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ON THE SOLUTION OF A BOUNDARY VALUE PROBLEM FOR A HYPERBOLIC EQUATION WITH FRACTIONAL LOADING

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Abstract. In this work investigates a boundary value problem for a fractional-loaded hyperbolic equation with one spatial variable that generalizes the classical equation of string vibrations. The equation contains the fractional derivative of Riemann Liouville, and the load is carried out at a time-dependent point. The purpose of the study is to construct an explicit solution to the problem. To find a solution, the initial problem is reduced to a Volterra type integral equation. A representation of the solution is obtained in the form of the sum of two integrals corresponding to different areas of variable definition. It is shown that there is a unique solution to the problem. The conditions for the functions included in the equation are given for which a solution exists. The results obtained can be used in modeling processes described by hyperbolic equations with fractional derivatives.

This paper studies a boundary value problem for a fractionally loaded hyperbolic equation with one spatial variable. The equation contains a Riemann-Liouville fractional derivative, and the load is applied at a time-dependent point [1, 2]. The aim of the research is to construct an explicit solution to the problem, which will provide deeper insight into the behavior of solutions to such equations and expand their applications in practical problems [3, 4].

Consider the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu D_t^{\alpha} u(x,t)\Big|_{x=0} + f(x,t), \quad 0 \le x < \infty, \quad t > 0 \tag{1}$$

with the initial conditions

$$u(x,t)\Big|_{t=0} = g_1(x), \tag{2}$$

$$\frac{\partial u(x,t)}{\partial t}\Big|_{t=0} = g_2(x), \tag{3}$$

and the boundary condition

$$u(x,t)\Big|_{x=0} = h(t). \tag{4}$$

where μ is a parameter, $\alpha \neq 0, \quad x_0 > 0, \quad 0 \leq \beta < 1$, and $D_t^{\alpha} u(x,t)$ is the Riemann-Liouville fractional derivative:

$$D_t^{\alpha} u(x,t) = \begin{cases} D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_t^{\frac{x}{a}} \frac{u(x,\tau)}{(t-\tau)^{\beta}} d\tau, & \text{for } t < \frac{x}{a}, \\ D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_{\frac{x}{a}}^{t} \frac{u(x,\tau)}{(t-\tau)^{\beta}} d\tau, & \text{for } t > \frac{x}{a} \end{cases}$$

The solution to problem (1)–(4) is given by:

1) For $t < \frac{x}{a}$,

$$u(x,t) = u_1(x,t) + \frac{\mu}{2a} \int_0^t \int_{\frac{x}{a}-(t-\tau)}^{\frac{x}{a}+(t-\tau)} D_{t\frac{x}{a}}^{\beta} u(\xi,\tau)\Big|_{\xi=x_0} d\xi d\tau + F_1(x,t) =$$

$$= u_1(x,t) + \mu \int_0^t (t-\tau) \cdot D_{t\frac{x}{a}}^{\beta} u(x,\tau)\Big|_{x=x_0} d\tau + F_1(x,t)$$

2) For $t > \frac{x}{a}$,

$$u(x,t) = u_2(x,t) + \frac{\mu}{2a} \int_0^{t-\frac{x}{a}} \int_{a(t-\tau)-x}^{\frac{x+a(t-\tau)}{a}} D_{\frac{x}{a}t}^{\beta} u(\xi,\tau)\Big|_{\xi=x_0} d\xi d\tau +$$

$$+\frac{\mu}{2a} \int_{t-\frac{x}{a}}^{t} \int_{x-a(t-\tau)}^{x+a(t-\tau)} D_{\frac{x}{a}t}^{\beta} u(\xi,\tau)\Big|_{\xi=x_0} d\xi d\tau + F_2(x,t) = \$\$ = u_2(x,t) + \frac{\mu}{a} \int_0^{t-\frac{x}{a}} x \cdot D_{\frac{x}{a}t}^{\beta} u(x,\tau)\Big|_{\xi=x_0} d\tau + \$$$

$$+\mu \int_{t-\frac{x}{a}}^{t} (t-\tau) \cdot D_{\frac{x}{a}t}^{\beta} u(x,\tau)\Big|_{x=x_0} d\xi d\tau + F_2(x,t)$$

where

$$u_1(x,t) = \frac{1}{2}[g_1(x+at) + g_1(x-at)] + \frac{1}{2a} \int_{x-at}^{x+at} g_2(\xi) d\xi,$$

$$u_2(x,t) = \frac{1}{2}[g_1(x+at) - g_1(x-at)] + \frac{1}{2a} \int_{x-at}^{x+at} g_2(\xi) d\xi + h\left(t - \frac{x}{a}\right),$$

$$F_1(x,t) = \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau) d\xi d\tau$$

$$F_2(x,t) = \frac{1}{2a} \int_0^{t-\frac{x}{a}} \int_{a(t-\tau)-x}^{x+a(t-\tau)} f(\xi,\tau) d\xi d\tau + \frac{1}{2a} \int_{t-\frac{x}{a}}^{t} \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau) d\xi d\tau.$$

The following theorem has been proved.

Theorem. Let the following conditions be satisfied:

1. The functions $g_1(x)$, $g_2(x)$, $h(t)$, and $f(x,t)$ are continuous and bounded on their domains $0 \leq x < \infty, \quad t > 0$..
2. The parameter μ, and constants $a > 0, x_0 > 0$ are such that the operators appearing in the integral equations for $\psi(t)$ are contraction mappings in the space of continuous functions on $[0,T]$ with the norm $\|\psi\| = \sup_{0 \leq t \leq T} |\psi(t)|$.

3. Then problem (1)–(4) has a unique solution $u(x,t)$ in the class of continuous functions satisfying the initial and boundary conditions, with the solution having different forms for $t > \frac{x}{a}$ and $t < \frac{x}{a}$

Remark. Instead of x_0, one can consider a continuous bounded function.

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АНАЛИЗ РАЗРЕШИМОСТИ И ПОСТРОЕНИЕ РЕШЕНИЯ ИНТЕГРАЛЬНОГО УРАВНЕНИЯ ФРЕДГОЛЬМА ПЕРВОГО РОДА

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Необходимое и достаточное условия существования решения интегрального уравнения Фредгольма первого рода и построение его решения является одной из актуальных нерешенных проблем математики (1; 2). Данная работа продолжение научных исследований из (3; 4). Предлагается решения интегрального уравнения Фредгольма первого рода:

$$Ku = \int_a^b K(t,\tau)u(\tau)\,d\tau = f(t), \quad t \in I_1 = [t_0, t_1], \quad \tau \in I_2 = [a,b] \tag{1}$$

где $K(t,\tau) = \|K_{ij}(t,\tau)\|$, $i = \overline{1,n}$, $j = \overline{1,m}$ – известная матрица порядка $n \times m$, элементы матрицы $K(t,\tau)$ функции $K_{ij}(t,\tau)$ измеримы и принадлежат классу L_2 на множестве $S = \{(t,\tau) \in R^2 \mid t \in I_1, \tau \in I_2\}$, функция $f(t) \in L_2(I_1, R^n)$ – заданная функция, $u(\tau) \in L_2(I_2, R^m)$ – искомая функция.