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Stability and existence of multiperiodic solutions for second-order linear equations with a diagonal differentiation operator

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The stability of differential equations with periodic and quasiperiodic coefficients is a central topic in modern stability theory, with important applications in mechanics, physics, and dynamical systems. A classical result in this area is the Lyapunov integral criterion, which provides stability conditions for linear second-order equations with periodic coefficients. In this paper, we extend this criterion to equations with quasiperiodic coefficients. Our analysis is based on the method of periodic characteristics, which has proven effective in the study of multiperiodic solutions for systems with a diagonal differentiation operator. Within this framework, the multiperiodicity condition is reduced to a functional equation, and a Floquet-type representation of the matricant of the associated system is derived. This representation shows that multiperiodicity of solutions follows from the purely imaginary nature of the characteristic multipliers and the periodicity of the helical characteristics. The obtained results confirm that the Lyapunov integral criterion remains valid for equations with quasiperiodic coefficients. More generally, they demonstrate the effectiveness of the characteristic method for analyzing stability in complex dynamical systems, thereby extending the scope of classical stability theory.

Keywords: Lyapunov integral criterion, stability analysis, periodic coefficients, quasiperiodic coefficients, periodic characteristics method, multiperiodic solutions, Floquet theory, differential equations.

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Introduction

The Lyapunov integral criterion for the stability of linear second-order differential equations with periodic coefficients is a fundamental result in stability theory. In this paper, we extend this criterion to the case where the coefficients are quasiperiodic. Thus, our study addresses the stability of linear equations with both periodic and quasiperiodic coefficients.

The analysis is based on the method of periodic characteristics developed in [1, 2], which has been successfully applied to the study of multiperiodic solutions of systems with the diagonal differentiation operator. Classical results on the theory of stability and periodic solutions of differential equations, including the Lyapunov integral criterion, are presented in [3, 4]. Fundamental results on systems of partial differential equations and characteristic methods are discussed in [5]. The theory of almost periodic and almost multiperiodic solutions for ordinary, partial, and evolutionary differential equations is developed in [6–8]. These works provide the theoretical background for the stability analysis and the formulation of multiperiodicity conditions in terms of functional–difference equations.

Unlike traditional approaches, the method proposed in [1, 2] represents the multiperiodicity condition for D-equations as a functional equation, enabling the use of a Floquet representation for the

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matricant of the associated system. The multiperiodicity of solutions then follows from the purely imaginary nature of the multipliers and the periodicity of the helical characteristics.

Research on multiperiodic solutions of systems with differentiation along characteristics has been further advanced in [9–11]. Further developments concerning periodic and impulsive evolution equations are discussed in [12–14]. The existence and multiplicity of periodic and multiperiodic solutions for second-order and parameter-dependent equations are investigated in [15–18]. Methods based on differentiation along characteristics and diagonal operators have received international recognition through publications in leading scientific journals [19, 20]. Related solvability and boundary value problems for parabolic, nonlocal, loaded, and hyperbolic equations are considered in [21–23], as well as in more recent studies [24].

Beyond its theoretical significance, stability analysis of systems with periodic and quasiperiodic coefficients has a wide range of applications. These include oscillatory processes in mechanics, signal propagation in physics and engineering, and models of dynamical systems with multiple interacting frequencies. In such contexts, stability criteria are essential for predicting long-term behavior, preventing undesirable oscillations, and ensuring reliable system performance.

The purpose of this article is twofold: (i) to establish the applicability of the Lyapunov integral criterion to linear second-order equations with quasiperiodic coefficients, and (ii) to demonstrate the effectiveness of the periodic characteristics method in stability analysis of such equations.

1 Main results

Consider the equation with respect to $z = z(\tau, t)$, $\tau \in (-\infty, +\infty) = R$, $t = (t_1, ..., t_m) \in R\times ...\times R = R^m$ of the form

$$D^2 z + a(\tau, t)Dz + b(\tau, t)z = 0, \quad D = \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} \frac{\partial}{\partial t_j}, \tag{1}$$

$$a(\tau+\theta, t+\omega) = a(\tau, t) \in^{\theta,\omega} C^{1,e}_{\tau,t}(R\times R^m), \quad b(\tau+\theta, t+\omega) = b(\tau, t) \in C^{(0,e)}_{\tau,t}(R\times R^m),$$

$$\omega_0 = \theta, \quad \omega = (\omega_1, ..., \omega_m), \quad \omega_i/\omega_j \in Q(i, j = \overline{0, m}), \quad D^2 z = D(Dz),$$

where Q is the set of rational numbers.

${}^{\theta,\omega}C^{\alpha,\beta}_{\tau,t}(R\times R^m)$ is class (θ, ω)-periodic in $(\tau, t) \in R\times R^m$ function smoothness in them (α, β).

1) In the study of any problem for multiperiodic equations with a directional differentiation operator, the Cauchy characteristic method is used. The characteristic equations link the independent variables, and some of them lose their independence. For example, in our case, the variables $t_j, j = \overline{1, m}$ become functions of the variable $\tau : t_j = h_j(\tau)$, $j = \overline{1, m}$. Since the system is θ-periodic with respect to τ, it is desirable that $h_j(\tau)$ have the property of θ-periodicity: $h_j(\tau + \theta) = h_j$, $j = \overline{1, m}$.

If the characteristic equation of the operator D

$$\frac{dt}{d\tau} = e, \quad e = (1, ..., 1), \tag{2}$$

on a manifold of the type of Euclidean space with Cartesian coordinates, then the characteristics do not possess this periodicity property. Consequently, it is necessary to change the type of manifold where the field (2) defines θ-periodic characteristics. It turns out that the typical manifold is a multidimensional cylindrical surface [1, 3].

$$\frac{dt_j}{d\tau} = 1. \tag{3}$$

Consider the surface of a straight circular infinite cylinder with circumference S_θ of length $\theta = 2\pi r$. Then the solutions $t_j = \eta_j + \tau - \xi = \beta_j(\tau - \xi, \eta_j)$ along the cylinder perform a helical motion of the period θ, i.e., the point (τ, t_j) moves along a helical line. Consequently, the direct product of such motions is a motion on the surface m of an $\mathcal{M}^m = \mathcal{M} \times ... \times \mathcal{M}$, $(\tau, t_j) \in \mathcal{M} = R \times S_\theta$, $j = \overline{1, m}$ multidimensional cylindrical surface.

Thus, according to [1, 2], the equation

$$t = \beta(\tau - \xi, \eta) \equiv (\beta_1(\tau - \xi, \eta_1), ..., \beta_m(\tau - \xi, \eta_m)) \tag{4}$$

represents a multidimensional helical line defined by the equation (2)-(3), possessing the properties of periodicity and the group

$$\begin{gathered} \beta(\tau + \theta - \xi, \eta) = \beta(\tau - \xi, \eta), \\ \beta(\tau - \xi, \eta + \omega) = \beta(\tau - \xi, \eta) + \omega, \\ \beta(\xi - \sigma, \beta(\sigma - \tau, t)) = \beta(\xi - \tau, t), \end{gathered} \tag{5}$$

where $\eta = (\eta_1, ..., \eta_m)$.

It turns out that, for our purpose of multiperiodic solutions, it is expedient to consider the problem on m-dimensional cylindrical surface $(\tau, t) \in \mathcal{M}^m$.

Note that the equation (4) represents the characteristics of the operator D, and equation

$$\eta = \beta(\xi - \tau, t) \tag{6}$$

there is an equation of the first characteristic integrals:

$$D\beta(\xi - \tau, t) = 0. \tag{7}$$

Usually, the problem is investigated along the characteristics (4), and the results of the investigation are formulated in terms of the first integrals (6). The transition from characteristics to integrals and vice versa is realised by the property of the group from (5).

A sign of the appropriateness of the manifold is the equivalence of (θ, ω)-periodicity of the solution $x(\tau, t)$ of the system according to (τ, t) with (θ, θ, ω)-periodicity of the composition $x(\sigma, \beta(\sigma - \tau, t)) = x(\xi, \eta) \circ (\sigma, \beta(\sigma - \tau, t))$ according to (σ, τ, t) on this manifold at $\sigma \in R$.

It is obvious that this principle holds only on the cylindrical manifold $\mathcal{M}^m$.

2) It is widely known in the literature [3] that Lyapunov's integral criterion for the stability of periodic motions described by a second-order equation with an ordinary differentiation operator $D = \frac{d}{d\tau}$.

An interesting question is whether this property remains valid when the motions are multi-frequency, i.e., quasiperiodic.

This question will be explored below based on a rotational-linear modification [1,2] of Kharasahal's method [6].

To make it easier to control the equation (1), we should reduce the number of determining parameters a and b. To this end, let us set

$$z = e^{-\frac{1}{2}\int\limits_0^\tau a(\sigma, \beta(\sigma - \tau, t)) d\sigma} x, \quad (\tau, t) \in \mathcal{M}^m. \tag{8}$$

Taking into account (7), we write the equation (1) as

$$D^2 x + p(\tau, t) x = 0 \tag{9}$$

with a single coefficient $p(\tau, t)$ of the form

$$p(\tau + \theta, t + \omega) = p(\tau, t) = b(\tau, t) - \frac{1}{4} a(\tau, t)^2 - \frac{1}{2} Da(\tau, t) \in C^{(0,e)}_{\tau, t}(\mathcal{M}^m).$$

3) Next, we represent the equation (8) as a system

$$Dx_1 = x_2,$$

$$Dx_2 = -p(\tau,t)x_1, \quad D = \frac{\partial}{\partial\tau} + \sum_{j=1}^{m}\frac{\partial}{\partial t_j}, \tag{10}$$

$$p(\tau+\theta, t+\omega) = p(\tau,t) \in C_{\tau,t}^{(0,e)}(\mathcal{M}^m),$$

where $x = x_1$.

In order to construct a fundamental system of solutions

$$X(\tau,t) = \begin{pmatrix} \varphi(\tau,t) & \psi(\tau,t) \\ D\varphi(\tau,t) & D\psi(\tau,t) \end{pmatrix}. \tag{11}$$

From (10), we will construct solutions $\varphi(\tau,t)$ and $\psi(\tau,t)$ of equation (9) that satisfy the conditions

$$\varphi(\tau,t)|_{\tau=0} = 1, \quad D\varphi(\tau,t)|_{\tau=0} = 0, \tag{12}$$

$$\psi(\tau,t)|_{\tau=0} = 0, \quad D\psi(\tau,t)|_{\tau=0} = 1. \tag{13}$$

We will determine these solutions using the auxiliary equation

$$D^2\widetilde{x} = \mu p(\tau,t)\widetilde{x} \tag{14}$$

in the form of power series with respect to the parameter μ.

In accordance with this, we set

$$\widetilde{\varphi}(\tau,t,\mu) = \sum_{k=0}^{\infty}\varphi_k(\tau,t)\mu^k. \tag{15}$$

Substituting (15) into (14) and equating the coefficients with the same powers of the parameter μ, we have

$$D^2\varphi_0 = 0, \quad \varphi_0(0,t) = 1, \quad D\varphi_0(0,t) = 0, \tag{16}$$

$$D^2\varphi_k(\tau,t) = p(\tau,t)\varphi_{k-1}(\tau,t), \quad \varphi_k(0,t) = D\varphi_k(0,t) = 0, \quad k = 1,2,... \tag{17}$$

Then, from (16) and (17) we have

$$\varphi_0 = 1, \; \varphi_1(\tau,t) = \int_0^{\tau} d\sigma_1 \int_0^{\sigma_1} p(\sigma, \beta(\sigma-\sigma_1, \beta(\sigma_1-\tau,t)))d\sigma =$$

$$= \int_0^{\tau} d\sigma_1 \int_0^{\sigma_1} p(\sigma, \beta(\sigma-\tau,t))d\sigma =$$

$$= \int_0^{\tau} (\tau-\sigma)p(\sigma,\beta(\sigma-\tau,t))d\sigma, ...,$$

$$\varphi_k(\tau,t) = \int_0^{\tau} (\tau-\sigma)p(\sigma,\beta(\sigma-\tau,t)) \cdot \varphi_{k-1}(\sigma,\beta(\sigma-\tau,t))d\sigma, \; k = 1,2,... \tag{18}$$

Evaluating (18) at $|p(\tau,t)| \le \triangle = const \ge 1$, we obtain

$$|\varphi_1(\tau,t)| \le \triangle \left| \int\limits_0^\tau (\tau-\sigma)d\sigma \right| = \frac{\triangle|\tau|^2}{2!},$$

$$|\varphi_2(\tau,t)| \le \left| \int\limits_0^\tau (\tau-\sigma)\triangle \cdot \frac{\triangle\sigma^2}{2!} \right| \le \frac{\triangle}{2!} \left| \left[\frac{\tau\sigma^3}{3} - \frac{\sigma^4}{4} \right] \right|_0^\tau = \frac{\triangle^2\tau^4}{4!}, ...$$

If we set $|\varphi_k(\tau,t)| \le \frac{\triangle^k \tau^{2k}}{2k!}$, then $\varphi_{k+1}(\tau,t)$ satisfies the estimate

$$|\varphi_{k+1}(\tau,t)| \le \left| \int\limits_0^\tau (\tau-\sigma)\triangle \cdot \frac{\triangle^k\sigma^{2k}}{(2k)!} d\sigma \right| \le$$

$$\le \frac{\triangle^{k+1}}{(2k)!} \left| \frac{\tau\sigma^{2k+1}}{2k+1} - \frac{\sigma^{2k+2}}{2k+2} \right| |_0^\tau = \frac{\triangle^{k+1}|\tau|^{2k+2}}{(2k+2)!}, \quad k=0,1,2,... \tag{19}$$

Therefore, by virtue of (16)–(19), the series (15) converges at $|\mu| < \mu_0$, $|\tau| < T$ with any finite constants μ_0 and T.

Then, setting $\mu = -1$, we obtain the solution

$$\varphi(\tau,t) = \widetilde{\varphi}(\tau,t,-1) = \sum_{k=0}^{\infty} (-1)^k \varphi_k(\tau,t) \tag{20}$$

of equations (14) at $\mu = -1$, therefore, equations (10) satisfy the condition (12).

In a similar way, we determine the solution $\psi(\tau,t)$ of the initial problem (10)–(13), setting $\psi(\tau,t) = \widetilde{\psi}(\tau,t,\mu)|_{\mu=-1}$ and

$$\widetilde{\psi}(\tau,t,\mu) = \sum_{k=0}^{\infty} \psi_k(\tau,t)\mu^k, \quad \widetilde{\psi}(\tau,t,\mu)|_{\tau=0} = 0, \quad D\widetilde{\psi}(\tau,t,\mu)|_{\tau=0} = 1, \tag{21}$$

where $\widetilde{\psi}(\tau,t,\mu)$ is the solution to problem (14) with initial condition (21) with coefficients $\psi_k(\tau,t)$, $k=0,1,2,...$, which are determined sequentially by the formulas

$$\psi_0(\tau,t) = \tau, \quad \psi_k(\tau,t) = \int\limits_0^\tau (\tau-\sigma)p(\sigma,\beta(\sigma-\tau,t))\psi_{k-1}(\sigma,\beta(\sigma-\tau,t))d\sigma, \quad k=1,2,... \tag{22}$$

Thus, we have estimates

$$|\psi_0(\tau,t)| \le \frac{|\tau|}{1!}, \quad |\psi_1(\tau,t)| \le \left| \int\limits_0^\tau (\tau-\sigma)\sigma d\sigma \right| \le \triangle \left| \frac{\tau\sigma}{2} - \frac{\sigma^3}{3} \right| |_0^\tau = \frac{\triangle|\tau|^3}{3!}, ...,$$

$$|\psi_{k-1}(\tau,t)| \le \frac{\triangle^{k-1}|\tau|^{2k-1}}{(2k-1)!}, \quad |\psi_k(\tau,t)| \le \triangle \left| \int\limits_0^\tau (\tau-\sigma)\triangle^{k-1}\sigma^{2k-1}d\sigma \right| = \frac{\triangle^k|\tau|^{2k+1}}{(2k+1)!}, ...$$

for which the series (21) converges absolutely and uniformly in finite domains: $|\mu| < \mu_0$, $|\tau| < T$.

Then we obtain the second solution $\psi(\tau,t)$ of equation (10) in the form

$$\psi(\tau,t)=\sum_{k=0}^{\infty}(-1)^k\psi_k(\tau,t), \tag{23}$$

by setting $\mu=-1$ from (21) with coefficients (22).

It is not particularly difficult to show that these solutions (20) and (23) are differentiable.

Thus, we have

$$D\varphi(\tau,t)=\sum_{k=1}^{\infty}(-1)^k D\varphi_k(\tau,t),\quad D\varphi_k(\tau,t)=\int_0^{\tau}p(\sigma,\beta(\sigma-\tau,t))\varphi_{k-1}(\sigma,\beta(\sigma-\tau,t))d\sigma, \tag{24}$$

$$D\psi(\tau,t)=1+\sum_{k=1}^{\infty}(-1)^k D\psi_k(\tau,t),\quad D\psi_k(\tau,t)=\int_0^{\tau}p(\sigma,\beta(\sigma-\tau,t))\psi_{k-1}(\sigma,\beta(\sigma-\tau,t))d\sigma$$

and based on them we obtain the matrix $X(\tau,t)$ of system (10).

4) It is easy to show that the determinant

$$w(\tau,t)=detX(\tau,t)$$

of the matrix $X(\tau,t)$ of system (10) satisfies the equation

$$Dw=SpP(\tau,t)\cdot w,\quad w|_{\tau=0}=detX(0,t),$$

where $SpP(\tau,t)$ is the trace of the matrix. $P(\tau,t)$ of the form

$$P(\tau,t)=\begin{pmatrix}0 & 1\\ -p(\tau,t) & 0\end{pmatrix}$$

of system (10) and is equal to the sum of its diagonal elements:

$$SpP(\tau,t)=0.$$

Therefore,

$$w(\tau,t)=1.$$

The proof of this Ostrogradsky-Jacobi formula is similar to the proof based on linear characteristics [6]. Therefore, we will not dwell on the proof.

Statement 1. For the matricant (11) of system (10), the Ostrogradsky–Jacobi formula holds true

$$detX(\tau,t)=e^{\int_0^{\tau}SpP(\sigma,\beta(\sigma,\tau,t))d\sigma}=1.$$

5) The matricant X of the system

$$Dx=P(\tau,t)x,\quad P(\tau+\theta,t+\omega)=P(\tau,t)\in^{\theta,\omega}C^{0,e}_{\tau,t}(\mathcal{M}^m) \tag{25}$$

has the properties

$$\begin{gathered}DX(\tau,t)=P(\tau,t)X(\tau,t),\quad X(0,t)=E,\\ X(\tau,t+\omega)=X(\tau,t)\in^{0,\omega}C^{(1,e)}_{(\tau,t)}(\mathcal{M}^m),\\ X(\tau+\theta,t)=X(\tau,t)X(\theta,\beta(-\tau,t)),\quad(\tau,t)\in\mathcal{M}^m,\end{gathered} \tag{26}$$

where E is the identity matrix.

Then, by virtue of (26), we have

$$X(\tau + k\theta, t) = X(\tau, t)X^k(\theta, \beta(-\tau, t)), \quad k = 0, 1, 2, ...$$

Hence, setting $\tau = 0$, we obtain

$$X(k\theta, t) = X^k(\theta, t), \quad (k\theta, t) \in \mathcal{M}^m, \quad k = \overline{0, +\infty}.$$

For any fixed value of $t = \eta$, we have the logarithm

$$\frac{1}{k\theta} LnX(k\theta, \eta) = \frac{1}{\theta} LnX(\theta, \eta), \quad k = \overline{1, \infty}.$$

From here, moving to the limit at $\xi = k\theta \longrightarrow \infty$, we obtain

$$\frac{1}{\theta} LnX(\theta, \eta) = \lim_{\xi \longrightarrow \infty} \frac{1}{\xi} LnX(\xi, \eta), \quad (\xi, \eta) \in \mathcal{M}^m.$$

Further, setting

$$LnX(\theta, \eta) = \Lambda(\eta), \quad (\theta, \eta) \in \mathcal{M}^m,$$

we have the representation

$$X(\theta, \eta) = e^{\Lambda(\eta)}, \quad \eta \in S_\theta^m.$$

Due to the arbitrariness of $\eta \in S_\theta^m$, from the properties (26) we obtain

$$\Lambda(\eta + \omega) = \Lambda(\eta) \in^\omega C_\eta^{(e)}(S_\theta^m).$$

For the first integral $\eta = \beta(-\tau, t)$ corresponding to the value η from the last two equalities, the properties follow

$$X(\theta, \beta(-\tau, t)) = e^{\Lambda(\beta(-\tau, t))},$$

$$\Lambda(\beta(-\tau - \theta, t)) = \Lambda(\beta(-\tau, t)),$$

$$\Lambda(\beta(-\tau, t + \omega)) = \Lambda(\beta(-\tau, t) + \omega) = \Lambda(\beta(-\tau, t)), \quad (\tau, t) \in \mathcal{M}^m.$$

As a result, due to last equalities from (26), we have the Floquet representation for the system (25) of the form

$$X(\tau, t) = Y(\tau, t)e^{\frac{\tau}{\theta}\Lambda(\beta(-\tau, t))}, \quad (\tau, t) \in \mathcal{M}^m, \tag{27}$$

$$Y(\tau, t) = X(\tau, t)e^{-\frac{\tau}{\theta}\Lambda(\beta(-\tau, t))}, \quad Y(\tau + \theta, t + \omega) = Y(\tau, t) \in^{\theta, \omega} C_{\tau, t}^{(1, e)}(\mathcal{M}^m).$$

Then we can formulate the reducibility theorem.

Theorem 1. The linear system (25) can be reduced to the form

$$x = Y(\tau, t)y, \quad (\tau, t) \in \mathcal{M}^m \tag{28}$$

using the transformation can be reduced to the system

$$Dy = \frac{1}{\theta}\Lambda(\beta(-\tau, t)) \cdot y, \quad (\tau, t) \in \mathcal{M}^m. \tag{29}$$

Corollary 1. A quasiperiodic linear system

$$\frac{dx}{d\tau} = P(\tau,t)x, \quad \frac{dt}{d\tau} = e, \quad P(\tau+\theta, t+\omega) = P(\tau,t) \in {}^{\theta,\omega}C^{(0,e)}_{\tau,t}(R \times R^m) \tag{30}$$

by means of a quasiperiodic substitution

$$x = Y(\tau, \delta(\tau,\eta))y, \quad \tau \in R, \quad \eta \in R^m \tag{31}$$

can be reduced to the system

$$\frac{dy}{d\tau} = \frac{1}{\theta}\Lambda(\eta)y \tag{32}$$

with any fixed $\eta \in R^m$, where η is the value t in the neighbourhood of which the question of reduction is considered, $t = \delta(\tau,\eta)$ is the diagonal characteristic originating from the point $(0,\eta)$.

Theorem 1 remains valid for rectilinear motions, therefore, the characteristic $t = \beta(\tau,\eta)$ can be replaced by $t = \delta(\tau,\eta)$. Then, from the theorem follows consequence 1 regarding equations (30), (32) and transformation (31) in accordance with systems (25), (29) and substitution (28).

Note that if we wrap an isosceles right triangle with side θ around a straight circular cylinder with circumference θ, then the hypotenuse of the triangle becomes a harmonic spiral with pitch $\varphi = 45^0$: $\operatorname{tg}\varphi = \frac{dt_j}{d\tau} = 1$.

$$u = \tau, \quad v = r\sin\frac{2\pi\tau}{\theta} = r\sin\frac{\tau}{r}, \quad \omega = r\cos\frac{2\pi\tau}{\theta} = r\cos\frac{\tau}{r}.$$

$S_{\breve{AC}} = S_{\overline{CE}} = \tau$ the length of the arc $\breve{AC}$ and the length of the rise $\overline{CE}$ are equal to τ.

These geometric interpretations clearly show the connection between rectilinear and circular motions. The measurement of the lengths of circular arcs of a helical line is automatically transferred to the measurement of segment lengths. The main barrier to understanding is psychological in nature and related to the topology of surface lines.

Next, we will examine the question of the stability of the motions described by the system of equations (10) based on the multipliers of the monodromy matrix $X(\theta,\eta)$.

To this end, let us consider the characteristic equation of the matrix $X(\theta,\eta)$ and, based on (10)–(13), (20) and (23), we have

$$det[X(\theta,\eta) - \rho E] = \begin{vmatrix} \varphi(\theta,\eta) - \rho & \psi(\theta,\eta) \\ D\varphi(\theta,\eta) & D\psi(\theta,\eta) - \rho \end{vmatrix} = 0.$$

Obviously, from the characteristic equation $h(\eta,\rho) \equiv det[X(\theta,\eta) - \rho E]$ of the monodromy matrix $X(\theta,\eta)$, it follows that the function $h(\eta,\rho)$ is ω-periodic in η and continuously differentiable with respect to its arguments; moreover, in our case, the roots are distinct: $\rho_1(\eta) \neq \rho_2(\eta)$ and nonzero. When the multiplicities of the roots do not depend on the independent variable η, they can be determined using the implicit function theorem, according to which the properties of the coefficients are inherited by the roots of the equation. Consequently, we obtain $\rho_i(\eta+\omega) = \rho_i(\eta) \in C^{(e)}_\eta(R^m)$, $i = \overline{1,n}$ and the existence of the logarithm of the monodromy matrix $X(\theta,\eta)$.

From this, taking into account Statement 1, we obtain

$$\rho^2 - a\rho + 1 = 0, \tag{33}$$

where $a = a(\eta)$ is determined by the relation

$$a(\eta) = \varphi(\theta,\eta) + D\psi(\theta,\eta), \quad a(\eta+\omega) = a(\eta) \in C^{(e)}_\eta(S^m_\theta), \tag{34}$$

which can be called the Lyapunov's multiperiodic characteristic coefficient with respect to the parameter η for the system (10).

By virtue of (18) from (20) we have

$$\varphi(\theta,\eta)=1-\int\limits_0^\theta(\theta-\sigma_1)p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1+$$

$$+\int\limits_0^\theta(\theta-\sigma_1)p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1\int\limits_0^{\sigma_1}(\sigma_1-\sigma_2)p(\sigma_2,\beta(\sigma_2,\eta))d\sigma_2+...+$$

$$+(-1)^k\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2...\int\limits_0^{\sigma_{k-1}}(\theta-\sigma_1)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta)))...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k+...$$

Similarly, due to (23) and (24), we have

$$D\psi(\theta,\eta)=1-\int\limits_0^\theta\sigma_1p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1+$$

$$+\int\limits_0^\theta p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1\int\limits_0^{\sigma_1}(\sigma_1-\sigma_2)\sigma_2p(\sigma_2,\beta(\sigma_2,\eta))d\sigma_2+...+(-1)^k\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2...$$

$$...\int\limits_0^{\sigma_{k-1}}(\theta-\sigma_1+\sigma_2)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))p(\sigma_2,\beta(\sigma_2,\eta))...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k+...$$

Due to last two equalities, from (34) we define the Lyapunov characteristic coefficient of equation (33) for the second-order multiperiodic system (10) in the form

$$a(\eta)=2-\theta\int\limits_0^\theta p(\sigma_1,\beta(\sigma_1,\eta))d\sigma_1+$$

$$+\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}(\theta-\sigma_1+\sigma_2)(\sigma_1-\sigma_2)p(\sigma_1,\beta(\sigma_1,\eta))p(\sigma_2,\beta(\sigma_2,\eta))d\sigma_2+...+$$

$$+(-1)k\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2...\int\limits_0^{\sigma_{k-2}}d\sigma_{k-1}\int\limits_0^{\sigma_{k-1}}(\theta-\sigma_1+\sigma_k)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))...\times$$

$$\times...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k+... \tag{35}$$

Next, let us assume that $p(\tau,t)\leq 0$ on $\mathcal{M}^m$, i.e., a sign-negative function, and there exists a point $(\xi,\eta)\in\mathcal{M}^m$ such that

$$p(\xi,\eta)<0,\quad(\xi,\eta)\in\mathcal{M}^m. \tag{36}$$

Then $\int\limits_0^\theta p(\sigma,\beta(\sigma,\eta))d\sigma<0$ and

$$(-1)^k\int\limits_0^\theta d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2...\int\limits_0^{\sigma_{k-2}}d\sigma_{k-1}\int\limits_0^{\sigma_{k-1}}(\theta-\sigma_1+\sigma_k)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))...\times$$

$$\times ... p(\sigma_k, \beta(\sigma_k, \eta)) d\sigma_k \geq 0, \quad k = 1, 2, ...$$

Consequently, from (35) and (36) we have

$$a(\eta) > 2$$

and the roots $\rho_{1,2}$ of equation (33) are distinct and real, and

$$\rho_1 = \frac{1}{2}(a - \sqrt{a^2 - 4}) < 1, \quad \rho_2 = \frac{1}{2}(a + \sqrt{a^2 - 4}) > 1. \tag{37}$$

Since the solution $x(\tau, t)$ of system (10) with the initial condition

$$x(\tau, t)|_{\tau=\xi} = x^0(t) \in^{\omega} C_t^{(e)}(S_\theta^m)$$

by virtue of (27) at $\eta = \beta(\xi - \tau, t)$, in accordance with Theorem 1, defined by the relation

$$x(\tau, t) = Y(\tau, t) e^{\frac{\tau}{\theta} ln X(\theta, \beta(\xi - \tau, t))} x^0(\beta(\xi - \tau, t)),$$

then from the properties of the multipliers $\rho_{1,2}$ (37) we have the instability of the system (10), and therefore of the equation (9).

Thus, we can formulate Theorem 2.

Theorem 2. Equation (9) with (θ, ω)-periodic $(0, e)$-smooth sign-negative function $p(\tau, t) \neq 0$ at $(\tau, t) \in \mathcal{M}^m$ is unstable, and its multipliers are positive, with one of them greater than unity and the other less than unity.

Now let us consider the case where $p(\tau, t) \neq 0$ is positive, i.e. there exists a point (ξ, η) where $p(\xi, \eta) > 0$.

Next, we estimate the multiperiod Lyapunov characteristic coefficient $a = a(p)$, which depends on the parameter $p \in S_\theta^m$, and we have the inequality

$$I_1 = \theta \int_0^\theta p(\sigma_1, \beta(\sigma_1, \eta)) d\sigma_1 > 0, \quad p \in S_\theta^m.$$

Along with this, let us assume that $p(\tau, t)$ satisfied the condition

$$\theta \int_0^\theta p(\sigma_1, \beta(\sigma_1, \eta)) d\sigma_1 \leq 4, \quad p \in S_\theta^m. \tag{38}$$

Then, taking into account $0 \leq \sigma_k < \sigma_1$, $\theta - \sigma_1 + \sigma_k < \theta$ and

$$(\theta - \sigma_1 + \sigma_{k+1})(\sigma_k - \sigma_{k+1}) \leq \frac{1}{4}(\theta - \sigma_1 + \sigma_k)^2 < \frac{\theta}{4}(\theta - \sigma_1 + \sigma_k)$$

by virtue of the estimate $xy \leq (\frac{x+y}{2})^2$, we obtain the following estimate:

$$I_{k+1} = \int_0^\theta d\sigma_1 \int_0^{\sigma_1} d\sigma_2 ... \int_0^{\sigma_{k-1}} d\sigma_k \int_0^{\sigma_k} (\theta - \sigma_1 + \sigma_{k+1})(\sigma_1 - \sigma_2)...(\sigma_k - \sigma_{k+1}) p(\sigma_1, \beta(\sigma_1, \eta)) ... \times$$

$$\times ... p(\sigma_{k+1}, \beta(\sigma_{k+1}, \eta)) d\sigma_{k+1} = \int_0^\theta d\sigma_1 \int_0^{\sigma_1} d\sigma_2 ... \int_0^{\sigma_{k-2}} d\sigma_{k-1} \int_0^{\sigma_{k-1}} (\theta - \sigma_1 + \sigma_{k+1})(\sigma_1 - \sigma_2) ... \times$$

$$\times ...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k \cdot \int\limits_0^{\sigma_k}(\sigma_k-\sigma_{k+1})p(\sigma_{k+1},\beta(\sigma_{k+1},\eta))d_{k+1} <$$

$$< \int\limits_0^{\theta}(\theta-\sigma_1+\sigma_{k+1})(\sigma_k-\sigma_{k+1})d\sigma_{k+1}\int\limits_0^{\sigma_1}d\sigma_1 ... \int\limits_0^{\sigma_{k-2}}d\sigma_{k-1}\int\limits_0^{\sigma_{k-1}}(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)\times$$

$$\times p(\sigma_1,\beta(\sigma_1,\eta))...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k\int\limits_0^{\theta}p(\sigma_{k+1},\beta(\sigma_{k+1},\eta))d\sigma_{k+1} <$$

$$< \int\limits_0^{\theta}d\sigma_1\int\limits_0^{\sigma_1}d\sigma_2 ... \int\limits_0^{\sigma_{k-2}}d\sigma_{k-1}\int\limits_0^{\sigma_{k-1}}\frac{\theta}{4}(\theta-\sigma_1+\sigma_k)(\sigma_1-\sigma_2)...(\sigma_{k-1}-\sigma_k)p(\sigma_1,\beta(\sigma_1,\eta))...\times$$

$$\times ...p(\sigma_k,\beta(\sigma_k,\eta))d\sigma_k \cdot \int\limits_0^{\theta}p(\sigma,\beta(\sigma,\eta))d\sigma = \frac{\theta}{4}\int\limits_0^{\theta}p(\sigma,\beta(\sigma,\eta))d\sigma \cdot I_k.$$

From this, by virtue of (38), we have

$$0 < I_{k+1}(\eta) \le I_k(\eta), \quad I_k(\eta) \to 0 \ \ at \ \ k \to \infty.$$

It is obvious that the series

$$a(\eta) = 2 - I_1(\eta) + I_2(\eta) - I_3(\eta) + ... + (-1)^k I_k(\eta) + ..., \quad \eta \in S_\theta^m,$$

are series. Leibniz, however, the estimate

$$a(\eta) - \theta\int\limits_0^{\theta}p(\sigma,\beta(\sigma,\eta))d\sigma < a < 2, \quad \eta \in S_\theta^m,$$

i.e., $-2 < a < 2$.

Consequently, the roots $p_{1/2}(\eta) = \frac{1}{2}[a(\eta) \pm \sqrt{a^2(\eta)-4}]$ are distinct and complex conjugate, and $|p_{1/2}(\eta)| = 1$.

Thus, the solution of the system (10), and therefore the equations (9), are bounded for all $\eta \in S_\theta^m$, i.e., they are stable.

Theorem 3. Equations (9) with (θ,ω)-periodic smooth positive function $p(\tau,t) \neq 0$ multipliers $\rho_{1,2}(\eta)$ under the condition of Lyapunov

$$0 < \theta\int\limits_0^{\theta}p(\sigma,\beta(\sigma,\eta))d\sigma < 4, \quad \eta \in S_\theta^m \tag{39}$$

are distinct, complex conjugate, and their moduli $|\rho_{1/2}(\eta)| = 1$, and therefore equation (9) is stable.

In conclusion, considering that equation (9) is an extended representation of an ordinary differential equation with a quasi-periodic coefficient of the form

$$\frac{d^2}{d\tau^2}x(\tau,e\tau) + p(\tau,e\tau)x(\tau,e\tau) = 0, \tag{40}$$

$$p(\tau+\theta,t+\omega) = p(\tau,t) \in C_{\tau,t}^{(0,e)}(R\times R^m), \quad p(\tau,t) \neq 0,$$

the Theorem 3 is also valid in the case of (40), and we have the following important corollary, where $e\tau = \beta(\tau,0)$.

Corollary 2. If the coefficient $p(\tau, t)$ of equation (40) is positive and satisfies inequality

$$0 < \theta \int\limits_0^\theta p(\sigma, e\sigma) d\sigma < 4,$$

then all solutions of this equation are limited along with their first-order derivatives on the numerical axis R, therefore, equation (40) is stable, and the multipliers $\rho_{1/2}(0)$ are different, complex conjugate and $|\rho_{1/2}(0)| = 1$.

The proof of Corollary 2 follows from Theorem 3 at $\eta = 0$, and last condition is derived from (39) at zero η. The limitation of the first derivatives of solutions is due to the fact that equation (40) is equivalent to a system of two first-order equations.

Conclusion

It should be noted that the Lyapunov method is closely related to the approach proposed by V. Kharasakhal concerning the transition from the ordinary differentiation operator to the D-differentiation operator along the diagonal, defined on a multidimensional cylindrical manifold. Within this framework, it becomes possible to investigate the stability of the Mathieu equation with quasiperiodic coefficients, which represents an important applied aspect of the present study.

If the multiperiodic coefficient of the considered equation is given in a stepwise form with respect to each variable, then, in the purely periodic case, methods of numerical analysis can be effectively employed for qualitative investigations.

Overall, the conducted research provides promising prospects for the further development of the theory of multifrequency oscillations described by Lyapunov-type equations.

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Author Contributions

G.M. Aitenova drafted the manuscript and integrated the research findings into a coherent paper. Zh.A. Sartabanov conceived the central idea of the study, secured the research funding, and provided overall supervision of the project. B.Zh. Omarova contributed to data collection and analysis, supporting the validation of results. A.Kh. Zhumagaziyev participated in data acquisition and analysis and coordinated the manuscript revision process. All authors critically reviewed the manuscript, provided intellectual input at various stages of the research, and approved the final version for submission.

Conflict of Interest

The authors declare no conflict of interest.

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