

Source identification problems for the neutron transport equations

A. Taskin*

ENKA Schools, Istanbul, Turkey;
 Yildiz Technical University, Istanbul, Turkey
 (E-mail: gafurtaskin@hotmail.com, abdulgafur.taskin@yildiz.edu.tr)

In this study, the time-dependent source identification problem for the two-dimensional neutron transport equation was studied. For the approximate solution of this problem a first order of accuracy difference scheme was presented. Stability estimates for the solution of these differential and difference problems were established. Numerical results were given.

Keywords: identification problem, neutron transport equation, difference scheme, differential equation, stability inequality.

2020 Mathematics Subject Classification: 65J22, 39A14, 82D75.

Introduction

The neutron transport equation describes the distribution of neutrons in terms of their positions in space and time, their energies and their travel directions. The various neutron transport equations are studied by many researchers (see, [1–4] and the references given therein). Identification problems play an important role in applied sciences and engineering applications and have been investigated in various papers (see, e.g., [5–27] and the references given therein). In the present paper, we consider the time-dependent source identification problem for two dimensional neutron transport equation

$$\left\{ \begin{array}{l} \frac{\partial u(t,x,y)}{\partial t} = \frac{\partial u(t,x,y)}{\partial x} + \frac{\partial u(t,x,y)}{\partial y} + p(t)q(x,y) + f(t,x,y), \\ t \in (0, T), x, y \in (0, L), \\ u(0, x, y) = \varphi(x, y), \quad x, y \in [0, L], \\ u(t, 0, y) = 0, \quad u(t, x, 0) = 0, \quad t \in [0, T], \quad x, y \in [0, L], \\ u(t, l, y) = \alpha(t, y), \quad t \in [0, T], \quad y \in [0, L], \quad l \in (0, L). \end{array} \right. \quad (1)$$

Here, $u(t, x, y)$ and $p(t)$ are unknown functions, $f(t, x, y)$, $q(x, y)$, $\varphi(x, y)$, and $\alpha(t, y)$ are given sufficiently smooth functions and all compatibility conditions are satisfied.

In the rest of paper, the theorem on the stability of differential problem (1) is established. For the approximate solution of problem (1), a first order of accuracy difference scheme is proposed. The theorem on stability of this difference scheme is established. Some results of numerical experiment are presented.

*Corresponding author. E-mail: gafurtaskin@hotmail.com

Received: 29 February 2024; Accepted: 29 May 2024.

© 2024 The Authors. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>)

1 Stability of differential equation

To formulate our results, we introduce the Banach space $C(E) = C([0, T], E)$ of all abstract continuous functions $\phi(t)$ defined on $[0, T]$ with values in E equipped with the norm

$$\|\phi\|_{C(E)} = \max_{0 \leq t \leq T} \|\phi(t)\|_E.$$

Let $E = C_{[0, L] \times [0, L]}$ be the space of all continuous functions $\psi(x, y)$ defined on $[0, L] \times [0, L]$ equipped with norm

$$\|\psi\|_{C_{[0, L] \times [0, L]}} = \max_{0 \leq x, y \leq L} |\psi(x, y)|$$

and $C_{[0, L] \times [0, L]}^{(1)}$ be the space of all continuously differentiable functions $\psi(x, y)$ defined on $[0, L] \times [0, L]$ equipped with norm

$$\|\psi\|_{C_{[0, L] \times [0, L]}^{(1)}} = \|\psi\|_{C_{[0, L] \times [0, L]}} + \max_{0 < x, y < L} |\psi_x(x, y)| + \max_{0 < x, y < L} |\psi_y(x, y)|.$$

We introduce the positive operator A , defined by formula

$$Au = - \left(\frac{\partial u(x, y)}{\partial x} + \frac{\partial u(x, y)}{\partial y} \right)$$

with the domain

$$D(A) = \{u : u, u_x, u_y \in C_{[0, L] \times [0, L]}, u(0, y) = u(x, 0) = 0, 0 \leq x, y \leq L\}.$$

Throughout the present paper, M denotes positive constants, which may differ in time and thus are not a subject of precision. However, we will use $M(\alpha, \beta, \gamma, \dots)$ to stress the fact that the constant depends only on $\alpha, \beta, \gamma, \dots$.

We have the following theorem on the stability of problem (1):

Theorem 1. Assume that $\varphi \in C_{[0, L] \times [0, L]}^{(1)}$, $f(t, x, y)$ is a continuously differentiable function in t and continuous in x and y , and $\alpha(t, y)$ is a continuously differentiable function in t and continuous in y . Then, for the solution of problem (1) the following stability estimates hold:

$$\begin{aligned} & \left\| \frac{\partial u}{\partial t} \right\|_{C(C_{[0, L] \times [0, L]})} + \|u\|_{C(C_{[0, L] \times [0, L]}^{(1)})} + \|p\|_{C[0, T]} \leq M(q) \left[\|\varphi\|_{C_{[0, L] \times [0, L]}^{(1)}} + \right. \\ & \left. + \|f(0, \cdot)\|_{C_{[0, L] \times [0, L]}} + \left\| \frac{\partial f}{\partial t} \right\|_{C(C_{[0, L] \times [0, L]})} + \|\alpha(0, \cdot)\|_{C[0, L]} + \|\alpha_t\|_{C(C[0, L])} \right]. \end{aligned}$$

Proof. We will use the following substitution

$$u(t, x, y) = w(t, x, y) + \eta(t) q(x, y),$$

where $\eta(t)$ is the function defined by formula

$$\eta(t) = \int_0^t p(s) ds, \quad \eta(0) = 0. \tag{2}$$

It is clear that $w(t, x, y)$ is the solution of the following initial boundary value problem

$$\left\{ \begin{array}{l} \frac{\partial w(t,x,y)}{\partial t} = \frac{\partial w(t,x,y)}{\partial x} + \frac{\partial w(t,x,y)}{\partial y} + \eta(t) (q_x(x,y) + q_y(x,y)) + f(t,x,y), \\ t \in (0,T), \quad x,y \in (0,L), \\ w(0,x,y) = \varphi(x,y), \quad x,y \in [0,L], \\ w(t,0,y) = 0, \quad t \in [0,T], \quad y \in [0,L], \\ w(t,x,0) = 0, \quad t \in [0,T], \quad x \in [0,L], \\ q(x,0) = 0, \quad q(0,y) = 0, \quad q(l,y) \neq 0, \\ w(t,l,y) = \alpha(t,y) - \eta(t) q(l,y), \quad t \in [0,T], \quad y \in [0,L]. \end{array} \right. \quad (3)$$

Applying the over determined condition $u(t,l,y) = \alpha(t,y)$ at substitution (2), we get

$$w(t,l,y) + \eta(t) q(l,y) = \alpha(t,y),$$

$$\eta(t) = \frac{\alpha(t,y) - w(t,l,y)}{q(l,y)}.$$

From that and $p(t) = \eta'(t)$, it follows

$$p(t) = \frac{\alpha_t(t,y) - w_t(t,l,y)}{q(l,y)}. \quad (4)$$

From identity (4) and the triangle inequality, we get the estimate

$$\begin{aligned} |p(t)| &= \left| \frac{\alpha_t(t,y) - w_t(t,l,y)}{q(l,y)} \right| \leq M(q) [|\alpha_t(t,y)| + |w_t(t,l,y)|] \leq \\ &\leq M(q) \left[\max_{0 \leq t \leq T} |\alpha_t(t,y)| + \max_{0 \leq t \leq T} \max_{0 \leq y \leq L} |w_t(t,l,y)| \right]. \end{aligned}$$

From that it follows

$$\|p\|_{C[0,T]} \leq M(q) \left[\|\alpha_t\|_{C(C[0,T], C[0,L])} + \|w_t\|_{C(C[0,T], C[0,L])} \right]. \quad (5)$$

Using operator A with the domain $D(A)$ we can rewrite problem (3) in the abstract form as an initial value problem

$$\left\{ \begin{array}{l} \frac{dw}{dt} + Aw = -\frac{\alpha(t,\cdot) - w(t,l,\cdot)}{q(l,\cdot)} Aq + f(t), \\ w(0) = \varphi. \end{array} \right.$$

By the Cauchy formula, the solution can be written as

$$w(t) = e^{-tA} \varphi + \int_0^t e^{-(t-s)A} \left\{ -\frac{\alpha(s,\cdot) - w(s,l,\cdot)}{q(l,\cdot)} Aq + f(s) \right\} ds.$$

Taking derivative with respect to t and using Leibniz integral rule, we obtain

$$w_t(t) = -Ae^{-tA}\varphi + \left\{ -\frac{\alpha(t,\cdot) - w(t,l,\cdot)}{q(l,\cdot)}Aq + f(t) \right\} + \int_0^t -Ae^{-(t-s)A} \left\{ -\frac{\alpha(s,\cdot) - w(s,l,\cdot)}{q(l,\cdot)}Aq + f(s) \right\} ds.$$

Applying the integration by parts formula, we get

$$w_t(t) = -Ae^{-tA}\varphi + e^{-tA} \left\{ -\frac{\alpha(0,\cdot) - w(0,l,\cdot)}{q(l,\cdot)}Aq + f(0) \right\} + \int_0^t e^{-(t-s)A} \left\{ -\frac{\alpha_s(s,\cdot) - w_s(s,l,\cdot)}{q(l,\cdot)}Aq + f'(s) \right\} ds = \sum_{k=1}^3 G_k(t),$$

where

$$G_1(t) = -Ae^{-tA}\varphi,$$

$$G_2(t) = e^{-tA} \left\{ -\frac{\alpha(0,\cdot) - w(0,l,\cdot)}{q(l,\cdot)}Aq + f(0) \right\},$$

$$G_3(t) = \int_0^t e^{-(t-s)A} \left\{ -\frac{\alpha_s(s,\cdot) - w_s(s,l,\cdot)}{q(l,\cdot)}Aq + f'(s) \right\} ds.$$

Now, we estimate, G_1 , G_2 , and G_3 , separately. Using the triangle inequality, we obtain

$$\|w_t\|_E \leq \|G_1(t)\|_E + \|G_2(t)\|_E + \|G_3(t)\|_E.$$

It is known (see [20]) that for any $t \in [0, T]$,

$$\|e^{-tA}\|_{E \rightarrow E} \leq Me^{-\delta t}, \quad M > 0, \quad \delta > 0. \tag{6}$$

Applying the definition of norm of the spaces E and estimate (6), we get

$$\|G_1(t)\|_E = \|-Ae^{-tA}\varphi\|_E \leq \|e^{-tA}\|_{E \rightarrow E} \|A\varphi\|_E \leq M_1(\delta) \|A\varphi\|_E. \tag{7}$$

Let us estimate $G_2(t)$. Using the triangle inequality, we get

$$\begin{aligned} \|G_2(t)\|_E &= \left\| e^{-tA} \left\{ -\frac{\alpha(0,\cdot) - w(0,l,\cdot)}{q(l,\cdot)}Aq + f(0) \right\} \right\|_E \leq \\ &\leq \|e^{-tA}\|_{E \rightarrow E} \left[\left| \frac{\alpha(0,\cdot)}{q(l,\cdot)} \right| + \left| \frac{w(0,l,\cdot)}{q(l,\cdot)} \right| \right] \|Aq\|_E + \|f(0)\|_E, \\ \|G_2(t)\|_E &\leq \|e^{-tA}\|_{E \rightarrow E} \left\{ \|Aq\|_{E \rightarrow E} \frac{\|\alpha(0,\cdot)\|_E + \|w(0,l,\cdot)\|_E}{\min_{0 \leq y \leq L} |q(l,\cdot)|} + \|f(0)\|_E \right\}. \end{aligned}$$

Hence,

$$\|G_2(t)\|_E \leq M_2(\delta, q) [\|\alpha(0,\cdot)\|_E + \|\varphi\|_E + \|f(0)\|_E] \tag{8}$$

for any $t, t \in [0, T]$.

Let us estimate $G_3(t)$. Using the triangle inequality, we get

$$\begin{aligned} \|G_3(t)\|_E &\leq \int_0^t \|e^{-(t-s)A}\|_{E \rightarrow E} \left\{ \frac{\max_{0 \leq s \leq T} |\alpha_s(s,\cdot)|_{E'} + \|w_s(s,\cdot)\|_E}{\min_{0 \leq y \leq L} |q(l,\cdot)|} \|Aq\|_E + \|f'(s)\|_E \right\} ds \leq \\ &\leq M_3(\delta, q) \int_0^t \left[\max_{0 \leq s \leq T} \|f'(s)\|_E + \max_{0 \leq s \leq T} |\alpha_s(s,\cdot)| \right] ds + \int_0^t M_4(\delta, q) \|w_s(s)\|_E ds, \end{aligned} \tag{9}$$

where $E' \subset E$.

Combining estimates (7), (8), and (9), we get

$$\begin{aligned} \|w_t\|_E &\leq M_1(\delta) \|A\varphi\|_E + M_2(\delta, q) [\|\alpha(0, \cdot)\|_E + \|\varphi\|_E + \|f(0)\|_E] + \\ &+ M_3(\delta, q) \int_0^t \left[\max_{0 \leq s \leq T} \|f'(s)\|_E + \max_{0 \leq s \leq T} \|\alpha_s(s, \cdot)\|_{E'} \right] ds + \int_0^t M_4(\delta, q) \|w_s(s)\|_E ds. \end{aligned}$$

Using Grönwall's inequality, we can write

$$\|w_t\|_E \leq M_5 e^{M_4(\delta, q)T},$$

where

$$M_5 = M_6(\delta, q) \left[\|A\varphi\|_E + \|\alpha(0, \cdot)\|_E + \|\varphi\|_E + \|f(0)\|_E + \max_{0 \leq s \leq T} \|f'(s)\|_E + \max_{0 \leq s \leq T} |\alpha_s(s, \cdot)| \right]. \tag{10}$$

Finally, combining estimates (10) and (5) it completes the proof of Theorem 1.

2 Stability of difference scheme

For the approximate solution of problem (1) we present the first order of accuracy difference scheme

$$\left\{ \begin{aligned} \frac{u_{n,m}^k - u_{n,m}^{k-1}}{\tau} &= \frac{u_{n+1,m+1}^k - u_{n,m+1}^k}{h} + \frac{u_{n,m+1}^k - u_{n,m}^k}{h} + p^k q_{n,m} + f_{n,m}^k, \\ f_{n,m}^k &= f(t_k, x_n, y_m), \quad q_{n,m} = q(x_n, y_m), \quad x_n = nh, \quad y_m = mh, \\ t_k &= k\tau, \quad 1 \leq k \leq N, \quad 1 \leq n, m \leq M-1, \quad Mh = L, \quad N\tau = T, \\ u_{n,m}^0 &= \varphi(x_n, y_m), \quad 0 \leq n, m \leq M, \\ u_{0,m}^k &= 0, \quad u_{n,0}^k = 0, \quad 0 \leq k \leq N, \quad 0 \leq n, m \leq M, \\ u_{s,m}^k &= \alpha(t_k, y_m), \quad 0 \leq k \leq N, \quad 0 \leq m \leq M, \quad s = \lfloor \frac{t}{h} \rfloor. \end{aligned} \right. \tag{11}$$

To formulate the results on difference problem, we introduce the Banach space

$$C_\tau(E) = C([0, T]_\tau, E)$$

of all grid functions

$$\phi^\tau = \{\phi(t_k)\}_{k=0}^N$$

defined on

$$[0, T]_\tau = \{t_k : t_k = k\tau, 0 \leq k \leq N, N\tau = T\}$$

with values in E equipped with the norm

$$\|\phi^\tau\|_{C_\tau(E)} = \max_{0 \leq k \leq N} \|\phi(t_k)\|_E.$$

Let $C_h = C_{[0, L]_h \times [0, L]_h}$ and $C_h^{(1)} = C_{[0, L]_h \times [0, L]_h}^{(1)}$ be spaces of all grid functions $\psi^h = \{\psi_{n,m}\}_{m,n=1}^M$ defined on $[0, L]_h \times [0, L]_h = \{x_n = nh, y_m = mh, 0 \leq n, m \leq M\}$ equipped with the norms

$$\|\psi^h\|_{C_h} = \max_{0 \leq n, m \leq M} |\psi_{n,m}|,$$

$$\|\psi^h\|_{C_h^{(1)}} = \|\psi^h\|_{C_h} + \frac{1}{h} \max_{0 \leq n \leq M} \max_{1 \leq m \leq M} |\psi_{n,m} - \psi_{n,m-1}| + \frac{1}{h} \max_{1 \leq n \leq M} \max_{0 \leq m \leq M} |\psi_{n,m} - \psi_{n-1,m}|,$$

respectively.

Moreover, we introduce difference neutron transport operator A_h

$$A_h u^h = - \left\{ \frac{u_{n+1,m+1} - u_{n,m+1}}{h} + \frac{u_{n,m+1} - u_{n,m}}{h} \right\}_{n,m=1}^{M-1}$$

acting in the space of grid functions $u^h = \{u_{n,m}\}_{n,m=1}^M$, $u_{0,m} = 0$, $u_{n,0} = 0$, $0 \leq n, m \leq M$.

Then, the following theorem on stability of problem (11) is established.

Theorem 2. For the solution of problem (11), the following stability estimates hold

$$\begin{aligned} & \left\| \left\{ \left\{ \frac{u_{n,m}^k - u_{n,m}^{k-1}}{\tau} \right\}_{k=1}^N \right\}_{n,m=0}^M \right\|_{C_\tau(C_h)} + \left\| \left\{ \{u_{n,m}^k\}_{k=1}^N \right\}_{n,m=0}^M \right\|_{C_\tau(C_h^{(1)})} + \|p^\tau\|_{C_\tau} \leq \\ & \leq M_1(q) \left[\|\varphi^h\|_{C_h^{(1)}} + \|f^{1,h}\|_{C_h} + \left\| \left\{ \left\{ \frac{f_{n,m}^k - f_{n,m}^{k-1}}{\tau} \right\}_{k=2}^N \right\}_{n,m=0}^M \right\|_{C_\tau(C_h)} + \right. \\ & \left. + \left\| \left\{ \left\{ \frac{\alpha_{s,m}^k - \alpha_{s,m}^{k-1}}{\tau} \right\}_{k=2}^N \right\}_{m=0}^M \right\|_{C_\tau(C[0,L]_h)} + \left\| \{q_m^1\}_{m=0}^M \right\|_{C[0,L]_h} \right]. \end{aligned}$$

Proof. For the solution of difference scheme (11), we consider substitution

$$u_{n,m}^k = \eta^k q_{n,m} + w_{n,m}^k, \tag{12}$$

where

$$q_{n,m} = q(x_n, y_m),$$

and η^k is the grid function determined by

$$\eta^k = \sum_{i=1}^k p_i \tau, \quad \eta^0 = 0, \quad p_k = \frac{\eta^k - \eta^{k-1}}{\tau}, \quad 0 \leq k \leq N.$$

It is easy to see that grid function $\left\{ \{w_{n,m}^k\}_{k=1}^N \right\}_{n,m=0}^M$ is the solution of difference scheme

$$\left\{ \begin{aligned} & \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau} = \frac{w_{n+1,m+1}^k - w_{n,m+1}^k}{h} + \frac{w_{n,m+1}^k - w_{n,m}^k}{h} \\ & + \eta^k \left[\frac{q_{n+1,m+1} - q_{n,m+1}}{h} + \frac{q_{n,m+1} - q_{n,m}}{h} \right] + f(t_k, x_n, y_m), \\ & f_{n,m}^k = f(t_k, x_n, y_m), \quad q_{n,m} = q(x_n, y_m), \quad x_n = nh, \quad y_m = mh, \\ & t_k = k\tau, \quad 1 \leq k \leq N, \quad 1 \leq n, m \leq M-1, \quad Mh = L, \quad N\tau = T, \\ & w_{n,m}^0 = \varphi(x_n, y_m), \quad 0 \leq n, m \leq M, \\ & w_{0,m}^k = 0, \quad w_{n,0}^k = 0, \quad 0 \leq k \leq N, \quad 0 \leq n, m \leq M, \\ & w_{s,m}^k = \alpha(t_k, y_m), \quad 0 \leq k \leq N, \quad 0 \leq m \leq M, \quad s = \lfloor \frac{l}{h} \rfloor. \end{aligned} \right. \tag{13}$$

Difference derivative of (12) can be written as

$$\frac{u_{n,m}^k - u_{n,m}^{k-1}}{\tau} = \frac{\eta^k - \eta^{k-1}}{\tau} q_{n,m} + \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau} = p_k q_{n,m} + \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau}. \tag{14}$$

Hence,

$$p_k = \frac{\frac{u_{n,m}^k - u_{n,m}^{k-1}}{\tau} - \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau}}{q_{n,m}} \tag{15}$$

for n, m and k , $1 \leq n, m \leq M - 1$ and $1 \leq k \leq N$. Applying the overdetermined condition $u_{s,m}^k$ in (15), we obtain that

$$p_k = \frac{\frac{u_{s,m}^k - u_{s,m}^{k-1}}{\tau} - \frac{w_{s,m}^k - w_{s,m}^{k-1}}{\tau}}{q_{s,m}}.$$

Using the triangle inequality, we obtain

$$|p_k| \leq M_7(q) \left[\left| \frac{u_{s,m}^k - u_{s,m}^{k-1}}{\tau} \right| + \left| \frac{w_{s,m}^k - w_{s,m}^{k-1}}{\tau} \right| \right]$$

for all $0 \leq k \leq N$. From that it follows,

$$\begin{aligned} \left\| \{p_k\}_{k=1}^N \right\|_{C[0,T]_\tau} &\leq M_7(q) \left[\left\| \left\{ \frac{u_{s,m}^k - u_{s,m}^{k-1}}{\tau} \right\}_{k=1}^N \right\|_{C[0,T]_\tau} + \right. \\ &\left. + \left\| \left\{ \frac{w_{s,m}^k - w_{s,m}^{k-1}}{\tau} \right\}_{k=1}^N \right\|_{C_\tau(C([0,L]_h \times [0,L]_h, E))} \right]. \end{aligned} \tag{16}$$

Now using substitution (14) we get

$$\frac{u_{n,m}^k - u_{n,m}^{k-1}}{\tau} = \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau} + p_k q_{n,m}.$$

Applying the triangle inequality, we obtain

$$\begin{aligned} \left\| \left\{ \frac{u_{n,m}^k - u_{n,m}^{k-1}}{\tau} \right\}_{k=1}^N \right\|_{C[0,T]_\tau} &\leq \left\| \left\{ \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau} \right\}_{k=1}^N \right\|_{C_\tau(C([0,L]_h \times [0,L]_h))} + \\ &+ \left\| \{p_k\}_{k=1}^N \right\|_{C[0,T]_\tau} \left\| \{q_{n,m}\}_{n=1}^M \right\|_{C([0,L]_h \times [0,L]_h)} \end{aligned} \tag{17}$$

for all $0 \leq k \leq N$. We can rewrite difference scheme (13) in the abstract form as

$$\begin{cases} \frac{w_h^k - w_h^{k-1}}{\tau} + A_h w_h^k + \eta^k A q = f^h(t_k), \\ w_h^0 = \varphi^h, \eta^0 = 0, t_k = k\tau, 1 \leq k \leq N, N\tau = T \end{cases} \tag{18}$$

in a Banach space $C_\tau(E) = C([0, T]_\tau, E)$ with the positive operator A_h defined by

$$A_h w^h = - \left\{ \frac{u_{n+1,m+1} - u_{n,m+1}}{h} + \frac{u_{n,m+1} - u_{n,m}}{h} \right\}_{n,m=1}^{M-1},$$

acting on grid functions u^h such that satisfies the condition $u^h = \{u_{n,m}\}_{n,m=1}^M, u_{0,m} = 0, u_{n,0} = 0, 0 \leq n, m \leq M$.

For equation (18) we have that

$$w_h^k = R w_h^{k-1} + R\tau \left(Aq \frac{\alpha(t_k) - w_s^k}{q_s} + f^h(t_k) \right),$$

for all $k, 1 \leq k \leq N$, where $R = (I + \tau A_h)^{-1}$. By recurrence relations, we get

$$w_h^k = R^k \varphi^h + \sum_{i=1}^k R^{k-i+1} \frac{\tau}{q_s} \alpha(t_i) Aq - \sum_{i=1}^k R^{k-i+1} \frac{\tau}{q_s} w_s^i Aq + \sum_{i=1}^k R^{k-i+1} \tau f^h(t_i)$$

for any $k, 1 \leq k \leq N$. Taking the difference derivative of both sides, we obtain that

$$\begin{aligned} \frac{w_h^k - w_h^{k-1}}{\tau} &= \frac{R^k - R^{k-1}}{\tau} \varphi^h + \frac{1}{q_s} \alpha(t_k) Aq + \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) \frac{1}{q_s} \alpha(t_i) Aq - \\ &- \frac{1}{q_s} w_s^k Aq - \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) \frac{1}{q_s} w_s^i Aq + f^h(t_k) + \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) f^h(t_i). \end{aligned}$$

Applying the formula,

$$\begin{aligned} \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) w_s^i &= \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) \varphi(x_s, y_m) + \\ &+ \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) \sum_{j=1}^i \frac{w_s^j - w_s^{j-1}}{\tau} \tau \end{aligned}$$

and changing the order of summation, we get

$$\begin{aligned} \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) w_s^i &= \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) \varphi(x_s, y_m) + \\ &+ \sum_{j=1}^k \sum_{i=j}^k (R^{k-i+1} - R^{k-i}) \frac{w_s^j - w_s^{j-1}}{\tau} \tau. \end{aligned}$$

Consequently, we obtain the following presentation for the solution of equation (13)

$$\begin{aligned} \frac{w_h^k - w_h^{k-1}}{\tau} &= \frac{R^k - R^{k-1}}{\tau} \varphi^h + \frac{1}{q_s} \alpha(t_k) Aq + \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) \frac{1}{q_s} \alpha(t_i) Aq - \\ &- \frac{1}{q_s} w_s^k Aq - \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) \frac{1}{q_s} Aq \varphi^h(x_s, y_m) - \sum_{j=1}^k \sum_{i=j}^k (R^{k-i+1} - R^{k-i}) \frac{w_s^j - w_s^{j-1}}{\tau} \tau + \\ &+ f^h(t_k) + \sum_{i=1}^k (R^{k-i+1} - R^{k-i}) f^h(t_i). \end{aligned}$$

Applying the definition of norm of the spaces $C_\tau(E) = C([0, T]_\tau, E)$ and methods of monograph [20],

we can write,

$$\begin{aligned} & \left\| \left\{ \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau} \right\}_{k=1}^N \right\|_{C_\tau(C([0,L]_h \times [0,L]_h))} \leq M_8(q) \left[\|\varphi^h\|_{C_h^{(1)}} + \left\| \left\{ \left\{ \frac{f_{n,m}^k - f_{n,m}^{k-1}}{\tau} \right\}_{k=2}^N \right\}_{n,m=0}^M \right\|_{C_\tau(C_h)} \right. \\ & \left. + \|f^{1,h}\|_{C_h} + \left\| \left\{ \left\{ \frac{\alpha_{s,m}^k - \alpha_{s,m}^{k-1}}{\tau} \right\}_{k=2}^N \right\}_{m=0}^M \right\|_{C_\tau(C[0,L]_h)} + \left\| \{\alpha_m^1\}_{m=0}^M \right\|_{C[0,L]_h} \right]. \end{aligned} \tag{19}$$

Finally, combining estimates (16), (17), and (19), it completes the proof of Theorem 2.

3 Numerical experiments

In this section, we study the numerical solution of the neutron transport identification problem with initial condition

$$\left\{ \begin{aligned} & \frac{\partial u(t,x,y)}{\partial t} = \frac{\partial u(t,x,y)}{\partial x} + \frac{\partial u(t,x,y)}{\partial y} + p(t) \sin \pi x \sin \pi y + f(t,x,y), \\ & f(t,x,y) = -e^{-2t}(3 \sin \pi x \sin \pi y + \pi \cos \pi x \sin \pi y + \pi \sin \pi x \cos \pi y), \\ & t \in (0, 1], \quad x, y \in (0, 1], \\ & u(0, x, y) = \sin \pi x \sin \pi y, \quad x, y \in [0, 1], \\ & u(t, 0, y) = 0, \quad t \in [0, 1], \quad y \in [0, 1], \\ & u(t, x, 0) = 0, \quad t \in [0, 1], \quad x \in [0, 1], \\ & u(t, \frac{1}{2}, y) = e^{-2t} \sin \pi y, \quad t \in [0, 1], \quad y \in [0, 1]. \end{aligned} \right. \tag{20}$$

The exact solution of problem is $u(t,x,y) = e^{-2t} \sin \pi x \sin \pi y$ and for the control parameter $p(t) = e^{-2t}$.

For the approximate solution of problem (20), we get the following first order of accuracy difference scheme

$$\left\{ \begin{aligned} & \frac{u_{n,m}^k - u_{n,m}^{k-1}}{\tau} = \frac{u_{n+1,m+1}^k - u_{n,m+1}^k}{h} + \frac{u_{n,m+1}^k - u_{n,m}^k}{h} + p^k q_{n,m} + f_{n,m}^k, \\ & f_{n,m}^k = -e^{-2t_k}(3 \sin \pi x_n \sin \pi y_m + \pi \cos \pi x_n \sin \pi y_m + \pi \sin \pi x_n \cos \pi y_m), \\ & q_{n,m} = \sin \pi x_n \sin \pi y_m, \quad x_n = nh, \quad y_m = mh, \quad t_k = k\tau, \\ & 1 \leq k \leq N, \quad 0 \leq n, m \leq M - 1, \quad Mh = 1, \quad N\tau = 1, \\ & u_{n,m}^0 = \sin \pi x_n \sin \pi y_m, \quad 0 \leq n, m \leq M, \\ & u_{0,m}^k = 0, \quad u_{n,0}^k = 0, \quad 0 \leq k \leq N, \quad 0 \leq n, m \leq M, \\ & u_{s,m}^k = e^{-2t_k} \sin \pi y_m, \quad 0 \leq k \leq N, \quad 0 \leq m \leq M, \quad s = \lfloor \frac{M}{2} \rfloor. \end{aligned} \right. \tag{21}$$

For the solution of difference scheme (21), we consider the substitution

$$u_{n,m}^k = \eta^k q_{n,m} + w_{n,m}^k, \tag{22}$$

where

$$\eta^k = \sum_{i=1}^k p_i \tau, \quad \eta^0 = 0, \tag{23}$$

$w_{n,m}^k$ is the solution of difference scheme

$$\left\{ \begin{array}{l} \frac{w_{n,m}^k - w_{n,m}^{k-1}}{\tau} = \frac{w_{n+1,m+1}^k - w_{n,m+1}^k}{h} + \frac{w_{n,m+1}^k - w_{n,m}^k}{h} + \\ + \eta^k \left[\frac{q_{n+1,m+1} - q_{n,m+1}}{h} + \frac{q_{n,m+1} - q_{n,m}}{h} \right] + f_{n,m}^k, \\ 1 \leq k \leq N, \quad 1 \leq n, m \leq M, \\ w_{n,m}^0 = \sin \pi x_n \sin \pi y_m, \quad 0 \leq n, m \leq M, \\ w_{0,m}^k = 0, \quad w_{n,0}^k = 0, \quad 0 \leq k \leq N, \quad 0 \leq n, m \leq M. \end{array} \right. \tag{24}$$

Applying (21) and formulas (22), (23), we get

$$\eta^k = \frac{u_{s,m}^k - w_{s,m}^k}{q_{s,m}} = \frac{e^{-2t_k} \sin \pi y_m - w_{s,m}^k}{q_{s,m}}, \tag{25}$$

$$p^k = \frac{1}{\tau} \left[\frac{(e^{-2t_k} - e^{-2t_{k-1}}) \sin \pi y_m - (w_{s,m}^k - w_{s,m}^{k-1})}{\sin \pi x_s \sin \pi y_m} \right] \tag{26}$$

for any $k, 1 \leq k \leq N$.

It is easy to see that (24) and (25) can be written in the matrix form

$$A w^k + B w^{k-1} = \varphi^k, \quad 1 \leq k \leq N, \quad w^0 = \{\sin \pi x_n \sin \pi y_m\}_{n,m=0}^M,$$

where

$$\left\{ \begin{array}{l} \varphi_{n,m}^k = e^{-2t_k} \left[\frac{\sin \pi x_{n+1} \sin \pi y_{m+1} - \sin \pi x_n \sin \pi y_{m+1}}{h} + \frac{\sin \pi x_n \sin \pi y_{m+1} - \sin \pi x_n \sin \pi y_m}{h} \right] - \\ - e^{-2t_k} (3 \sin \pi x_n \sin \pi y_m + \pi \cos \pi x_n \sin \pi y_m + \pi \sin \pi x_n \cos \pi y_m), \\ 1 \leq n, m \leq M, \quad \varphi_{0,m}^k = 0, \quad \varphi_{n,0}^k = 0, \quad 1 \leq n, m \leq M. \end{array} \right.$$

Here A and B are $(M+1) \times (M+1) \times (N+1)$ square matrices, w^k and φ^k are $(M+1) \times (M+1) \times 1$ column matrices. First, we obtain w^k by formula

$$w^k = -A^{-1} B w^{k-1} + A^{-1} \varphi^k, \quad 1 \leq k \leq N, \quad w^0 = \{\sin \pi x_n \sin \pi y_m\}_{n,m=0}^M.$$

Second, applying formulas (22) and (26), we get p^k and u^k .

4 Error analysis

Now, we will give the results of the numerical analysis. In order to get the solution of (21), we used MATLAB program. The errors are computed by

$$E_M^N u = \max_{0 \leq k \leq N} \max_{0 \leq n, m \leq M} |u(t_k, x_n, y_m) - u_{n,m}^k|, \quad E^N p = \max_{1 \leq k \leq N} |p(t_k) - p^k|$$

of the numerical solutions for different values of M and N , where $u(t_k, x_n, y_m)$ represents the exact solution, $u_{n,m}^k$ represents the numerical solution at (t_k, x_n, y_m) , $p(t_k)$ represents the exact solution, and p^k represents the numerical solution at t_k . Now, let us give the obtained numerical results (Table).

Table

Error analysis of first order DS

Error	$N = M = 10$	$N = M = 20$	$N = M = 40$	$N = M = 80$
$E_M^N u$	0.1813	0.0952	0.0488	0.0247
$E^N p$	0.0698	0.0481	0.0264	0.0137

The obtained results indicate that when the numerical parameters N and M are multiplied by two, the errors in the solution for first order difference scheme (21) decrease by approximately half.

Conclusion

In this study, we consider an inverse problem related to the two-dimensional neutron transport equation with a time-dependent source control parameter. For the approximate solution of this problem, a first-order accuracy difference scheme is constructed. A finite difference scheme is presented for identifying the control parameter. Stability inequalities for the solution of this problem are established. The results of a numerical experiment are presented, and the accuracy of the solution for this inverse problem is discussed.

Acknowledgments

The authors would like to thank Prof. Allaberen Ashyralyev for his valuable comments and suggestions to improve the quality of the present paper.

Conflict of Interest

The author declare no conflict of interest.

References

- 1 Lewis, E.E., & Miller, W.F. (1984). *Computational methods of neutron transport*. John Wiley and Sons United States.
- 2 Marchuk, G., & Lebedev, V.I. (1986). *Numerical methods in the theory of neutron transport*. Harwood Academic Publishers United States.
- 3 Mokhtar-Kharroubi, M. (1997). *Mathematical Topics In Neutron Transport Theory: New Aspects*. World Scientific Singapore.
- 4 Ashyralyev, A., & Taskin, A. (2019). The structure of fractional spaces generated by a two-dimensional neutron transport operator and its applications. *Advances in Operator Theory*, 4(1), 140–155. <https://doi.org/10.15352/aot.1711-1261>

- 5 Ashyralyev, A., & Agirseven, D. (2014). On source identification problem for a delay parabolic equation. *Nonlinear Analysis: Modelling and Control*, 19(3), 335–349. <https://doi.org/10.15388/NA.2014.3.2>
- 6 Ashyralyev A., Agirseven, D., & Agarwal, R.P. (2020). On source identification problem for delay parabolic differential and difference equations. *Applied Computational Mathematics*, 19(2), 175–204.
- 7 Ashyralyev, A., & Ashyralyyev, C. (2014). On the problem of determining the parameter of an elliptic equation in a Banach space. *Nonlinear Analysis: Modelling and Control*, 19(3), 350–366. <https://doi.org/10.15388/NA.2014.3.3>
- 8 Ashyralyev, A., & Erdogan, A.S. (2014). Well-Posedness of the Right-Hand Side Identification Problem for a Parabolic Equation. *Ukrainian Mathematical Journal*, 66, 165–177. <https://doi.org/10.1007/s11253-014-0920-0>
- 9 Ashyralyyev, C. (2017). Numerical solution to Bitsadze-Samarskii type elliptic overdetermined multipoint NBVP. *Boundary Value Problems*, 74, 1–22. <https://doi.org/10.1186/s13661-017-0804-y>
- 10 Ashyralyyev, C. (2017). A fourth order approximation of the Neumann type overdetermined elliptic problem. *Filomat*, 31(4), 967–980. <https://doi.org/10.2298/FIL1704967A>
- 11 Ashyralyev, A., & Taskin, A. (2021). Source identification problems for two dimensional neutron transport differential and difference equations. *AIP Conference Proceedings*, 2334, 060017. <https://doi.org/10.1063/5.0042296>
- 12 Ashyralyev, A., & Urun, M. (2021). Time-dependent source identification Schrodinger type problem. *International Journal of Applied Mathematics*, 34(2), 297–310. <https://doi.org/10.12732/ijam.v34i2.7>
- 13 Ashyralyev, A., & Emharab, F. (2019). Source identification problems for hyperbolic differential and difference equations. *Journal of Inverse and Ill-posed Problems*, 27(3), 301–315. <https://doi.org/10.1515/jiip-2018-0020>
- 14 Ashyralyev, A., & Emharab, F. (2022). A note on the time identification nonlocal problem. *Advanced Mathematical Models & Applications*, 7(2), 105–120.
- 15 Ashyralyev, A., Al-Hazaimeh, H., & Ashyralyyev, C. (2023). Absolute stability of a difference scheme for the multidimensional time-dependently identification telegraph problem. *Computational and Applied Mathematics*, 42, 333. <https://doi.org/10.1007/s40314-023-02478-5>
- 16 Ashyralyev, A., Erdogan, A., & Sazaklioglu, A. (2019). Numerical solution of a source identification problem: Almost coercivity. *Journal of Inverse and Ill-posed Problems*, 27(4), 457–468. <https://doi.org/10.1515/jiip-2017-0072>
- 17 Kabanikhin, S. (2011). *Inverse and Ill-posed Problems: Theory and Applications*. Berlin, Boston: De Gruyter. <https://doi.org/10.1515/9783110224016>
- 18 Borukhov, V.T., & Vabishchevich, P.N. (2000). Numerical solution of the inverse problem of reconstructing a distributed right-hand side of a parabolic equation. *Computer Physics Communications*, 126(1), 32–36. [https://doi.org/10.1016/S0010-4655\(99\)00416-6](https://doi.org/10.1016/S0010-4655(99)00416-6)
- 19 Dehghan, M. (2001). An inverse problem of finding a source parameter in a semilinear parabolic equation. *Applied Mathematical Modelling*, 25(9), 743–754. [https://doi.org/10.1016/S0307-904X\(01\)00010-5](https://doi.org/10.1016/S0307-904X(01)00010-5)
- 20 Ashyralyev, A., & Sobolevskii, P.E. (2004). *New Difference Schemes for Partial Differential Equations, Operator Theory Advances and Applications*. Birkhäuser Verlag, Basel, Boston, Berlin.
- 21 Cholli, M., & Yamamoto, M. (1999). Generic well-posedness of a linear inverse parabolic

- problem with diffusion parameters. *Journal of Inverse and Ill-posed Problems*, 7(3), 241–254. <https://doi.org/10.1515/jiip.1999.7.3.241>
- 22 Ashyralyev, A., & Al-Hammouri, A. (2021). Stability of the space identification problem for the elliptic telegraph differential equation. *Mathematical Methods in the Applied Sciences*, 44(1), 945–959. <https://doi.org/10.1002/mma.6803>
- 23 Isakov, V. (2021). *Inverse Problems for Partial Differential Equations*. Applied Mathematical Sciences, 127. Springer, Cham. <https://doi.org/10.1007/978-3-319-51658-5>
- 24 Taskin, A. (2016). *Structure of Fractional Spaces Generated by Differential and Difference Neutron Transport Operators*. PhD Thesis, Istanbul University, Istanbul.
- 25 Krein, S.G. (1971). *Linear Differential Equations in Banach Space*. American Mathematical Society, Providence, R.I.
- 26 Kabanikhin, S. (2011). *Inverse and Ill-posed Problems: Theory and Applications*. Berlin, Boston: De Gruyter. <https://doi.org/10.1515/9783110224016>
- 27 Gryazin, Y.A., Klivanov, M.V., & Lucas, T.R. (1999). Imaging the diffusion coefficient in a parabolic inverse problem in optical tomography. *Inverse Problems*, 15(2), 373–397. <https://doi.org/10.1088/0266-5611/15/2/003>

*Author Information**

Abdulgafur Taskin (*corresponding author*) — PhD, Department of Mathematics and Science Education, Yildiz Technical University, Istanbul, Turkey; e-mail: gafurtaskin@hotmail.com; <https://orcid.org/0000-0001-7432-7450>

*The author's name is presented in the order: First, Middle and Last Names.