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Two theorems on estimates for solutions of one class of nonlinear equations in a finite-dimensional space

The need to study boundary value problems for elliptic parabolic equations is dictated by numerous practical applications in the theoretical study of the processes of hydrodynamics, electrostatics, mechanics, heat conduction, elasticity theory and quantum physics. In this paper, we obtain two theorems on a priori estimates for solutions of nonlinear equations in a finite-dimensional Hilbert space. The work consists of four items. In the first subsection, the notation used and the statement of the main results are given. In the second subsection, the main lemmas are given. The third section is devoted to the proof of Theorem 1. In the fourth section, Theorem 2 is proved. The conditions of the theorems are such that they can be used in studying a certain class of initial-boundary value problems to obtain strong a priori estimates in the presence of weak a priori estimates. This is the meaning of these theorems.

Keywords: finite-dimensional Hilbert space, nonlinear equations, invertible operator, differentiable vector-functions, a priori estimate of solutions.

Introduction

The problem of describing the dynamics of an incompressible fluid, due to its theoretical and applied importance, attracts the attention of many researchers. In mid-2000, the Clay Mathematics Institute formulated this problem as The Millennium Prize Problems on the existence and smoothness of solutions to the Navier-Stokes equations for an incompressible viscous fluid [1].

Countless works were devoted to the solution of this problem even before it was declared the problem of the millennium. Since there are an infinite number of them, we simply do not list them. The given article provides an incomplete list of works [2].

Many first-class mathematicians who managed to solve other important mathematical problems, including those in problems of gas-hydrodynamics considered this problem. Such prominent mathematicians of the 20th century as A.N. Kolmogorov, J. Leray, E. Hopf, J.-L. Lions provided significant results in their works. Complete solution to the problem for two-dimensional case given by O.A. Ladyzhenskaya [3]. In [4], a complete analysis of the current state of the problem and a review of the available literature, as well as proposed methods for solving the problem, are given. In particular, the main problem of the global unique solvability of the three-dimensional Navier-Stokes problem is reduced to the question of finding a strong a priori estimate for all possible solutions. Works [5]–[12] are devoted to the study of the solvability in the whole of equations of the Navier-Stokes type, the continuous dependence of the solution to a parabolic equation and the smoothness of the solution. In papers [13], [14] questions about the formulation and their solvability of boundary value problems for high-order quasi-hyperbolic equations were studied.

In this article, we obtain two theorems on a priory estimates for solutions of nonlinear equations in a finite-dimensional space. These theorems are proved under certain conditions, which are borrowed

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from the conditions that are satisfied by finite-dimensional approximations of one class of nonlinear initial-boundary value problems, rewritten in "restricted notation".

1 Used conditions and designations. Formulation of the main results

Let H be a finite-dimensional Hilbert space ($10 \leq \dim H = N < \infty$) and G is an invertible operator in H such that $\|G\| \leq 1$, $\|G^{-1}\| < \infty$. We will be interested in the following equation

$$f\left(u\right) := u + L\left(u\right) = g \in H.$$

Throughout this paper, $f\left(u\right)$ will mean an operation of the form $u + L\left(u\right)$, where $L\left(\cdot\right)$ is a non-linear transformation.

If ξ is a parameter from $[0, +\infty)$ and the vector $u\left(\xi\right)$ is a vector function that is continuously differentiable with respect to the parameter ξ, then we assume that the vector function $L\left(u\left(\xi\right)\right)$ is also continuously differentiable (as well as the expressions arising from $L(u)$ and $f(u)$ below).

We introduce the notation L_u :

$$\left(L\left(u(\xi)\right)\right)_{\xi} = L_u u_{\xi}.$$

It is obvious that L_u (for every $u \in H$) will be a linear operator

$$L_u v = \left(L\left(u\left(\xi\right)\right)\right)_{\xi}\Big|_{u_{\xi}=v}.$$

We have

$$\left(f\left(u(\xi)\right)\right)_{\xi} = u_{\xi} + L_u u_{\xi} = \left(E + L_u\right) u_{\xi}.$$

Here and throughout follows, E is the identity transformation.

Operator adjoint to L_u denote by L_u^*, that is $L_u^* = \left(L_u\right)^*$. Denote

$$D_u^* = E + L_u^* D_u^* f\left(u\right) = f\left(u\right) + L_u^* f\left(u\right).$$

If u is a differentiable vector function, then we set

$$\left(D_u^* f(u)\right)_{\xi} = M_u u_{\xi}.$$

Here M_u is a linear operator for fixed is defined by the formula

$$M_u v = \left(M_u u_{\xi}\right)\big|_{u_{\xi}=v}.$$

We will use the following conditions C1–C4.

Condition 1. If u, $v \in H$, then the transformation L_u and L_u^* continuous in H, $L\left(0\right) = 0$ and the conditions are met

$$\|L(u) - L(v)\| \leq \psi(\|u\|)\|u - v\|,$$

$$\|L_u - L_v\|_{H \to H} + \|L_u^* - L_v^*\|_{H \to H} \leq \psi(\|u\|)\|u - v\|,$$

$$\|D_u^* v\| \leq \psi(\|u\|)\|v\|,$$

$$\|M_u v\| \leq \psi(\|u\|)\|v\|,$$

where $\|\cdot\| = \|\cdot\|_H$, $\psi(\cdot)$ strictly monotonously increasing on $[0, \infty)$ positive continuous function.

This condition is natural, since H is certainly, as a rule, it is performed. Therefore, we will use this condition often without stipulating. We will sometimes use the above designations without reservations. In addition to them, we give often the frequently used designations

$$\gamma(u) = \langle D_u^* f(u), u\rangle \|u\|^{-2},$$

$$\mu(u) = \|Gu\|^2 \|u\|^{-2},$$

$$S(u) = D_u^* f(u) - \gamma(u)u - K(u)R(u),$$

$$R(u) = G^* Gu - \mu(u)u,$$

$$K(u) = \frac{\left\langle G^* Gu - \mu(u)u, D_u^* f(u) - \frac{u}{\|u\|^2} \right\rangle}{\|G^* Gu\|^2},$$

$$J(u) = \|u\|^2 \exp(-\|f(u)\|^2).$$

Condition 2. If u-operator's own vector G^*G, then the inequality has been fulfilled

$$\|u\|^2 \le \left(\|f(u)\|^2 + 2\right)^m,$$

where m–integer number, $m \ge 2$.

Condition 3. For any $u \in H$ evaluation is made

$$\|Gu\|^2 \le d\,\|f(u)\|^2.$$

For some $0 \ne u \in H$

$$\widetilde{K} = \inf \frac{\langle M_u a, a\rangle - \frac{a}{\|u\|^2}}{\|Ga\|^2},$$

where the infinum takes on all such $a \in H$, that

$$\|a\| = 1,\ \langle G^* Gu, a\rangle = \langle u, a\rangle = 0.$$

Condition 4. If $0 \ne u \in H$, $S(u) = 0$, $K(u) \ge 0$, then

$$\widetilde{K}d < 1 - \delta,$$

fair, where $\delta \in (0, 1)$.

Theorem 1. If the conditions C1- C4 are met then for any $u \in H$ fair assessment

$$\|u\|^2 \le C \exp(\|f(u)\|^2), \tag{1}$$

where C-does not depend on u and depends only on the conditions C2, C3, C4.

Remark 1. Since G is an invertible operator, we immediately have from condition C3 that the following estimate holds:

$$\|u\|^2 \le \|G^{-1}\|^2\,\|Gu\|^2 \le d\,\|f(u)\|^2. \tag{2}$$

When approximating an infinite-dimensional problem, the finite-dimensional quantity $\|G^{-1}\|$ can tend to ∞. Therefore, from (2) it is impossible to obtain the estimate for $\|u\|$.

Theorem 1 is extended to infinite-dimensional problems, and this is its meaning.

We present one more result.

Theorem 2. Let H be a finite-dimensional Hilbert space. Assume that $L(\cdot)$ is a continuous transformation in H and D is a linear invertible operator. Let us pretend that $L(0) = 0$ and for any H we have the inequality

$$\langle Du,\, DL\,(u)\rangle \geq -\delta \big\|Du\big\|^2$$

at some $0 < \delta < \frac{1}{2}$. Then for any $g \in H$ equation

$$u + L\,(u) = g$$

has a solution satisfying the estimate

$$\|Du\|^2 \leq (1 - 2\delta)^{-1}\, \|Dg\|^2\,.$$

Various forms of this theorem are well-known.

Basic lemmas

Lemma 1. If $0 \neq u \in H$, then the orthogonality equalities

$$\langle u, R(u)\rangle = \langle u, S(u)\rangle = \langle R(u), S(u)\rangle = 0.$$

Proof. These equalities are consequences of the definitions $R(u)$ and $S(u)$.

Lemma 2. For any $C > 0$

$$M_{C,\delta} = \{u:\ \|u\|^2\, e^{-\delta \|f(u)\|^2} \geq C\}.$$

Proof. Since G is an invertible operator and condition C3 is satisfied, then for $u \in M_{C,\delta}$ we have

$$C \leq \|u\|^2\, e^{-\delta\|f(u)\|^2} \leq \|u\|^2\, e^{-\delta d^{-1}\|Gu\|^2} \leq \|u\|^2\, e^{-\delta d^{-1}\|G^{-1}\|^{-2}\|u\|^2}.$$

This implies the boundedness of the set $M_{C,\delta}$. But then, since H is non-dimensional, we obtain the compactness of the set $M_{C,\delta}$. Lemma 2 is proved.

Let us put

$$b(\mathring{u}) = \sup \|Gu\|^2, \tag{3}$$

where the supremum is taken over all such $a \in H$, then

$$J(u) \geq J(\mathring{u}). \tag{4}$$

Lemma 3. If $0 \neq \mathring{u} \in H$. Then there is a vector $\widetilde{u}$, such that

$$\|G\widetilde{u}\|^2 = b(\mathring{u}) \geq \|Gu\|^2,\ \ J(\widetilde{u}) \geq J(\mathring{u}).$$

Proof. The existence of the vector $\widetilde{u}$ follows from Lemma 2, since over a compact set is achieved on some element of this compact space, and the suprenum set over which is taken is compact by Lemma 2 (see (3) suprenum and (4)). The lemma is proven.

We define a vector function as a solution to the problem

$$\begin{cases} u_\xi = x\, G^*\, Gu + y\, S(u), \\ u(\xi)|_{\xi=0} = \widetilde{u}. \end{cases} \tag{5}$$

here $\widetilde{u}$ is the vector constructed in Lemma 3 for some $0 \neq \mathring{u} \in H$. For functionality $J(u(\xi))$ and for the norm $\|Gu(\xi)\|$ using the orthogonality equalities of lemma 1 we have

$$(\|Gu(\xi)\|^2)_\xi = 2\,\langle G^*Gu, u_\xi\rangle = 2\,x\,\|G^*Gu\|^2, \tag{6}$$

$$
\begin{aligned}
J_\xi(u(\xi)) = 2\,J(u(\xi))\,\Big\langle \frac{u}{\|u\|^2} - D^*_{u(\xi)} f(u(\xi)), u_\xi \Big\rangle = \\
= -2\,J(u(\xi))\,\langle [K(u(\xi))G^*Gu(\xi) + S(u(\xi))], u_\xi\rangle = \\
= 2\,J(u(\xi))\,\big[- x\,K(u(\xi))\|G^*Gu(\xi)\|^2 - y\,\|S(u(\xi))\|^2\big].
\end{aligned} \tag{7}
$$

Lemma 4. Let $\mathring{u} \in H$, $\widetilde{u}$ be from Lemma 3. Then the conditions a), b), c), d), e):

$$a)\ S(\widetilde{u}) = 0,$$

$$b)\ K(\widetilde{u}) \geq 0,$$

$$c)\ K(\widetilde{u}) \leq \widetilde{K}(\widetilde{u}),$$

$$d)\ G(\widetilde{u}) = G(\mathring{u}),$$

$$e)\ J(\widetilde{u}) = J(\mathring{u}).$$

Proof. Let $S(\widetilde{u}) \neq 0$. Then in (5) we choose

$$x = 1,\ y = -\|S(\widetilde{u})\|^{-2}\big[1 + K(\widetilde{u})\|G^*Gu(\xi)\|^2\big].$$

Then from (6) and from (7) we find

$$(\|Gu(\xi)\|^2)_\xi\ > 0,\ J_\xi(u(\xi)) > 0. \tag{8}$$

This implies the existence of a number $\xi_0 > 0$ such that the strict inequalities

$$\|Gu(\xi_0)\| > \|G\widetilde{u}\|,\ \ J(u(\xi_0)) > J(\widetilde{u}). \tag{9}$$

These inequalities contradict the origin of the vector $\widetilde{u}$. Therefore, $S(\widetilde{u}) = 0$ and done a).

Let us pretend that $K(\widetilde{u}) < 0$. If we choose $x = 1$, $y = 0$, then it follows from (6) and (7) that (8) is satisfied. From (8) it follows that there is a small $\xi_0 > 0$, such that (9) is satisfied. We obtain a contradiction with the definition of $\widetilde{u}$. Therefore, b) is satisfied.

We define a vector function as a solution to the problem

$$
\begin{cases}
u_\xi = a - \frac{\langle a, G^*Gu\rangle}{\|G^*Gu\|^2}\, G^*Gu, \\
u(\xi)|_{\xi=0} = \widetilde{u},
\end{cases}
$$

where $a \in H$ and $\langle a, G^*Gu\rangle = 0$. Because $\dim \widetilde{H} \geq 3$, such a vector $e \in \widetilde{H},\ \|e\| = 1$ exists. Thus,

$$(\|Gu(\xi)\|^2)_\xi = 2\,\langle e, G^*Gu\rangle = 0 \tag{10}$$

$$
\begin{aligned}
J_\xi(u(\xi)) = 2\,J(u(\xi))\,\langle -S(u(\xi)), u_\xi\rangle = 2\,J(u(\xi))\,\Big\langle -\int_0^\xi S_\eta(u(\eta))d\eta, u_\xi \Big\rangle = \\
= -\,J(u(\xi))\,\Big[\xi\langle S_\eta(u(\eta)), u_\xi\rangle|_{\eta=0}\xi + \xi^2\mathrm{O}(1)\Big] = \\
= 2\,J(u(\xi))\,\Big[\Big\langle \frac{u_\eta}{\|u\|^2} - M_u u_\eta + K(u)G^*Guu_\eta, a\Big\rangle|_{\eta=0}\xi + \xi^2\mathrm{O}(1)\Big] = \\
= 2\,J(u(\xi))\,\Big[\frac{\|a\|^2}{\|\widetilde{u}\|^2} - \langle M_{\widetilde{u}}a, a\rangle + K(\widetilde{u})\|Ga\|^2\Big]\xi + \xi^2\mathrm{O}(1).
\end{aligned} \tag{11}
$$

In the last transition, we used the condition C3 and the equality $u_\eta|_{\eta=0} = e$. By definition $K(\widetilde{u})$ from (10) and from (11) it follows that if $K(\widetilde{u}) > \widetilde{K}(\widetilde{u})$, then there exists a vector a and $\xi_0 > 0$ such that

$$\|Gu(\xi_0)\| = \|G\widetilde{u}\|,\ \ J(u(\xi)) > J(\widetilde{u}). \tag{12}$$

Now, we define the vector function $g(\xi)$ from the problem

$$\begin{cases} g_\xi(\xi) = G^*Gg(\xi), \\ g(\xi)|_{\xi=0} = u(\xi_0). \end{cases}$$

Then for $g(u(\xi))$ we have

$$\|Gg(\xi)\|^2 = \|Gu(\xi_0)\|^2 + 2\int_{\xi_0}^{\xi} \|G^*Gg(\eta)\|^2 d\eta.$$

From here and from (12) it follows that there exists ξ_1 such, that $\xi_1 > \xi_0$ as for $g(\xi_1)$ relations (9) are fulfilled, in which instead of ξ_0 taken ξ_1. We get a contradiction. Therefore item d) of the lemma is proved.

Suppose $J(\widetilde{u}) > J(\mathring{u})$ and define the vector function $g(\xi)$ how to solve a problem

$$\begin{cases} g_\xi(\xi) = G^*Gg(\xi), \\ g(\xi)|_{\xi=0} = \widetilde{u}. \end{cases}$$

For $\|Gg(\xi)\|$ we have

$$\|Gg(\xi)\|^2 = \|G\widetilde{u}\|^2 + 2\int_0^{\xi} \|G^*Gg(\eta)\|^2 d\eta \geq \|G\dot{u}\|^2 + 2\int_0^{\xi} \|G^*Gg(\eta)\|^2 d\eta.$$

Since for small ξ the strict inequality $J(\widetilde{u}) > J(\mathring{u})$ will not get spoiled, then from the inequality for $\|Gg(\xi)\|$ we obtain that there exists $\xi_0 > 0$ such that the strict inequalities $J(u(\xi_0)) > J(\mathring{u})$, $\|Gg(\xi)\| > 0$, which contradict the origin of the vector $\widetilde{u}$. That's why $J(\widetilde{u}) = J(\mathring{u})$. Item e) of the lemma is proved. The lemma is completely proved.

Lemma 5. Let $0 \neq \mathring{u} \in H$, $\widehat{u}$ be a vector constructed from $\mathring{u}$ according to Lemma 3. Let us pretend that $R(\widetilde{u}) \neq 0$ and define the vector function $u(\xi)$ as a solution to the problem

$$\begin{cases} u_\xi = R(u), \\ u(\xi)|_{\xi=0} = \widetilde{u}. \end{cases} \tag{13}$$

Then relations (15)–(17) are satisfied for $0 < \xi < 1$

$$e^{-4\xi}\|R(\widetilde{u})\|^2 \leq \|R(u(\xi))\|^2 \leq e^{4\xi}\|R(\widetilde{u})\|^2, \tag{14}$$

$$\|Gu\|^2 \geq \|G\widetilde{u}\|^2 + 2\int_0^{\xi} \|R(u(\eta))\|^2 d\eta \geq \|G\widetilde{u}\|^2 + 2\xi\|R(\widetilde{u})\|^2 - \xi^2 8e^{4\xi}\|R(\widetilde{u})\|, \tag{15}$$

$$\begin{gathered} J(\widetilde{u}) > J(u(\xi)) = J(\widetilde{u})exp\Big(-2\int_0^{\xi} K(u(\eta))\|R(u(\eta))\|^2 d\eta\Big) \\ J(\mathring{u})\exp\Big(-2\int_0^{\xi} K(u(\eta))\|R(u(\eta))\|^2 d\eta\Big) \geq \\ J(\mathring{u})\exp\Big(-2\,\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2 - \xi^2\|R(\widetilde{u})\|^2 C_1(\|\widetilde{u}\|)\Big), \end{gathered} \tag{16}$$

$$J(\widetilde{u}) \geq J(\mathring{u})\exp\Big[\widetilde{K}(\widetilde{u})(\|G(\dot{u})\|^2 - \|G(u(\xi))\|^2) - \xi_n^2 C_2(\|\widetilde{u}\|)\Big]. \tag{17}$$

where $C_1(\cdot)$, $C_2(\cdot)$ – functions continuous on $[0;\infty)$.

Proof. Let $R(u(\xi))$ then we have

$$\Big|(\|R(u(\xi))\|^2)_\xi\Big| = 2|\langle R(u(\xi)), (G^*G - \mu(u(\xi)))u_\xi - \mu_\xi(u(\xi))u(\xi)\rangle| = \\ 2|\langle R(u(\xi)), (G^*G - \mu(u(\xi)))Ru(\xi)\rangle| \leq 4\|R(u(\xi))\|^2. \tag{18}$$

This implies estimates (14). Further

$$(\|G(u(\xi))\|^2))_\xi = 2\,\langle G^*Gu, u_\xi\rangle = 2\,\langle G^*Gu - \mu(u)u + \mu(u)u, R(u)\rangle = 2(\|R(u)\|^2 \tag{19}$$

$$J_\xi(u(\xi)) = 2J(u(\xi))\Big[\langle -K(u)G^*Gu - S(u), R(u)\rangle\Big] = \\ 2J(u(\xi))\Big[-K(u)\|R(u)\|^2 - \langle S(u), R(u)\rangle\Big]. \tag{20}$$

Integrating (19), using (18) and already proven inequalities (14), we obtain (15). Now we integrate (20), and then using the definitions $R(\cdot)$, $K(\cdot)$, $S(\cdot)$ and the results of Lemma 4, we get

$$J(\widetilde{u}) \geq J(\widetilde{u})\exp\Big[-2\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \\ \int_0^\xi\int_0^\tau \Big(K(u(\eta))\|R(u(\eta))\|^2 + \langle S(u(\eta)), R(u(\eta))\rangle\Big)_\eta d\eta d\tau\Big] = \\ J(\widetilde{u})\exp\Big[-2\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \int_0^\xi\int_0^\tau \Big(K(u(\eta))\|R(u(\eta))\|^2 + \\ \langle -\alpha u(\eta) + D^k_{u(\eta)}f(u(\eta)) - K(u(\eta))G^kGu(\eta), R(u(\eta))\rangle\Big)_\eta d\eta d\tau\Big] = \\ J(\widetilde{u})\exp\Big[-2\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \int_0^\xi\int_0^\tau \langle D^k_{u(\eta)}f(u(\eta)), R(u(\eta))\rangle\Big] = \\ J(\widetilde{u})\exp\Big[-2\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \int_0^\xi\int_0^\tau \langle M_{u(\eta)}R(u(\eta)), R(u(\eta))\rangle\Big] \geq \\ J(\widetilde{u})\exp 2\Big[-\widetilde{K}(\widetilde{u})\|R(\widetilde{u})\|^2\xi - \xi^2\|R(\widetilde{u})\|^2 C_1(\|\widetilde{u}\|^2)\Big], \tag{21}$$

where $C_1(\cdot)$ – functions continuous on $[0;\infty)$.

To estimate the factor at ξ^2, we used the equality $\|u(\xi)\| = \|\widetilde{u}\|$, which follows from the following equality

$$\big(\|u(\xi)\|^2\big)_\xi = 2\langle u(\xi), R(u(\xi))\rangle = 0.$$

From (21) follows (16), from (15) and (16) follows (17). The lemma is proven.

Lemma 6. Let $0 \neq \mathring{u} \in H$, $\widetilde{u}$ – the vector constructed from $\mathring{u}$ in accordance with Lemma 3. Let us pretend that $R(\widetilde{u}) = 0$ and define the vector function $u(\xi)$ as a solution to the problem

$$\begin{cases} u_\xi = G^*Gu, \\ u(\xi)|_{\xi=0} = \widetilde{u}. \end{cases} \tag{22}$$

Then at $0 < \xi < 1$ relations (23)-(26).

$$e^{-2\xi}\|G^*G\widetilde{u}\|^2 \leq \|G^*Gu(\xi)\|^2 \leq e^{2\xi}\|G^*G\widetilde{u}\|^2 \tag{23}$$

$$\|G\dot{u}\|^2 \geq \|G\widetilde{u}\|^2 \geq \|G\widetilde{u}\|^2 = \|G\widetilde{u}\|^2 + \\ +2\int_0^\xi \|G^*Gu(\eta)\|^2 d\eta \leq \|G\widetilde{u}\|^2 + 2\,\xi\|G^*G\widetilde{u}\|^2 + 2\xi^2 8e^{2\xi}\|G^*G\widetilde{u}\| \tag{24}$$

$$J(u(\xi)) \geq J(\widetilde{u}) \exp\left[-2\xi \widetilde{K}(\widetilde{u})\|G^*G\widetilde{u}\|^2 - \xi^2 C_3(\|\widetilde{u}\|^2) l(\xi)\right] = \\ = J(\overset{\circ}{u}) \exp\left[-2\xi \widetilde{K}(\widetilde{u})\|G^*G\widetilde{u}\|^2 - \xi^2 C_3(\|\widetilde{u}\|^2) l(\xi)\right], \tag{25}$$

$$J(u(\xi)) = J(\overset{\circ}{u}) \exp\left[\lambda_0^{-1}(1-\alpha)(\|G\widetilde{u}\|^2 - \|Gu\|^2) - \xi^2 C_4(\|\widetilde{u}\|^2) l(\xi)\right], \tag{26}$$

where $C_3(\cdot)$, $C_4(\cdot)$ functions continuous on $[0;\infty)$ a $l(\cdot)$ function with values from the interval $[-1;1]$.

Proof. For $\|G^*Gu\|$ and $\|Gu\|$ we have

$$\left|(\|G^*Gu\|^2)_\xi\right| = 2|\langle G^*Gu, G^*Gu_\xi\rangle| \leq 2\|G^*Gu\|^2, \\ (\|Gu\|^2|)_\xi = 2\langle G^*Gu, u_\xi\rangle = 2\|G^*Gu\|^2.$$

Integrating these inequalities and using Lemmas 3 and 4, we obtain (23) and (24).

For $J_\xi(u(\xi))$ we have

$$J_\xi(u(\xi)) = 2J(u(\xi))\left[\langle -K(u)G^*Gu - S(u), G^*Gu\rangle\right] = 2J(u(\xi))\left[-K(u)\|G^*Gu\|^2\right].$$

Hence, using Lemma 4, we find

$$J(u(\xi)) = J(\widetilde{u}) \exp\left(-2\int_0^\xi K(u(\eta))\|G^*Gu(\eta)\|^2 d\eta\right) \geq \\ \geq J(\widetilde{u}) \exp\left(-2\xi \widetilde{K}(\widetilde{u})\|G^*G\widetilde{u}\|^2 - \xi^2 C_3(\|\widetilde{u}\|^2) l(\xi)\right). \tag{27}$$

Here $C_3(\cdot)$ – continuous on $[0;\infty)$ functions and $l(\cdot)$ function with values from the segment $[-1;1]$. When estimating the factor at ξ^2, we used the equalities

$$|\left(\|u\|^2\right)_\xi| = 2|\langle u, u_\xi\rangle| = 2\langle u, G^*Gu\rangle = 2\|Gu\|^2 \leq 2\|u\|^2.$$

From which it follows that

$$e^{-2\xi}\|\widetilde{u}\|^2 \leq \|u\|^2 \leq e^{2\xi}\|\widetilde{u}\|^2.$$

From (27) and from $J(\widetilde{u}) = J(\dot{u})$ (25) follows, and (24) implies (26). The lemma is proven.

Proof Theorem 1. Let $0 \neq \overset{\circ}{u} \in H$. If $R(\overset{\circ}{u}) = 0$,, then $\overset{\circ}{u}$ will be an eigenvector of the operator G^*G. Therefore, from condition C2 we have

$$J(\dot{u}) = \|\dot{u}\|^2 \exp\left(-\|f(\dot{u})\|^2\right) = \left(\|f(\dot{u})\|^2 + 2\right)^m e^{-\|f(\dot{u})\|^2} \leq \\ \leq \sup_{x\geq 2} x^m e^{-x+2} = m^m e^{-m+2}. \tag{28}$$

If $R(\overset{\circ}{u}) \neq 0$, construct the vector $\overset{\circ}{u}$ then by the vector $\widetilde{u}_0$. If $R(\widetilde{u}_0) = 0$, then for $J(\widetilde{u}_0)$ we obtain an inequality similar to (28) $J(\widetilde{u}_0) \leq m^m e^{-m+2}$. Therefore, since by construction $J(\widetilde{u}_0) \geq J(\overset{\circ}{u})$ we have that (1) holds. From this we draw the following conclusion.

Thus, if at least one of the conditions $R(\overset{\circ}{u}) = 0$ and $R(\widetilde{u}) = 0$, is satisfied, then Theorem 1 will be proven.

If $R(\overset{\circ}{u}) \neq 0$ and $R(\widetilde{u}_0) \neq 0$, then we construct a sequence of pairs according to the following algorithm.

Let the pairs be built $(\widetilde{u}_0, \overset{0}{u}), ..., (\widetilde{u}_n, \overset{n}{u})$, $0 \leq n$, $R(\widetilde{u}_j) \neq 0$, $j = 0, ..., n$.

Let us build a pairs $(\widetilde{u}_{n+1}, \overset{n+1}{u})$.

In Lemma 3, instead of the vector $\overset{\circ}{u}$, we take the vector $\overset{n}{u}$ and construct the vector $\widetilde{u}$, which we take as $\widetilde{u}_{n+1}$.

For $\widetilde{u}_{n+1}$, two cases are possible:

$$(I)\ R(\widetilde{u}_{n+1}) = 0,$$

$$(II)\ R(\widetilde{u}_{n+1}) \neq 0.$$

If (I) is satisfied, then we construct the vector $\overset{n+1}{u}$ using Lemma 6. To do this, in (22) as $\widetilde{u}$ we take the vector $\widetilde{u}_{n+1}$ and for $\overset{n+1}{u}$ we take the value $u(\xi)$ at point $\widehat{\xi}_n$:

$$\widehat{\xi}_n = \widetilde{\xi},\ \overset{n+1}{u} = u(\xi_n).$$

Then from Lemma 6, using condition C4, we have

$$\begin{aligned} J(\overset{n+1}{u}) &\geq J(\overset{n}{u}) \exp\left[d^{-1}(1-\delta)(\|G\widetilde{u}_{n+1}\|^2 - \|G\overset{n+1}{u}\|^2 - \xi_n^2 G(\widetilde{u}_{n+1}))\right] \geq \\ &\geq J(\overset{n}{u}) \exp\left[d^{-1}(1-\delta)(\|G\overset{n}{u}\|^2 - \|G\overset{n+1}{u}\|^2) - \xi_n^2 C_4(\widetilde{u}_{n+1})\right]. \end{aligned} \tag{29}$$

When deriving (29), relations $J(u_{n+1}) = J(\overset{n}{u})$, $\|G\widetilde{u}_{n+1}\| \geq \|G\overset{n}{u}\|$ were used, which follow from Lemmas 3 and 4 and the definition of $\widetilde{u}_{n+1}$. Let's choose ξ_n in the right place.

In the case (I) pair $(\widetilde{u}_{n+1}, \overset{n+1}{u})$ is constructed.

In the this case, we stop the process of constructing a sequence of pairs.

In case (II), we construct $\overset{n+1}{u})$ using Lemma 5. To do this, in (13) as $\widetilde{u}$ we take the vector $\widetilde{u}_{n+1}$ and for $\overset{n+1}{u}$ we take the value (13) at point $\widehat{\xi}_n$:

$$\xi_n = \frac{\delta_0 \|R(\widetilde{u}_{n+1})\|^2}{(n+1)[C_1(\|\widetilde{u}_{n+1}\|) + C_2(\|\widetilde{u}_{n+1}\|) + 1]\,[\|R(\widetilde{u}_{n+1})\|^4 + 1]}, \tag{30}$$

here $C_1(\cdot)$, $C_2(\cdot)$– continuous on $[0;\infty)$ functions from Lemma 5, δ_0 is a small number. From Lemma 5 (17) and from (30) we find

$$\begin{cases} \|G\overset{n+1}{u}\|^2 \geq \|G\overset{n+1}{u}\|^2 + 2\xi\|R(\widetilde{u}_{n+1})\|^2 - 20\frac{\delta_0^2}{(n+1)^2}, \\ J(\overset{n+1}{u}) \geq J(\overset{n}{u}) \exp\left[d^{-1}(1-\delta)(\|G\overset{n}{u}\|^2 - \|G\overset{n+1}{u}\|^2) - 20\frac{\delta_0^2}{(n+1)^2}\right]. \end{cases} \tag{31}$$

Pair $(\widetilde{u}_{n+1}, \overset{n+1}{u})$ built. Let $1 \leq n_0$ be an integer number, which holds for all $j \leq n_0$

$$R(\widetilde{u}_j) \neq 0,\ \ R(\widetilde{u}_{n_0+1}) = 0.$$

Then from (31) we deduce

$$\begin{cases} \|G\overset{n_0}{u}\|^2 \geq \|G\overset{0}{u}\|^2 + 2\sum_{j=0}^{n_0} \xi_j \|R(\widetilde{u}_j)\|^2 - 10^2\delta_0^2, \\ J(\overset{n_0}{u}) \geq J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)(\|G\overset{0}{u}\|^2 - \|G\overset{n_0}{u}\|^2) - 10^2\delta_0^2\right]. \end{cases} \tag{32}$$

From the second inequality (32) we have

$$\begin{aligned} J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)\|G\overset{0}{u}\|^2 - 10^2\delta_0^2\right] &\leq J(\overset{n_0}{u}) \exp\left[d^{-1}(1-\delta)\|G\overset{n_0}{u}\|^2\right] = \\ = \|\overset{n_0}{u}\|^2 \exp\left[-\|f(\overset{n_0}{u})\|^2 + d^{-1}(1-\delta)\|G\overset{n_0}{u}\|^2\right] &\leq \|\overset{n_0}{u}\|^2 \exp\left[\delta\|f(\overset{n_0}{u})\|^2\right]. \end{aligned} \tag{33}$$

In the last transition, condition C3 is used.

From (33) and from Lemma 2 on compactness (see Lemma 2), since the left side of (33) does not depend on n_0 and the inequalities $0 < \delta < 1$, it follows that

$$\|\overset{n_0}{u}\| \leq J(\overset{0}{u}) < \infty, \tag{34}$$

where $J(\overset{0}{u})$ does not depend on n_0. From (34) and from the first inequality in (32), due to the choice of ξ_n, follows that only two cases (A) and (B) are possible:

(A) There is an $n \geq 0$ such that $R(\widetilde{u}_{n_1}) = 0$ and $R(\widetilde{u}_j) \neq 0$, if $0 < j < n_1$;

(B) For any $j = 0, 1, ...$ done $R(\widetilde{u}_j) \neq 0$ and $\underline{\lim}_{n\to\infty}\|R(\widetilde{u}_n)\| = 0$.

Here $\underline{\lim}$ means lower limit.

Indeed, if none of the conditions (A) is satisfied, then by virtue of (34) and the choice of A (see (30)) from the first inequality in (32) we obtain

$$J(\overset{0}{u}) \geq \|\overset{n+1}{u}\|^2 \geq \|G\overset{n+1}{u}\|^2 \geq \|G\overset{0}{u}\|^2 - 10^2\delta_0^2 + 2\sum_{j=0}^{n} \frac{\delta_0}{j+1} \inf_k \|R(\widetilde{u}_k)\|^2.$$

When $n \to \infty$, the right side tends to $+\infty$. So we got a contradiction. Therefore, at least one of conditions (A) and (B) is satisfied.

Let condition (B) be satisfied. Then, by virtue of (34), if necessary, passing to sequences can be considered

$$\lim_{j\to\infty} \overset{j}{u} = \widetilde{g}, \ \lim_{j\to\infty} R(\overset{j}{u}) = R(\widetilde{g}) = 0, \ \lim_{j\to\infty} f(\overset{j}{u}) = f(\widetilde{g}).$$

When deriving the equality for $R(\widetilde{g})$ and $f(\widetilde{g})$, we used the estimates for $R(\overset{j}{u})$ in terms of $R(\widetilde{u}_j)$ from Lemma 5 and choice ξ_j (see (30)), as well as the divergence of the harmonic series.
Letting go to infinity and then using the conditions C2 and C3, we obtain

$$\begin{gathered} J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)\|G\overset{0}{u}\|^2 - 10^2\delta_0^2\right] \leq \|\widetilde{g}\|^2 \exp\left[-\delta\|f(\widetilde{g})\|^2\right] \leq \\ \leq \left(\|f(\widetilde{g})\|^2 + 2\right)^m \exp\left[-\delta(\|f(\widetilde{g})\|^2 + 2) + 2\delta\right] \leq \sup_{x\geq 2} x^m e^{-x+2\delta} = \left(\frac{m}{\delta}\right)^m e^{-m+2\delta}. \end{gathered} \tag{35}$$

Now from the definition of A and from (35) we deduce

$$\|\overset{0}{u}\|^2 \leq \exp\left[\|f(\overset{0}{u})\|^2 - d^{-1}(1-\delta)\|G\overset{0}{u}\|^2 - 10^2\delta_0^2\right]\left(\frac{m}{\delta}\right)^m \leq \left(\frac{m}{\delta}\right)^m \exp\|f(\overset{0}{u})\|^2. \tag{36}$$

In the derivation, we used that the possibility of choosing δ_0 small and inequalities $0 < \delta < \frac{1}{2}, m \geq 1$. Theorem 1 follows from (36) in case (B).

If (A) is satisfied, then (29) is satisfied. From (29), since for all $j \leq n$ the inequalities $R(\widetilde{u}_j) \neq 0$, then choosing $\xi_n = \widetilde{\xi}$ small enough, we get

$$J(\overset{n+1}{u}) \geq J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)(\|G\overset{0}{u}\|^2 - \|G\overset{n+1}{u}\|^2) - 10^2\delta_0^2\right]. \tag{37}$$

Since $\overset{n+1}{u}$ is defined in terms of $\widetilde{u}_{n+1}$ by the equation (see (22))

$$\begin{cases} u_\xi = G^*Gu, \\ u(\xi)|_{\xi=0} = \widetilde{u}_{n+1}. \end{cases}$$

And $\overset{n+1}{u} = u(\widetilde{\xi}_n)$, then choosing the number $\widetilde{\xi}_n = \widetilde{\xi}$ small enough from (37) we obtain

$$J(\widetilde{u}_{n+1}) \geq J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)(\|G\overset{0}{u}\|^2 - \|G\widetilde{u}_{n+1}\|^2) - 10^2\delta_0^2\right].$$

Hence follows

$$J(\overset{0}{u}) \exp\left[d^{-1}(1-\delta)\|G\overset{0}{u}\|^2 - 10^2\delta_0^2\right] \leq \|\widetilde{u}_{n+1}\| exp\left[-\|f(\widetilde{u}_{n+1})\|^2 + d^{-1}(1-\delta)\|G\widetilde{u}_{n+1}\|^2\right]. \qquad (38)$$

From (38), since $\widetilde{u}_{n+1}$ is an eigenvector of the operator G^*G, we get the estimate

$$\|\overset{0}{u}\|^2 \leq \left(\frac{m}{\delta}\right)^m \exp\|f(\overset{0}{u})\|^2,$$

which are derived in the same way as we derived the estimate from (36). Theorem 1 is proved in case (A). Therefore, Theorem 1 is proved completely.

Proof Theorem 2. We use the notation of Theorem 2. If $g \in H$, then vector $u \equiv 0$ is a solution to the equation

$$u + L(u) = 0.$$

Let $0 \neq g \in H-$ arbitrary vector. Since D is an invertible operator, then $\|Dg\| > 0$.

Denote by M the set

$$M = \{u: \|Du\|^2 \leq \frac{2}{(1-2\delta)\delta}\|Dg\|^2\}.$$

Let's put

$$F(u) = -\frac{u + L(u) - g}{\|D(u + L(u) - g)\|}\eta,$$

where the number η is chosen as follows:

$$\eta = \sqrt{\frac{2}{(1-2\delta)\delta}\|Dg\|^2}.$$

Suppose equation $u + L(u) = g$ has no solution in M: Since equation $u + L(u) = g$ has no solution, the transformation $F(\cdot)$ continuously translates from M to M. But then by the Schauder fixed-point theorem u_0 we get that there exists such that

$$-\frac{u_0 + L(u_0) - g}{\|D(u_0 + L(u_0) - g)\|}\eta = u_0. \qquad (39)$$

From here and the choice of η we have

$$\|Du_0\|^2 = \eta^2 = \frac{2}{(1-2\delta)\delta}\|Dg\|^2.$$

We act on (39) with the operator D, and then multiply scalarly by Du_0. Then, using (39) and the condition of the theorem, we obtain

$$\eta^{-1}\|Du_0\|^2\|D(u_0 + L(u_0) - g)\| = -\|Du_0\|^2 - \langle DL(u_0), Du_0\rangle + \langle Dg, Du_0\rangle \leq$$

$$\leq -\|Du_0\|^2 + \delta\|Du_0\|^2 + \frac{\varepsilon^{-1}}{2} + \frac{1}{2}\varepsilon\|Du_0\|^2.$$

Let us take $\varepsilon = 2\delta$. Then from the last inequality and from (39) we deduce

$$0 \le -\|Du_0\|^2(1-\delta) + \frac{1}{4\delta}\|Dg\|^2 = -\frac{2}{(1-2\delta)\delta}(1-\delta)\|Dg\|^2 + \frac{1}{4\delta}\|Dg\|^2 = -\frac{7}{4\delta}\|Dg\|^2.$$

We got a contradiction. Therefore, the equation $u + L(u) = g$ has a solution. We act on the equation $u + L(u) = g$ by the operator D:

$$Du + DL(u) = Dg.$$

Multiplying the resulting equality scalarly by Du, we obtain

$$\|Du\|^2 + \langle Du, DL(u)\rangle = \langle Dg, DL(u)\rangle \le \frac{1}{2}\|Dg\|^2 + \frac{1}{2}\|Du\|^2.$$

Now, using condition $\langle Du, DL(u)\rangle \ge -\delta\|Du\|^2$, , we obtain the desired evaluation

$$(1-2\delta)\|Du\|^2 \le \|Dg\|^2.$$

Theorem 2 is proven.

Remark 2. Note that in Lemma 1 we can take K non-linear transformations as K.

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Ақырлыөлшемді кеңістіктегі сызықты емес теңдеулердің бір класының шешімін бағалаудың екі теоремасы

Эллиптикалық және параболалық теңдеулер үшін шеттік есептерді зерттеу қажеттілігі гидродинамика, электростатика, механика, жылу өткізгіштік, серпімділік теориясы, кванттық физика процестерін теориялық тұрғыдан зерттеуде көптеген практикалық қосымшалардың түсіндіруімен тікелей байланысты. Бұл жұмыста ақырлыөлшемді кеңістікте сызықтық емес теңдеулердің шешімдері үшін априорлық бағалаулары туралы екі теорема алынған. Жұмыс төрт бөлімнен тұрады. Бірінші бөлімде пайдаланылған белгілеулер мен негізгі нәтиженің тұжырымдамасы келтірілген. Екінші бөлімде негізгі леммалар берілген. Үшінші бөлім 1-ші теореманың дәлелдемесіне арналған. Төртінші бөлімде екінші теорема дәлелденген. Теореманың шарты мынадай, оны бастапқы-шекаралық есептердің белгілі бір класын зерттеу кезінде олардың шешімдеріне априорлық бағалау алу үшін қолдануға болады. Теореманың мәні осында.

Кілт сөздер: ақырлыөлшемді Гильберт кеңістігі, сызықтық емес теңдеулер, кері оператор, дифференциалданатын вектор-функциялар, шешімдерді априорлық бағалау.

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Две теоремы об оценках решений одного класса нелинейных уравнений в конечномерном пространстве

Необходимость исследования краевых задач для эллиптических и параболических уравнений продиктована с многочисленными практическими приложениями при теоретическом изучении процессов гидродинамики, электростатики, механики, теплопроводности, теории упругости, квантовой физики. В этой работе мы получили две теоремы об априорных оценках решений нелинейных уравнений в конечномерном гильбертовом пространстве. Работа состоит из четырех пунктов. В первом пункте приведены используемые обозначения и формулировка основных результатов. Во втором — основные леммы. Третий пункт посвящен доказательству теоремы 1. В четвертом — доказана теорема 2. Условия теорем таковы, что можно использовать при изучении некоторого класса начально-краевых задач для получения сильных априорных оценок при наличии слабых априорных оценок. В этом и состоит смысл этих теорем.

Ключевые слова: конечномерное гильбертово пространство, нелинейные уравнения, обратимый оператор, дифференцируемые вектор-функции, априорная оценка решений.

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