{\bar{p}} \leq 1 \right\}.$$

It is known that, for  $1 < \theta_1 < \theta_2 < \infty$ , the following imbeddings are true:

$$B_{\bar{p},1}^r \subset B_{\bar{p},\theta_1}^r \subset B_{\bar{p},\theta_2}^r \subset B_{\bar{p},\infty}^r = H_{\bar{p}}^r. \quad (1)$$

Consider a given class  $F \subset L_{\bar{p}}(\mathbb{T}^m)$ . A trigonometric  $n$ -width  $d_n^T(F, L_{\bar{p}})$  of the class  $F$  in the space  $L_{\bar{p}}(\mathbb{T}^m)$  is defined as

$$d_n^T(F, L_{\bar{p}}) = \inf_{\Omega_n} \sup_{f \in F} \inf_{t \in t(\Omega_n)} \|f - t(\Omega_n)\|_{\bar{p}}, \quad (2)$$

where

$$t(\Omega_n, \bar{x}) = \sum_{j=1}^n c_j e^{i(\bar{k}^{(j)}, \bar{x})},$$

$\Omega_n = \{\bar{k}^{(1)}, \bar{k}^{(2)}, \dots, \bar{k}^{(n)}\}$  is a family of vectors  $\bar{k}^{(j)} = (k_1^{(j)}, \dots, k_m^{(j)})$ ,  $j = 1, \dots, n$ , with integer-valued coordinates, and  $c_j$  are arbitrary numbers.

For the first time, the notion of a trigonometric width in the one-dimensional case was introduced by Nikol'skii in [3] who established estimates for some classes in the space of continuous functions. For functions of many variables, the trigonometric widths of the Sobolev classes  $W_p^{\bar{r}}$  and the Nikol'skii classes  $H_p^{\bar{r}}$  were investigated by Bugrov [4], Belinskii [5, 6], Maiorov [7], Magaril-II'yaev [8], and Temlyakov [9]. For the Besov classes  $B_{p,\theta}^{\bar{r}}$ , these widths were studied by Romanyuk [10]. For the generalized Nikol'skii–Besov classes, the trigonometric widths were investigated by Stasyuk [11, 12] and Bazarkhanov [13].

For the class  $B_{p,\theta}^r$ , where  $r$ ,  $p$ , and  $\theta$  are certain numerical parameters, the following theorem is proved in [14]:

**Theorem [14].** *Let  $1 \leq p < 2 \leq q < \frac{p}{p-1}$ ,  $1 \leq \theta \leq \infty$ , and  $r > m$ . Then*

$$d_n^T(B_{p,\theta}^r, L_q) \asymp n^{-\frac{r}{m} + \frac{1}{p} - \frac{1}{2}}.$$

The aim of the present paper is to determine the exact order of the trigonometric width of the class  $B_{\bar{p},\theta}^r$  in the space  $L_{\bar{q}}(\mathbb{T}^m)$ .

In the case where inequality  $B \geq C_1 A$  (or  $B \leq C_2 A$ ) is true, we often write  $B \gg A$  (or  $B \ll A$ , respectively). The notation  $A \asymp B$  means that  $A \ll B$  and  $B \ll A$ .

### Auxiliary Statements

Let  $f \in L_{\bar{p}}(\mathbb{T}^m)$  and let  $\{\bar{k}^{(j)}\}_{j=1}^M$  be a system of vectors  $\bar{k}^{(j)} = (k_1^{(j)}, \dots, k_m^{(j)})$  with integer-valued coordinates. Consider a quantity

$$e_M(f)_{\bar{p}} = \inf_{\bar{k}^{(j)}, b_j} \left\| f - \sum_{j=1}^M b_j e^{i\langle \bar{k}^{(j)}, \bar{x} \rangle} \right\|_{\bar{p}},$$

where  $b_j$  are arbitrary numbers. The quantity  $e_M(f)_{\bar{p}}$  is called the best  $M$ -term trigonometric approximation of the function  $f \in L_{\bar{p}}(\mathbb{T}^m)$ . For a given class  $F \subset L_{\bar{p}}(\mathbb{T}^m)$ , we set

$$e_M(F)_{\bar{p}} = \sup_{f \in F} e_M(f)_{\bar{p}}. \tag{3}$$

According to definitions (2) and (3), the quantities  $d_M^T(F, L_{\bar{p}})$  and  $e_M(F)_{\bar{p}}$  are related by the inequality

$$e_M(F)_{\bar{p}} \leq d_M^T(F)_{\bar{p}}. \tag{4}$$

To get a lower bound for the trigonometric width of the class  $B_{\bar{p}, \theta}^r$ , we need the following assertion:

**Theorem 1** [15]. Let  $\bar{p} = (p_1, \dots, p_m)$ ,  $\bar{q} = (q_1, \dots, q_m)$ ,  $1 < p_j \leq 2 < q_j < \infty$ ,  $j = 1, \dots, m$ ,  $1 \leq \theta \leq \infty$ . If

$$r > \sum_{j=1}^m \frac{1}{p_j},$$

then

$$e_M(B_{\bar{p}, \theta}^r)_{\bar{q}} \asymp M^{-\frac{1}{m}(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2}))}.$$

**Proof.** We restrict ourselves to finding the lower bound frequently is used in what follows. To do this, we need the relation (see [16, p. 25])

$$e_M(f)_{\bar{q}} = \inf_{\Omega_M} \sup_{P \in L^\perp, \|P\|_{\bar{q}'} \leq 1} \left| \int_{\mathbb{T}^m} f(\bar{x}) \bar{P}(\bar{x}) d\bar{x} \right|, \tag{5}$$

where  $\bar{q}' = (q'_1, \dots, q'_m)$ ,  $\frac{1}{q_j} + \frac{1}{q'_j} = 1$ ,  $j = 1, \dots, m$ ,  $L^\perp$  is the set of functions orthogonal to a subspace of trigonometric polynomials with numbers of harmonics from the set  $\Omega_M$ .

Since the estimate for the quantity  $e_M(B_{\bar{p}, \theta}^r)_{\bar{q}}$  is independent of  $\theta$ , by virtue of (1), it suffices to establish the lower bound for  $B_{\bar{p}, 1}^r$ .

For a natural number  $M$ , we choose a number  $n \in \mathbb{N}$  such that  $M \asymp 2^{nm}$  and  $2M \leq \#\rho(n)$ , where  $\#\rho(n)$  is the number of elements of the set  $\rho(n)$ .

Consider a function

$$f_1(\bar{x}) = 2^{-n(r + \sum_{j=1}^m (1 - \frac{1}{p_j}))} \sum_{\bar{k} \in \rho(n)} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

Then  $\|\sigma_s(f_1)\|_{\bar{p}} = 0$  if  $s \neq n$  and

$$\|\sigma_n(f_1)\|_{\bar{p}} = 2^{-n(r + \sum_{j=1}^m (1 - \frac{1}{p_j}))} \prod_{j=1}^m \left\| \sum_{k_j=2^{n-1}}^{2^n-1} e^{ik_j x_j} \right\|_{p_j}.$$

By virtue of the estimate for the norm of the Dirichlet kernel (see [17, p. 181]), we obtain

$$\left\| \sum_{k_j=2^{n-1}}^{2^n-1} e^{ik_j x_j} \right\|_{p_j} \ll 2^{n(1-\frac{1}{p_j})}$$

for  $p_j \in (1, \infty)$ ,  $j = 1, \dots, m$ . Hence,

$$\|\sigma_n(f_1)\|_{\bar{p}} \ll 2^{-nr}$$

and, therefore,

$$\sum_{s=1}^{\infty} 2^{sr} \|\sigma_s(f_1)\|_{\bar{p}} \leq C_1,$$

i.e., the function  $C_1^{-1} f_1 \in B_{\bar{p},1}^r$ .

Further, we consider two functions

$$v_1(\bar{x}) = \sum_{\bar{k} \in \rho(n)} e^{i\langle \bar{k}, \bar{x} \rangle},$$

$$u_1(\bar{x}) = \sum_{\bar{k} \in \rho(n) \cap \Omega_M} e^{i\langle \bar{k}, \bar{x} \rangle}$$

and set  $w_1(\bar{x}) = v_1(\bar{x}) - u_1(\bar{x})$ . By virtue of the Parseval equality, we obtain

$$\|u_1\|_2 = (\pi)^m \left( \sum_{\bar{k} \in \rho(n) \cap \Omega_M} 1 \right)^{1/2} \ll M^{1/2},$$

$$\|v_1\|_2 = (\pi)^m \left( \sum_{\bar{k} \in \rho(n)} 1 \right)^{1/2} \ll 2^{nm/2}.$$

According to the property of the norm, these relations yield the following inequality:

$$\|w_1\|_2 \leq \|v_1\|_2 + \|u_1\|_2 \leq C_2 2^{nm/2}.$$

Hence, the function  $P_1(\bar{x}) = C_2^{-1} 2^{-nm/2} w_1(\bar{x})$  satisfies (5) for  $q_j = 2$ ,  $j = 1, \dots, m$ . Since  $2 < q_j$ ,  $j = 1, \dots, m$ , we have  $e_M(f_1)_2 \ll e_M(f_1)_{\bar{q}}$ . By using relation (5), we get

$$e_M(f_1)_{\bar{q}} \gg e_M(f_1)_2 \gg \inf_{\Omega_M} \int_{\mathbb{T}^m} f_1(\bar{x}) \bar{P}_1(\bar{x}) d\bar{x}$$

$$= C_2^{-1} 2^{-nm/2} 2^{-n(r + \sum_{j=1}^m (1 - \frac{1}{p_j}))} \inf_{\Omega_M} [\#\rho(n) - \#(\rho(n) \cap \Omega_M)]$$

$$\begin{aligned} &\gg 2^{-nm/2} 2^{-n(r+\sum_{j=1}^m(1-\frac{1}{p_j}))} [\#\rho(n) - M] \\ &\gg 2^{-nm/2} 2^{-n(r+\sum_{j=1}^m(1-\frac{1}{p_j}))} \left[ \#\rho(n) - \frac{\#\rho(n)}{2} \right] \gg 2^{-nm/2} 2^{-n(r-\sum_{j=1}^m \frac{1}{p_j})}. \end{aligned}$$

In view of relations (1), the last inequalities, and the fact that  $2^{nm} \asymp M$ , we find

$$e_M(B_{\bar{p},\theta}^r)_q \geq e_M(B_{\bar{p},1}^r)_q \gg e_M(f_1)_q \gg M^{-\frac{1}{m}(r-\sum_{j=1}^m(\frac{1}{p_j}-\frac{1}{2}))}.$$

Theorem 1 is proved.

**Remark 1.** In the case where  $p_1 = \dots = p_m = p$  and  $q_1 = \dots = q_m = q$ , Theorem 1 is proved in [18]. The estimates for the quantity  $e_M(B_{p,\theta}^r)_q$  in the case

$$\frac{m}{p} - \frac{m}{q} < r < \frac{m}{p}$$

are established in [19]. In the one-dimensional case, Theorem 1 is proved in [6]. Moreover, Theorem 1 is presented in [15] for the other relations between the parameters  $p_j$  and  $q_j$ ,  $j = 1, \dots, m$ .

**Theorem B** [20]. Let  $\bar{n} = (n_1, \dots, n_m)$ ,  $n_j \in \mathbb{N}$ ,  $j = 1, \dots, m$ , and let

$$T_{\bar{n}}(\bar{x}) = \sum_{|k_j| \leq n_j, j=1, \dots, m} c_{\bar{k}} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

Then, for  $1 \leq p_j < q_j \leq \infty$ ,  $j = 1, \dots, m$ , the inequality

$$\|T_{\bar{n}}\|_{\bar{q}} \leq 2^m \prod_{j=1}^m n_j^{1/p_j-1/q_j} \|T_{\bar{n}}\|_{\bar{p}}$$

is true.

Let  $\Omega_M$  be a set that contains at most  $M$  vectors  $\bar{k} = (k_1, \dots, k_m)$  with integer-valued coordinates. The following lemma is true:

**Lemma A** [21]. Let  $2 \leq q < \infty$ . Then, for any trigonometric polynomial

$$P(\Omega_M, \bar{x}) = \sum_{j=1}^M e^{i\langle \bar{k}^{(j)}, \bar{x} \rangle}$$

and any natural number  $N \leq M$ , there exists a trigonometric polynomial  $P(\Omega_N, \bar{x})$  containing at most  $N$  harmonics such that

$$\|P(\Omega_M) - P(\Omega_N)\|_q \ll MN^{-1/2}.$$

Furthermore,  $\Omega_N \subset \Omega_M$ , all coefficients  $P(\Omega_N, \bar{x})$  are equal, and their absolute value does not exceed  $MN^{-1}$ .

## Main Results

First, we prove an auxiliary statement necessary in what follows in the proof of the main results. Consider a trigonometric polynomial

$$t_s(\bar{x}) = \sum_{\bar{k} \in \rho(s)} e^{i\langle \bar{k}, \bar{x} \rangle}.$$

Let  $t(\Omega_{n_s}; \bar{x})$  be a trigonometric polynomial approximating the polynomial  $t_s(\bar{x})$  according to Lemma A, i.e.,

$$\|t_s - t(\Omega_{n_s})\|_{\bar{q}} \leq \|t_s - t(\Omega_{n_s})\|_{\bar{q}} \ll 2^{sm} n_s^{-1/2}, \quad (6)$$

where  $\bar{q} = \max\{q_j : j = 1, \dots, m\}$ ,  $\Omega_{n_s} \subset \rho(s)$ , all coefficients of the polynomial  $t(\Omega_{n_s}; \bar{x})$  are equal, and their absolute value does not exceed  $2^{sm} n_s^{-1}$ .

Consider an operator  $T_s$  of the form

$$T_s f(\bar{x}) = f(\bar{x}) * (t_s(\bar{x}) - t(\Omega_{n_s}, \bar{x})),$$

where the symbol  $*$  denotes the operation of convolution of two functions, i.e.,

$$(\varphi * g)(\bar{x}) := (2\pi)^{-m} \int_{\mathbb{T}^m} \varphi(\bar{y}) g(\bar{x} - \bar{y}) d\bar{y}$$

for  $\varphi, g \in L(\mathbb{T}^m)$ .

The following statement is true:

**Lemma 1.** *Let  $1 < p_j < 2 < q_j < \frac{p_j}{p_j - 1} = p'_j$ ,  $j = 1, \dots, m$ . Then the norm of the operator  $T_s$  acting from  $L_{\bar{p}}(\mathbb{T}^m)$  into  $L_{\bar{q}}(\mathbb{T}^m)$  satisfies the inequality*

$$\|T_s\|_{\bar{p} \rightarrow \bar{q}} = \sup_{\|f\|_{\bar{p}} \leq 1} \|T_s f\|_{\bar{q}} \ll 2^{sm} n_s^{-\frac{1}{m} \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})}.$$

**Proof.** According to an analog of the Riesz–Thorin theorem in the Lebesgue space with mixed norm (see [22]), we can write

$$\|T_s\|_{\bar{p} \rightarrow \bar{q}} \leq \|T_s\|_{2 \rightarrow 2}^{1-\lambda} \|T_s\|_{1 \rightarrow \bar{q}^*}^{\lambda}, \quad (7)$$

where  $0 < \lambda < 1$  and the coordinates  $\bar{q}^* = (q_1^*, \dots, q_m^*)$  satisfy the equality

$$\frac{1}{q_j} = \frac{1-\lambda}{2} + \frac{\lambda}{q_j^*}, \quad j = 1, \dots, m.$$

We choose

$$\lambda = \frac{2}{m} \sum_{j=1}^m \left( \frac{1}{p_j} - \frac{1}{2} \right).$$

The coefficients of the polynomial  $t_s(\bar{x}) - t(\Omega_{n_s}, \bar{x})$  are equal and their absolute values do not exceed  $2^{(s+1)m}n_s^{-1} + 1$ . Hence, by virtue of the Parseval equality, we obtain

$$\|T_s\|_{2 \rightarrow 2} \ll 2^{sm}n_s^{-1}. \tag{8}$$

Further, by using the generalized Minkowski inequality (see [1, p. 27]) and (6), we get

$$\|T_s f\|_{\bar{q}^*} \leq \|f\|_1 \|t_s - t(\Omega_{n_s})\|_{\bar{q}^*} \ll \|f\|_1 2^{sm}n_s^{-1/2}.$$

Therefore,

$$\|T_s\|_{1 \rightarrow \bar{q}^*} \ll 2^{sm}n_s^{-1/2}. \tag{9}$$

Substituting (8) and (9) in (7), we obtain

$$\|T_s\|_{\bar{p} \rightarrow \bar{q}} \ll (2^{sm}n_s^{-1})^{1-\lambda} (2^{sm}n_s^{-1/2})^\lambda = C 2^{sm}n_s^{\frac{\lambda}{2}-1} = C 2^{sm}n_s^{-\frac{1}{m} \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})}.$$

Lemma 1 is proved.

**Remark 2.** In the case where  $p_1 = \dots = p_m = p$  and  $q_1 = \dots = q_m = q$ , Lemma 1 was proved in [14].

We now formulate and prove the main result of the present paper.

**Theorem 2.** Let  $1 < p_j < 2 \leq q_j < \frac{p_j}{p_j - 1}$ ,  $j = 1, \dots, m$ ,  $1 \leq \theta \leq \infty$ , and  $r > m$ . Then

$$d_n^l(B_{\bar{p}, \theta}^r, L_{\bar{q}}) \asymp n^{-\frac{r}{m} + \frac{1}{m} \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2})}.$$

**Proof.** In view of (4), the lower bound was, in fact, established in deducing the corresponding lower bound in Theorem 1.

We establish the upper bound. For a number  $n \in \mathbb{N}$ , we choose a natural number  $l$  such that  $2^{(l-1)m-1} \leq n < 2^{lm-1}$ . We set

$$\alpha = \frac{\frac{r}{m} - \frac{1}{m} \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{2}\right)}{\frac{r}{m} - \frac{1}{m} \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{q_j}\right)}.$$

For  $s = 0, 1, 2, \dots$ , we denote  $n_s = 2^{sm}$  if  $0 \leq s < l$ ,

$$n_s = \left[ 2^{lr} 2^{sm \left(1 - \frac{r}{m}\right)} \right]$$

if  $l \leq s \leq [\alpha l] + 1$ , and  $n_s = 0$  if  $s > [\alpha l] + 1$ , where  $[y]$  is the integer part of the number  $y$ .

Since  $r > m$ , we get

$$\sum_s n_s \leq \sum_{s=0}^{l-1} 2^{sm} + \sum_{s=l}^{[\alpha l]+1} 2^{lr} 2^{sm(1-\frac{r}{m})} \ll C \left\{ 2^{lm} + 2^{lr} 2^{lm(1-\frac{r}{m})} \right\} \leq 2C2^{lm} \asymp n. \quad (10)$$

Consider the sets

$$P = \bigcup_{0 \leq s < l} \rho(s) \quad \text{and} \quad Q = \bigcup_{l \leq s \leq [\alpha l]+1} \Omega_{n_s}.$$

We construct a subspace of trigonometric polynomials with harmonics from the set  $P \cup Q$  such that the approximation of the class  $H_{\bar{p}}^r$  in the space  $L_{\bar{q}}(\mathbb{T}^m)$  by this subspace realizes the order of the quantity  $d_n^T(H_{\bar{p}}^r, L_{\bar{q}})$ . Let  $f \in H_{\bar{p}}^r$ . Consider the approximation of the function  $f$  by polynomials of the form

$$t(\bar{x}) = \sum_{s=0}^{l-1} \sigma_s(f, \bar{x}) + \sum_{s=l}^{[\alpha l]+1} (t(\Omega_{n_s}; \bar{x}) * \sigma_s(f, \bar{x})).$$

By virtue of relation (10), the number of harmonics in the polynomial  $t(\bar{x})$  does not exceed  $n$ . Thus, according to the property of the norm, we find

$$\|f - t\|_{\bar{q}} \leq \left\| \sum_{s=l}^{[\alpha l]+1} \left( \sigma_s(f, \bar{x}) - (t(\Omega_{n_s}; \bar{x}) * \sigma_s(f, \bar{x})) \right) \right\|_{\bar{q}} + \left\| \sum_{s > [\alpha l]+1} \sigma_s(f) \right\|_{\bar{q}} = J_1 + J_2. \quad (11)$$

We now estimate  $J_2$ . Since  $f \in H_{\bar{p}}^r$ , we have

$$\|\sigma_s(f)\|_{\bar{p}} \leq 2^{-sr}, \quad s = 0, 1, \dots$$

By the property of the norm, the inequality of different metrics (see Theorem B), and the inequality  $r > m$ , we obtain

$$\begin{aligned} J_2 &= \left\| \sum_{s > [\alpha l]+1} \sigma_s(f) \right\|_{\bar{q}} \leq \sum_{s > [\alpha l]+1} \|\sigma_s(f)\|_{\bar{q}} \\ &\ll \sum_{s > [\alpha l]+1} 2^{s \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j})} \|\sigma_s(f)\|_{\bar{p}} \ll \sum_{s > [\alpha l]+1} 2^{s \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j})} 2^{-sr} \\ &\ll 2^{-\alpha l (r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{q_j}))}. \end{aligned}$$

In view of the choice of the number  $\alpha$ , we get

$$J_2 \ll 2^{-l(r - \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2}))} \asymp n^{-\left(\frac{r}{m} - \frac{1}{m} \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2})\right)}. \quad (12)$$

We now estimate  $J_1$ . For a natural number  $s$  satisfying the inequalities  $l \leq s \leq [\alpha l] + 1$ , we consider an operator  $T_s$  given by the formula

$$T_s f(\bar{x}) = f(\bar{x}) * (t_s(\bar{x}) - t(\Omega_{n_s}, \bar{x})).$$

Let  $p_j \in (1, 2)$ ,  $j = 1, \dots, m$ . By using the Minkowski inequality and Lemma 1, we get

$$\begin{aligned} J_1 &= \left\| \sum_{s=l}^{[\alpha l]+1} \sigma_s(f, \bar{x}) - (t(\Omega_{n_s}; \bar{x}) * \sigma_s(f, \bar{x})) \right\|_{\bar{q}} \\ &\leq \sum_{s=l}^{[\alpha l]+1} \left\| \sigma_s(f, \bar{x}) - (t(\Omega_{n_s}; \bar{x}) * \sigma_s(f, \bar{x})) \right\|_{\bar{q}} \ll \sum_{s=l}^{[\alpha l]+1} \|T_s \sigma_s(f)\|_{\bar{q}} \\ &\ll \sum_{s=l}^{[\alpha l]+1} \|T_s\|_{\bar{p} \rightarrow \bar{q}} \|\sigma_s(f)\|_{\bar{p}} \ll \sum_{s=l}^{[\alpha l]+1} 2^{sm} n_s^{-\frac{1}{m} \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})} \|\sigma_s(f)\|_{\bar{p}}. \end{aligned} \tag{13}$$

Substituting the values of the numbers  $n_s$  in (13), we obtain

$$\begin{aligned} J_1 &\ll \sum_{s=l}^{[\alpha l]+1} 2^{sm} \left( 2^{lr} 2^{sm(1-\frac{r}{m})} \right)^{-\frac{1}{m} \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})} \|\sigma_s(f)\|_{\bar{p}} \\ &\ll 2^{-l \frac{r}{m} \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})} \sum_{s=l}^{[\alpha l]+1} 2^{sm} 2^{-s(1-\frac{r}{m}) \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})} \|\sigma_s(f)\|_{\bar{p}} \\ &\ll 2^{-l \frac{r}{m} \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})} \sum_{s=l}^{[\alpha l]+1} 2^{2sm} 2^{-s(1-\frac{r}{m}) \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})} 2^{-sr}. \end{aligned} \tag{14}$$

As a result of simple calculations, we conclude that

$$m - \left(1 - \frac{r}{m}\right) \sum_{j=1}^m \left(\frac{1}{2} + \frac{1}{p'_j}\right) - r = \left(1 - \frac{r}{m}\right) \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{2}\right).$$

It follows from relation (14) that

$$J_1 \ll 2^{-l \frac{r}{m} \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})} \sum_{s=l}^{[\alpha l]+1} 2^{s(1-\frac{r}{m}) \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2})}.$$

Since  $\frac{2}{p_j} - 1 > 0$  and  $1 - \frac{r}{m} < 0$ , we get

$$J_1 \ll 2^{-l \frac{r}{m} \sum_{j=1}^m (\frac{1}{2} + \frac{1}{p'_j})} 2^{l(1-\frac{r}{m}) \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2})} = C 2^{-lm(\frac{r}{m} - \frac{1}{m} \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2}))} \asymp n^{-\left(\frac{r}{m} - \frac{1}{m} \sum_{j=1}^m (\frac{1}{p_j} - \frac{1}{2})\right)}.$$

Hence,

$$J_1 \ll n^{-\left(\frac{r}{m} - \frac{1}{m} \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{2}\right)\right)} \quad (15)$$

in the case where  $p_j \in (1, 2)$ ,  $j = 1, \dots, m$ .

It follows from inequalities (11), (12), and (15) that

$$\|f - t\|_{\bar{q}} \ll n^{-\left(\frac{r}{m} - \frac{1}{m} \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{2}\right)\right)}$$

in the case where  $p_j \in (1, 2)$ ,  $j = 1, \dots, m$ , for any function  $f \in H_{\bar{p}}^r$ . Thus,

$$d_n^T(H_{\bar{p}}^r, L_{\bar{q}}) \ll n^{-\left(\frac{r}{m} - \frac{1}{m} \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{2}\right)\right)}$$

for  $p_j \in (1, 2)$ ,  $j = 1, \dots, m$ . In view of the inclusion  $B_{\bar{p}, \theta}^r \subset H_{\bar{p}}^r$  [see (1)], we find

$$d_n^T(B_{\bar{p}, \theta}^r, L_{\bar{q}}) \ll n^{-\left(\frac{r}{m} - \frac{1}{m} \sum_{j=1}^m \left(\frac{1}{p_j} - \frac{1}{2}\right)\right)}$$

for  $p_j \in (1, 2)$ ,  $j = 1, \dots, m$ .

The theorem is proved.

**Remark 3.** In the case where  $p_1 = \dots = p_m = p$  and  $q_1 = \dots = q_m = q$ , Theorem 2 gives the results obtained by A. S. Romanyuk and V. S. Romanyuk in [14].

## REFERENCES

1. S. M. Nikol'skii, *Approximation of Functions of Many Variables and Imbedding Theorems* [in Russian], Nauka, Moscow (1977).
2. O. V. Besov, "Investigation of one family of functional spaces in connection with imbedding and continuation theorems," *Tr. Mat. Inst. Akad. Nauk SSSR*, **60**, 42–81 (1961).
3. R. S. Ismagilov, "Widths of the sets in normalized linear spaces and the approximation of functions by trigonometric polynomials," *Usp. Mat. Nauk*, **29**, No. 3, 161–178 (1974).
4. Ya. S. Bugrov, "Approximation of a class of functions with dominating mixed derivative," *Mat. Sb.*, **64(106)**, No. 3, 410–418 (1964).
5. É. S. Belinskii, "Approximation of periodic functions by a 'floating' system of exponents and trigonometric widths," in: *Investigation in the Theory of Functions of Many Real Variables* [in Russian], Yaroslavl' (1984), pp. 10–24.
6. É. S. Belinskii, "Approximation by a 'floating' system of exponents on classes of smooth periodic functions," *Mat. Sb.*, **132**, No. 1, 20–27 (1987).
7. V. E. Maiorov, "Trigonometric widths of the Sobolev classes  $W_p^r$  in the space  $L_q$ ," *Mat. Zametki*, **40**, No. 2, 161–173 (1986).
8. G. G. Magaril-II'yaev, "Trigonometric widths of the Sobolev classes of functions in  $R^n$ ," *Tr. Mat. Inst. Akad. Nauk SSSR*, **181**, 147–155 (1988).
9. V. N. Temlyakov, "Approximation of functions with bounded mixed derivative," *Tr. Mat. Inst. Akad. Nauk SSSR*, **178**, 3–112 (1986).
10. A. S. Romanyuk, "Kolmogorov and trigonometric widths of the Besov classes  $B_{p, \theta}^r$  of periodic functions of many variables," *Mat. Sb.*, **197**, No. 1, 71–96 (2006).
11. S. A. Stasyuk, "Trigonometric widths of the classes  $B_{p, \theta}^\Omega$  of periodic functions of many variables," *Ukr. Mat. Zh.*, **54**, No. 5, 700–705 (2002); **English translation:** *Ukr. Math. J.*, **54**, No. 5, 852–861 (2002).
12. S. A. Stasyuk, "Best approximations and Kolmogorov and trigonometric widths of the classes  $B_{p, \theta}^\Omega$  of periodic functions of many variables," *Ukr. Mat. Zh.*, **56**, No. 11, 1557–1568 (2004); **English translation:** *Ukr. Math. J.*, **56**, No. 11, 1849–1863 (2004).
13. D. B. Bazarkhanov, "Estimates for some approximate characteristics of the Nikol'skii–Besov spaces of generalized mixed smoothness," *Dokl. Ros. Akad. Nauk*, **426**, No. 1, 11–14 (2009).

14. A. S. Romanyuk and V. S. Romanyuk, “Trigonometric and orthoprojection widths of the classes of periodic functions of many variables,” *Ukr. Mat. Zh.*, **61**, No. 10, 1348–1366 (2009); **English translation:** *Ukr. Math. J.*, **61**, No. 10, 1589–1609 (2009).
15. G. Akishev, “On the  $M$ -term approximations of the Besov classes,” in: *Abstr. of the Internat. Conf. “Theory of Approximation of Functions and Its Applications” Dedicated to the 70th Birthday of A. I. Stepanets (1942–2007) (Kamenets-Podol'skii, 28.05–03.06, 2012)*, p. 12.
16. N. P. Korneichuk, *Extremal Problems of Approximation Theory* [in Russian], Nauka, Moscow (1976).
17. V. M. Tikhomirov, *Some Problems of Approximation Theory* [in Russian], Moscow University, Moscow (1976).
18. R. A. DeVore and V. N. Temlyakov, “Nonlinear approximation by trigonometric sums,” *J. Fourier Anal. Appl.*, **2**, No. 1, 29–48 (1995).
19. S. A. Stasyuk, “Best  $m$ -term trigonometric approximation for the classes  $B_{p,\theta}^r$  of functions of low smoothness,” *Ukr. Mat. Zh.*, **62**, No. 1, 104–111 (2010); **English translation:** *Ukr. Math. J.*, **62**, No. 1, 114–122 (2010).
20. A. P. Uninskii, “Inequalities in a mixed norm for trigonometric polynomials and entire functions of finite powers,” in: *Imbedding Theorems and Their Applications, Proc. of the Symp. on Imbedding Theorems (Baku, 1966)* [in Russian], Nauka, Moscow (1970), pp. 112–118.
21. É. S. Belinskii and É. M. Galeev, “On the least value of the norms of mixed derivatives of trigonometric polynomials with given number of harmonics,” *Vestn. Mosk. Univ., Ser. Mat. Mekh.*, No. 2, 3–7 (1991).
22. A. Benedek and R. Panzone, “The space  $L_p$  with mixed norm,” *Duke Math. J.*, **28**, No. 3, 301–324 (1961).