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Solving Volterra-Fredholm integral equations by natural cubic spline function

Using the natural cubic spline function, this paper finds the numerical solution of Volterra-Fredholm integral equations of the second kind. The proposed method is based on employing the natural cubic spline function of the unknown function at an arbitrary point and using the integration method to turn the Volterra-Fredholm integral equation into a system of linear equations concerning to the unknown function. An approximate solution can be easily established by solving the given system. This is accomplished with the help of a computer program that runs on Python 3.9.

Keywords: Volterra integral equation, Fredholm integral equation, spline function.

Introduction

Integral equations can be used to express a variety of mathematical physics topics. Some of these will be examined and treated explicitly as examples. It would be nearly impossible to compile a list of such applications. To say that integral equations play a role in practically every area of applied mathematics and mathematical physics is an understatement; because, the literature on integral equations and their applications is extensive.
Many researches have been conducted in recent years, with the results revealing the interaction of Fredholm integral equation, Volterra integral equation, Volterra-Fredholm integral equation, and numerical solutions of these three types of the integral equation.

The linear Volterra-Fredholm integral equations (VFIEs) of the following type were taken into consideration in this work:

$$u(x) = f(x) + \lambda_1 \int_a^x K(x,t)u(t)dt + \lambda_2 \int_a^b L(x,t)u(t)dt, \tag{1}$$

where the functions $f(x)$, and the kernels $K(x,t)$ and $L(x,t)$ are known L^2 analytic functions and λ_1, λ_2 are arbitrary constants, x is variable and $u(x)$ is the unknown continuous function to be determined. Numerous applications in the fields of physics, fluid dynamics, electrodynamics, and biology include the use of these equations. These integral equations are a reduction of several boundary value formulations using Neumann, Dirichlet, or boundary conditions. Additionally, they offer mathematical models for the spread of an epidemic as well as a host of other physical and biological problems.

Since the analytical solution to VFIEs generally does not exist outside of special cases, the numerical method is the most successful and efficient way to solve these issues. In order to solve VFIEs, a number of numerical and approximative techniques have been developed, including the linear and quadratic spline functions by Salim, et. al. [1, 2], Taylor polynomial by Yalcinbacs and Sezer [3]; Yalcinbas [4], the least square method and Chebyshev polynomials by Dastjerdi and Ghaini [5].

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Also, Lagrange collocation method by Wang [6], Series solution, successive approximation method and method of successive substitutions by Saeed and Berdawood [7], trigonometric Functions and Laguerre Polynomials by Hasan and Sulaiman [8], Touchard Polynomials (T-Ps) method by Al-Miah and Taie [9]. Some iterative numerical methods by Micula [10], Taylor polynomial by Didgara and Vahidi [11]. For additional information, the reader might turn to the following references and the references given there: (Jerry [12], Atkinson [13], Lange and Herbert [14], Kaminaka and Wadati [15], Ladopoulos [16], Corduneanu [17], Saeed and Aziz [18] and Jaber and Alrammahi [19]).

Equation (1) will be studied in this work using the natural cubic spline function. The rest of this paper is structured as follows. Our approach is introduced in Section 2 for solving Equation (1). We examine various numerical examples proving the viability of our method in Section 3. Some conclusions will be made in Section 4.

1 *Description of the method*

In this section, we solve Equation (1) by using the quadratic spline function (Cheney and Kincaid [20], Saeed *et. al.* [21]). The unknown function $u(x)$ in (1) is approximated by the quadratic spline function $S(x)$. In the interval $[x_i, x_{i+1}]$, the quadratic spline function is defined by the following formula:

$$S_i(x) = A_i(x)S_i + B_i(x)S_{i+1} + C_i(x)S_i' + D_i(x)S_{i+1}', \tag{2}$$

where $A_i(x) = 1-3\frac{(x-t_i)^2}{h^2}+2\frac{(x-t_i)^3}{h^3}$, $B_i(x) = 1-A_i(x)$, $C_i(x) = \frac{(x-t_i)(x-t_{i+1})^2}{h^2}$, $D_i(x) = \frac{(x-t_{i+1})(x-t_i)^2}{h^2}$, and $h = x_{i+1} - x_i$ for all $i = 0, 1, \cdots, n-1$. Now substituting (2) in (1) and letting $x = x_i$, we get

$$\begin{aligned}
S_i &= f_i + \lambda_1 \int_a^{x_i} K(x_i,t)S(t) + \lambda_2 \int_a^b L(x_i,t)S(t)dt \\
&= f(x_i) + \lambda_1 \left[\sum_{j=0}^{j=i-2} \int_{x_j}^{x_{j+1}} K(x_i,t)[A_j(t)S_j + B_j(t)S_{j+1} + C_j(t)S_j' + D_j(t)S_{j+1}']dt \right. \\
&\left. + \int_{x_{i-1}}^{x_i} K(x_i,t)[A_{i-1}(t)S_{i-1} + B_{i-1}(t)S_i + C_iS_{i-1}' + D_i(t)S_i']dt \right] \\
&+ \lambda_2 \left[\int_{x_0=a}^{x_1} L(x_i,t)S_0(t)dt + \int_{x_1}^{x_2} L(x_i,t)S_1(t)dt + \cdots + \int_{x_{n-1}}^{x_n=b} L(x_i,t)S_{n-1}(t)dt \right] \\
&= f(x_i) + \lambda_1 \left[\sum_{j=0}^{j=i-2} \int_{x_j}^{x_{j+1}} K(x_i,t)[A_j(t)S_j + B_j(t)S_{j+1} + C_j(t)S_j' + D_j(t)S_{j+1}']dt \right. \\
&\left. + \int_{x_{i-1}}^{x_i} K(x_i,t)[A_{i-1}(t)S_{i-1} + B_{i-1}(t)S_i + C_iS_{i-1}' + D_i(t)S_i']dt \right] \\
&+ \lambda_2 \left[\int_{x_0=a}^{x_1} L(x_i,t)[A_0(t)S_0 + B_0(t)S_1 + C_0(t)S_0' + D_0(t)S_1']dt \right. \\
&+ \int_{x_1}^{x_2} L(x_i,t)[A_1(t)S_1 + B_1(t)S_2 + C_1(t)S_1' + D_1(t)S_2']dt + \cdots \\
&\left. + \int_{x_{n-1}}^{x_n=b} L(x_i,t)[A_{n-1}(t)S_{n-1} + B_{n-1}(t)S_n + C_{n-1}(t)C_j(t){S'}_{n-1} + D_n - 1(t)S_n']dt \right].
\end{aligned}$$

By computing the integrals in the above equation using the trapezoidal rule, we get

$$S_i = f_i + \frac{h}{2}(\lambda_1 K_{i0} + \lambda_2 L_{i0})S_0 + h\sum_{j=1}^{i-1}(\lambda_1 K_{ij} + \lambda_2 L_{ij})S_j + \frac{h}{2}(\lambda_1 K_{ii} + \lambda_2 L_{ii})S_i, \tag{3}$$

for $i = 0, 1, \cdots, n$.

In this way, Equation (3) constructs a system of linear equations concerning to the unknown function S_i. Briefly, this system can be rewritten as follows:

$$CS = F, \tag{4}$$

where

$$S = \begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_n \end{bmatrix}, \quad F = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad C = \begin{bmatrix} C_0 & C_1 & C_2 & \dots & C_{n-1} & C_n \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1 - \frac{\lambda_2 h}{2} L_{00} \\ -\frac{h}{2}(2\lambda_1 K_{10} + \lambda_2 L_{10}) \\ -\frac{h}{2}(\lambda_1 K_{20} + \lambda_2 L_{20}) \\ -\frac{h}{2}(\lambda_1 K_{30} + \lambda_2 L_{30}) \\ \vdots \\ -\frac{h}{2}(\lambda_1 K_{n0} + \lambda_1 L_{n0}) \end{bmatrix}, \quad C_1 = \begin{bmatrix} -\lambda_2 h L_{01} \\ 1 - (\frac{h}{2}(\lambda_1 K_{11} + 2\lambda_2 L_{11}) \\ -\frac{h}{2}(3\lambda_1 K_{21} + 2\lambda_2 L_{21}) \\ -h(\lambda_1 K_{31} + \lambda_2 L_{31}) \\ \vdots \\ -h(\lambda_1 K_{n1} + \lambda_2 L_{n1}) \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -\lambda_2 h L_{02} \\ -\frac{h}{2}(2\lambda_2 L_{12} - \lambda_1 K_{10}) \\ 1 - \frac{h}{2}(\lambda_1 K_{22} + 2\lambda_2 L_{22}) \\ -\frac{h}{2}(3\lambda_1 K_{32} + 2\lambda_2 L_{32}) \\ \vdots \\ -h(\lambda_1 K_{n2} + \lambda_2 L_{n2}) \end{bmatrix}, \ \dots, \ C_{n-1} = \begin{bmatrix} -\lambda_2 h L_{0(n-1)} \\ -\lambda_2 h L_{1(n-1)} \\ -\lambda_2 h L_{2(n-1)} \\ \vdots \\ 1 - \frac{h}{2}\left(\lambda_1 K_{(n-1)(n-1)} + 2\lambda_2 L_{(n-1)(n-1)}\right) \\ -\frac{h}{2}\left(\lambda_1 K_{n(n-1)} + 2\lambda_2 L_{n(n-1)}\right) \end{bmatrix},$$

and

$$C_n = \begin{bmatrix} -\frac{\lambda_2 h}{2} L_{0n} \\ -\frac{\lambda_2 h}{2} L_{1n} \\ -\frac{\lambda_2 h}{2} L_{2n} \\ \vdots \\ -\frac{\lambda_2 h}{2} L_{(n-1)(n-1)} \\ 1 - \frac{h}{2}(\lambda_1 (K_{nn} + \lambda_2 L_{nn}) \end{bmatrix}.$$

In the sequel, using a standard rule to the resulting system yields an approximate solution of Equation (1) as $S_i(x)$ given by Equation (2).

2 Numerical examples

In this section, we present three examples to illustrate the efficiency and accuracy of the proposed method. The computed errors e_i are defined by $e_i = |u_i - S_i|$, where u_i is the exact solution of Equation

(1) and S_i is an approximate solution of the same equation. Also we compute Least square error (LSE) which is defined by formula $= \sum_{i=0}^{n}(u_i - S_i)^2$ and all computations are performed using the Python program.

Example 1. Consider the linear Volterra-Fredholm integral equation

$$u(x) = -\frac{x^2}{2} - \frac{7x}{2} + 2 + \int_0^x u(t)dt + \int_0^1 xu(t)dt.$$

The exact solution to this equation is given by $u(x) = x + 2$.
Table (1) demonstrates LSE obtained from applying our method to Example (1) for $n = 5$.

Table 1

The Numerical Results for Example (1) with $n = 5$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	2	2	0	0
0.2	2.2	2.167670	0.323298	0.00104522
0.4	2.4	2.369779	0.302205	0.00091328
0.6	2.6	2.533617	0.06738252	0.00454041
0.8	2.8	3.0875551	0.2875513	0.08268795
1	3	3.0788111	0.0788111	0.00621119
LSE				9.5398050×10^{-2}

Example 2. Consider the linear Volterra-Fredholm integral equation

$$u(x) = 2cos(x) - 1 + \int_0^x (x - t)u(t)dt + \int_0^\pi u(t)dt.$$

The exact solution to this equation is given by $u(x) = cos(x)$.
Table (2) demonstrates LSE obtained from applying our method to Example (2) for $n = 5$.

Table 2

The Numerical Results for Example (2) with $n = 5$

x_i	u_i	S_i	$\lvert u_i - S_i\rvert$	$\lvert u_i - S_i\rvert^2$
0	1	0.9020653	0.0979346	$9.59119752 \times 10^{-3}$
$\frac{\pi}{5}$	0.8090169	0.80093654	0.00808045	$6.52937193 \times 10^{-5}$
$\frac{2\pi}{5}$	0.3090169	0.38139274	0.07237575	$5.23824911 \times 10^{-3}$
$\frac{3\pi}{5}$	-0.3090169	-0.15642311	0.15259389	$2.32848948 \times 10^{-2}$
$\frac{4\pi}{5}$	-0.8090169	0.8466296	0.0376126	$1.41470781 \times 10^{-3}$
π	$-1.$	-0.9488828	0.0511172	$2.61296767 \times 10^{-3}$
LSE				$4.22073106838 \times 10^{-2}$

Example 3. Consider the linear Volterra-Fredholm integral equation

$$u(x) = -\frac{x^5}{10} + 2x^3 - \frac{x^2}{2} - \frac{3x}{2} + \frac{1}{10} + \int_0^x (x + t)u(t)dt + \int_0^1 (x - t)u(t)dt.$$

The exact solution to this equation is given by $u(x) = 2x^3 + 1$.
Table (3) demonstrates LSE obtained from applying our method to Example (3) for $n = 5$.

Table 3

The Numerical Results for Example (3) with $n = 5$

x_i	u_i	S_i	$\|u_i - S_i\|$	$\|u_i - S_i\|^2$
0	1	0.9320698	0.0679302	0.00461451
0.2	1.016	0.91835798	0.09764202	0.00953396
0.4	1.128	0.99225104	0.13574896	0.01842778
0.6	1.432	1.24719269	0.18480731	0.03415374
0.8	2.024	1.83381163	0.19018837	0.03617162
1	3	2.72882412	0.27117588	0.07353636
LSE				$1.76437975899 \times 10^{-1}$

Table 4

LSE for different values of n for Examples (1)–(3)

LSEn	10	20	30	40	50
Example 1	$2.02193681 \times 10^{-2}$	$5.21316981 \times 10^{-3}$	$2.36837446 \times 10^{-3}$	$1.348919901 \times 10^{-3}$	8.7012926×10^{-4}
Example 2	$3.03911660 \times 10^{-3}$	$1.87835246 \times 10^{-4}$	$3.79028932 \times 10^{-5}$	$1.24158506 \times 10^{-5}$	$5.28945050 \times 10^{-6}$
Example 3	$1.95064939 \times 10^{-2}$	$2.31413450 \times 10^{-3}$	$6.752544397 \times 10^{-4}$	$2.82826976 \times 10^{-4}$	$1.44203795 \times 10^{-4}$

3 Conclusion

In this paper the cubic spline function is used to solve linear Volterra-Fredholm integral equations, and it is a powerful numerical approach. The numerical results in the present section demonstrate that the proposed method can successfully tackle the Volterra-Fredholm type problem. Table (4) shows that the proposed method has extremely good stability; as n increases, the error decreases at first and then stabilizes. We also conclude that we have high accuracy when the exact solution is a trigonometric function. The present method can be easily extended to systems of Volterra-Fredholm integral equations and systems of Volterra-Fredholm integro-differential equations. The current method may be simply extended to Volterra-Fredholm integral equations and Volterra-Fredholm integro-differential equations.

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Вольтерра-Фредгольм интегралдық теңдеулерiн кубтық сплайн-функциясымен шешу

Мақалада табиғи кубтық сплайн-функциясын қолданып екiншi тектi Вольтерра-Фредгольм аралас интегралдық теңдеулерiнiң сандық шешiмi табылған. Ұсынылған әдiс еркiн нүктеде белгiсiз функцияның табиғи кубтық сплайн-функциясын қолдануға және Вольтерра-Фредгольм интегралдық теңдеуiн белгісіз функцияға қатысты сызықтық теңдеулер жүйесіне түрлендіру үшін интегралдау әдісін қолдануға негізделген. Бұл жүйені шешу арқылы жуық шешімді табу оңай. Бұған Python 3.9-да жұмыс істейтін компьютерлік бағдарлама арқылы қол жеткізіледі.

Кілт сөздер: Вольтерра интегралдық теңдеуі, Фредгольм интегралдық теңдеуі, сплайн-функциясы.

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Решение интегральных уравнений Вольтерра-Фредгольма с помощью естественной кубической сплайн-функции

В статье с использованием функции натурального кубического сплайна найдено численное решение смешанных интегральных уравнений Вольтерра-Фредгольма второго рода. Предлагаемый метод основан на применении естественной кубической сплайн-функции неизвестной функции в произвольной точке и метода интегрирования для преобразования интегрального уравнения Вольтерра-Фредгольма в систему линейных уравнений относительно неизвестной функции. Приближенное решение легко получить, решив данную систему. Это достигается с помощью компьютерной программы, работающей на Python 3.9.

Ключевые слова: интегральное уравнение Вольтерра, интегральное уравнение Фредгольма, сплайн-функция.