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## Spectral properties of local and nonlocal problems for the diffusion-wave equation of fractional order

The paper investigates the issues of solvability and spectral properties of local and nonlocal problems for the fractional order diffusion-wave equation. The regular and strong solvability to problems stated in the domains, both with characteristic and non-characteristic boundaries are proved. Unambiguous solvability is established and theorems on the existence of eigenvalues or the Volterra property of the problems under consideration are proved.

*Keywords:* diffusion-wave equations, fractional order equations, boundary value problems, strong solution, Volterra property, eigenvalue.

### 1 Introduction

The theory of derivatives and integrals of non-integer (fractional) order, called fractional calculus, is becoming increasingly important both for the development of modern mathematics and for applications in various fields of natural science. Both ordinary and partial differential equations of fractional order have been used over the past few decades to model many physical and chemical processes and in engineering [1–7].

Fractional partial differential equations have become especially important for modeling the so-called anomalous diffusion processes in nature and the theory of complex systems [1]. Such equations are also associated with fractional Brownian motions, the continuous random walk in time (CTRW) method, stable Levy distributions, etc. [2, 7]. Fractional differential equations also make it possible to study the long-term and nonlocal dependence of many anomalous processes.

Since the fractional order equation generalizes the integer order equation, as well as a relatively small number of systematized analytical and numerical methods for such equations, make this direction a priority in the general theory of differential equations.

The mathematical theory of fractional differential equations is more or less fully investigated for ordinary equations [1], whereas for partial differential equations it differs from the situation for the equation of one variable. In the scientific literature, analogs of the initial data problem and initial boundary value problems for the simplest partial differential equations of fractional order were considered mainly. Methods for solving such problems are considered in [1, 8–10].

The issues of solvability of local and non-local problems for various fractional order equations are considered in [11–16].

Spectral properties, including Volterra property and the existence of eigenvalues, for a mixed fractional order equation, as far as we know, are almost not studied. Note that the solvability issues and spectral properties of local and nonlocal problems for a mixed parabolic-hyperbolic equation of the second and third orders are studied in [17–24].

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The work is devoted to the study of the solvability and spectral properties of local and nonlocal problems for the diffusion-wave equation of fractional order. The regular and strong solvability of the tasks set in the domains with both characteristic and non-characteristic boundaries of the domain is proved. The unambiguous solvability of the problem is established, theorems on the existence of eigenvalues are proved, or the Volterra nature of the problems under consideration.

Consider equation

$$Lu(x, y) = f(x, y), \quad (1)$$

where

$$Lu(x, y) = \begin{cases} {}_cD_{0x}^\alpha u(x, y) - u_{yy}(x, y), & y > 0, \\ u_{xx}(x, y) - u_{yy}(x, y), & y < 0, \end{cases} \quad (2)$$

$${}_cD_{0x}^\alpha u(x, y) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{u_x(t, y)}{(x-t)^\alpha} dt, \quad 0 < \alpha < 1.$$

$\Gamma(x)$  is Euler's gamma-function, (2) is an integral-differential operator of fractional order  $\alpha$  in the sense of Caputo [1; 92],  $f(x, y)$  is a given function.

## 2 Solvability and Volterra property of local and nonlocal problems for the diffusion-wave equation

Let  $\Omega = \Omega_0 \cup \Omega_1 \cup AB$  be a domain, where  $\Omega_0$  is a rectangle  $ABB_0A_0$  with vertices  $A(0, 0)$ ,  $B(1, 0)$ ,  $B_0(1, 1)$ ,  $A_0(0, 1)$ ,  $\Omega_1$  is a domain bounded by segments  $AB$  and smooth curve  $AD : y = -\gamma(x)$ ,  $0 < x < l$ , where  $0, 5 < l \leq 1$ ;  $\gamma(0) = 0$ ,  $l + \gamma(l) = 1$ , and characteristic  $BD : x - y = 1$  of equation (1), if  $l < 1$  and  $\gamma(l) = 0$ , if  $l = 1$  (when  $D = B$ ), located inside the characteristic triangle  $0 < x + y \leq x - y < 1$ .

With respect to the curve  $\gamma(x)$ , we suppose that  $\gamma(x)$  is twice continuously differentiable function and  $x \pm \gamma(x)$  are monotonically increasing functions, and  $0 < \gamma'(x) < 1$ ,  $\gamma(x) > 0$ ,  $x > 0$ .

**Problem  $M_1A$ .** Find a solution to equation (1) satisfying conditions:

$$u(0, y) = 0, \quad 0 \leq y \leq 1, \quad (3)$$

$$u(x, 1) = 0, \quad 0 \leq x \leq 1, \quad (4)$$

$$(u_x - u_y)|_{AD} = 0. \quad (5)$$

*Definition 1.* The regular solution to the problem  $M_1A$  in the domain  $\Omega$  will be called the function  $u(x, y) \in V$ , where

$$V = \{u(x, y) : u(x, y) \in C(\bar{\Omega}) \cap C^{1,1}(\Omega \cup AC), D_{0x}^\alpha u(x, y), u_{yy}(x, y) \in C(\Omega_0), u(x, y) \in C^{2,2}(\Omega_1)\},$$

satisfying the equation (1) in  $\Omega_0 \cup \Omega_1$  and conditions (3)–(5).

In domain  $\Omega_0$  consider the following auxiliary problem:

**Problem  $C_1$ .** Find a solution to equation (1) for  $y > 0$  satisfying conditions (3), (4) and

$$u_x(x, 0) - u_y(x, 0) = \delta(x), \quad 0 < x < 1, \quad (6)$$

where  $\delta(x)$  is a given function.

*Lemma 1.* Let be  $\delta(x) \in C^1[0, 1]$ . Then for any function  $f(x, y) \in C^1(\bar{\Omega}_0)$  is a solution to problem  $C_1$  allows a priori estimates.

$$D_{0x}^{\alpha-1} \|u(x, y)\|_{L_2(0,1)}^2 + 2 \int_0^x \|u_y(t, y)\|_{L_2(0,1)}^2 dt \leq C \left[ \int_0^x \|f(t, y)\|_{L_2(0,1)}^2 dt + \int_0^x \delta^2(t) dt \right], \quad (7)$$

where  $\|f(x, y)\|_{L_2(0,1)}^2 = \int_0^1 f^2(x, y)dy$ . Hereinafter symbol will denote a positive constant that does not depend on  $u(x, y)$ , not necessarily the same.

*Proof of Lemma 1.* We multiply equation (1) for  $y > 0$  by  $u(x, y)$  and integrating from 0 to 1 over  $y$  and taking into account conditions (3),(4) after some transformations we have

$$\int_0^1 u(x, y)D_{0x}^\alpha u(x, y)dy + \int_0^1 u_y^2(x, y)dy + \tau(x)\nu(x) = \int_0^1 f(x, y)u(x, y)dy, \quad (8)$$

where

$$\tau(x) = u(x, 0), \quad 0 \leq x \leq 1, \quad (9)$$

$$\nu(x) = u_y(x, 0), \quad 0 < x < 1. \quad (10)$$

It is known [10], that

$$\int_0^1 u(x, y) \cdot D_{0x}^\alpha u(x, y)dy \geq \frac{1}{2} \int_0^1 D_{0x}^\alpha u^2(x, y)dy.$$

By virtue of the last inequality, from (8), taking into account (6) and the notations (9), (10) we obtain

$$\int_0^1 D_{0x}^\alpha u^2(x, y)dy + 2 \int_0^1 u_y^2(x, y)dy + 2\tau(x)\tau'(x) \leq 2 \int_0^1 u(x, y)f(x, y)dy + 2\tau(x)\delta(x). \quad (11)$$

Integrating (11) over  $t$  from 0 to  $x$ , taking into account  $\tau(0) = 0$  and using known inequalities we have

$$D_{0x}^{\alpha-1} \|u(x, y)\|_{L_2(0,1)}^2 + 2 \int_0^x \|u_y(t, y)\|_{L_2}^2 dt + \tau^2(x) \leq \int_0^x [\|u(t, y)\|_{L_2(0,1)}^2 + \|f(t, y)\|_{L_2(0,1)}^2 + \tau^2(t) + \delta^2(t)] dt. \quad (12)$$

In the left part of (12), omitting the first two terms and applying the Gronwall-Bellman inequality, we will have

$$\int_0^x \tau^2(t)dt \leq C \int_0^x [\|u(t, y)\|_{L_2(0,1)}^2 + \|f(t, y)\|_{L_2(0,1)}^2 + \delta^2(t)] dt.$$

Taking into account the last from (12) we have

$$D_{0x}^{\alpha-1} \|u(x, y)\|_{L_2(0,1)}^2 + 2 \int_0^x \|u_y(t, y)\|_{L_2(0,1)}^2 \leq C \int_0^x [\|u(t, y)\|_{L_2(0,1)}^2 + \|f(t, y)\|_{L_2(0,1)}^2 + \delta^2(t)] dt. \quad (13)$$

Similarly as above, omitting the second term of the left part in (13) and applying Lemma 1 in [10] we have

$$D_{0x}^{-\alpha-1} \|f(x, y)\|_{L_2(0,1)}^2 \leq \frac{x^\alpha}{\Gamma(1 + \alpha)} \int_0^x \|f(t, y)\|_{L_2(0,1)}^2 dt$$

we have

$$\int_0^x \|u(t, y)\|_{L_2(0,1)}^2 dt \leq C \int_0^x [\|f(t, y)\|_{L_2(0,1)}^2 + \delta^2(t)] dt. \quad (14)$$

From (12)–(14) it is followed the validity of the a priori estimate (7). *Lemma 1 is proved.*

Now consider equation (1) in the domain  $\Omega_1$ . By virtue of the unambiguous solvability of the Cauchy problem (1), (9), (10) for the wave equation, any regular solution of the B problem in the domain  $\Omega_1$  is represented as

$$u(x, y) = \frac{1}{2} \left[ \tau(\xi) + \tau(\eta) - \int_{\xi}^{\eta} \nu(t) dt \right] - \int_{\xi}^{\eta} d\xi_1 \int_{\xi_1}^{\eta} f_1(\xi_1, \eta_1) d\eta_1, \quad (15)$$

where  $\xi = x + y$ ,  $\eta = x - y$ ,  $4f_1(\xi, \eta) = f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)$ . Due to the conditions imposed on the function  $\gamma(x)$ , equation of the curve  $AD$  in characteristic variables  $\xi, \eta$  allows representation

$$\xi = \lambda(\eta), \quad 0 \leq \eta \leq 1, \quad \text{and } \lambda(\eta) < \eta.$$

In (15) satisfying condition (5) after some simple transformations we have

$$\nu(x) = \tau'(x) - 2 \int_{\lambda(x)}^x f_1(\xi, x) d\xi, \quad 0 < x < 1. \quad (16)$$

The ratio (16) is the main functional relationship between  $\tau(x)$  and  $\nu(x)$  brought to the segment  $AB$  from hyperbolic domain  $\Omega_1$ .

Substituting obtained expression  $\nu(x)$  into (15), after some transformations we get presentation of the solution  $u(\xi, \eta)$  in domain  $\Omega_1$ .

$$u(x, y) = \tau(\xi) + \int_{\xi}^{\eta} d\eta_1 \int_{\lambda(\eta_1)}^{\xi} f_1(\xi_1, \eta_1) d\xi_1. \quad (17)$$

Now in (7) assuming that  $\delta(x) = 2 \int_{\lambda(x)}^x f_1(\xi, x) d\xi$  it is not difficult to establish the validity of the following lemma.

*Lemma 2.* For any function  $f(x, y) \in C^1(\bar{\Omega})$ ,  $f(0, 0) = 0$  the solution to problem  $M_1B$  allows a priori estimate

$$D_{0x}^{\alpha-1} \|u(x, y)\|_{L_2(0,1)}^2 + \int_0^x \|u_y(t, y)\|_{L_2(0,1)}^2 dt \leq C \left[ \int_0^x \|f(t, y)\|_{L_2(0,1)}^2 dt + \int_0^x d\xi \int_{\xi}^x |f(\xi, t)|^2 dt \right]. \quad (18)$$

Lemma 2 implies the validity of the following estimate

$$\|u(x, y)\|_{L_2(\Omega_0)} + \|u_y(x, y)\|_{L_2(\Omega_0)} \leq C \|f(x, y)\|_{L_2(\Omega)}, \quad (19)$$

where  $L_2(\Omega)$  is quadratically summable functions in  $\Omega$ .

Consider the following auxiliary problem  $C_2$ . In domain  $\Omega_0$  find a solution of equation (1), satisfying conditions (3), (4) and (9).

The solution of equation (1), satisfying conditions (3), (4) and (9) in domain  $\Omega_0$  can be presented in a form [8]

$$u(x, y) = \int_0^x E_{y_1}(x - x_1, y, 0) \tau(x_1) dx_1 + \int_0^x dx_1 \int_0^1 E(x - x_1, y, y_1) f(x_1, y_1) dy_1, \quad (20)$$

where

$$E(x, y, y_1) = \frac{x^{\beta-1}}{2} \sum_{n=-\infty}^{+\infty} \left[ e_{1,\beta}^{1,\beta} \left( -\frac{|y - y_1 + 2n|}{x^\beta} \right) - e_{1,\beta}^{1,\beta} \left( -\frac{|y + y_1 + 2n|}{x^\beta} \right) \right], \quad \beta = \frac{\alpha}{2},$$

$e_{1,\beta}^{1,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n! \Gamma(\beta - \beta n)}$  is Wright type function [8]. Differentiating (20) over  $y$  we have

$$u_y(x, y) = \int_0^x E_{y_1 y}(x - x_1, y, 0) \tau(x_1) dx_1 + \int_0^x dx_1 \int_0^1 E_y(x - x_1, y, y_1) f(x_1, y_1) dy_1 \quad (21)$$

and using known formulas [8], [18] after some calculations, going to limit in (21) for  $y \rightarrow 0$  we have:

$$\nu(x) = - \int m(x - x_1) \tau'(x_1) dx_1 + \int_0^x dx_1 \int_0^1 E_y(x - x_1, 0, y_1) f(x_1, y_1) dy_1, \quad (22)$$

where

$$m(x) = \sum_{n=-\infty}^{+\infty} x^{-\beta} e_{1,\beta}^{1,1-\beta} \left( -\frac{|2n|}{x^\beta} \right) = \frac{1}{\Gamma(1-\beta)} x^{-\beta} + 2x^{-\beta} \sum_{n=1}^{+\infty} e_{1,\beta}^{1,1-\beta} \left( -\frac{2n}{x^\beta} \right). \quad (23)$$

Note that (22) is the main functional rate between  $\tau'(x)$  and  $\nu(x)$ , brought to the segment from domain  $\Omega_0$ .

Excluding from the functional relations (16) and (22) the function  $\nu(x)$ , with respect to  $\tau'(x)$  we obtain the equation

$$\tau'(x) + \int_0^x m(x-t) \tau'(t) dt = Q(x), \quad 0 \leq x \leq 1, \quad (24)$$

where

$$Q(x) = 2 \int_{\lambda(x)}^x f_1(\xi, x) d\xi + \int_0^x dx_1 \int_0^1 E_y(x - x_1, 0, y_1) f(x_1, y_1) dy_1. \quad (25)$$

*Lemma 3.* [8] Let be  $0 < \theta \leq 1$ . Then for functions  $E(x, y, y_1)$  and  $E_y(x, y, y_1)$  the following estimates take place

$$|E(x, y, y_1)| \leq Cx^{(2+\theta)\beta-1}, \quad 0 < x \leq 1, \quad 0 \leq y_1 < y \leq 1, \quad 0 < \theta \leq 1, \quad (26)$$

$$|E_y(x, y, y_1)| \leq Cx^{\beta(1+\theta)-1}, \quad 0 < x \leq 1, \quad 0 \leq y_1 < y \leq 1, \quad 0 < \theta \leq 1. \quad (27)$$

The proof of Lemma 3 is carried out using the inequality

$$\left| y^{p-1} t^{\delta-1} e_{\omega,\tau}^{p,\delta}(-y^\omega t^{-\tau}) \right| < C y^{p-\omega\theta-1} \cdot t^{\delta+\theta\tau-1}, \quad 0 < \theta \leq 1.$$

By virtue of Lemma 3 and  $\gamma(x) \in C^2[0, l]$ ,  $f(x, y) \in C^1(\bar{\Omega})$ ,  $f(0, 0) = 0$  from (25) it is not difficult to establish that

$$Q(x) \in C^1[0, 1] \quad \text{and} \quad Q(0) = 0. \quad (28)$$

Thus, by virtue of (23), the problem  $M_1 A$  is equivalently (in the sense of unambiguous solvability) reduced to a Volterra type integral equation of the second kind with a weak singularity (24). Therefore,

by virtue of (28), there is a unique solution of equation (24) from the class  $C^1 [0, 1]$  and it is representable as

$$\tau'(x) = Q(x) + \int_0^x R(x-t)Q(t)dt, \quad (29)$$

where  $R(x)$  is the resolvent of the integral equation (24)

$$R(x) = \sum_{n=1}^{\infty} (-1)^n m_n(x), \quad m_1(x) = m(x), \quad m_{n+1}(x) = \int_0^x m_1(x-t)m_n(t)dt.$$

From (29) taking into account  $\tau(0) = 0$ , we have

$$\tau(x) = \int_0^x R_1(x-t)Q(t)dt, \quad \text{where } R_1(x) = 1 + \int_0^x R(t)dt. \quad (30)$$

Substituting (30) in (17) and (20), taking into account (25) after some transformations we have

$$u(x, y) = \iint_{\Omega} M_1(x, y, x_1, y_1)f(x_1, y_1)dx_1dy_1, \quad (31)$$

where

$$M_1(x, y, x_1, y_1) = \theta(x - x_1)[\theta(y)M_{01}(x, y, x_1, y_1) + \theta(-y)M_{11}(x, y, x_1, y_1)], \quad (32)$$

$$M_{01}(x, y, x_1, y_1) = \theta(y_1) \left[ E(x - x_1, y, y_1) + \int_{x_1}^x dz \int_{x_1}^z E_{y_1}(x - z, y, 0)R_1(z - t)E_y(t - x_1, 0, y_1)dt \right] +$$

$$+ \theta(-y_1) \int_{\eta_1}^x E_{y_1}(x - t, y, 0)R_1(t - \eta_1)dt,$$

$$M_{11}(x, y, x_1, y_1) = \theta(y_1) \int_0^{\xi} R_1(\xi - t)E_y(t - x_1, 0, y_1)dt +$$

$$+ \frac{1}{2}\theta(-y_1)[\theta(\xi - \eta_1)R_1(\xi - \eta_1) + \theta(-y_1)\theta(\eta - \eta_1)\theta(\eta_1 - \xi)\theta(\xi - \xi_1)],$$

where  $\xi_1 = x_1 + y_1$ ,  $\eta_1 = x_1 - y_1$ ,  $\xi = x + y$ ,  $\eta = x - y$ ,  $\theta(y) = 1$ ,  $y > 0$  and  $\theta(y) = 0$ ,  $y < 0$ .

Taking into account explicit types of functions

$$M_{01}(x, y, x_1, y_1), M_{11}(x, y, x_1, y_1)$$

it is not difficult to establish that in (32) all terms are bounded, with the exception of the first –  $M_{01}(x, y, x_1, y_1)$ , in which by virtue of Lemma 3, the summand may not be limited  $E(x - x_1, y, y_1)$ . Therefore, it is enough to show that

$$\theta(x - x_1)\theta(y_1)\theta(y)E(x - x_1, y, y_1) \in L_2(\Omega \times \Omega).$$

By virtue of Lemma 3 from estimation (26) by direct calculation we have

$$\|\theta(x - x_1)E(x - x_1, y, y_1)\|_{L_2(\Omega \times \Omega)}^2 \leq C\{(2 + \theta)\beta[1 + (2 + \theta)\beta]\}^{-1}.$$

Therefore,  $M_1(x, y, x_1, y_1) \in L_2(\Omega \times \Omega)$ .

*Lemma 4.* If  $f(x, y) \in L_2(\Omega)$ , then  $Q(x) \in L_2[0, 1]$  and  $\|Q(x)\|_{L_2(0,1)}^2 \leq C \|f(x, y)\|_{L_2(\Omega)}^2$ .

*Proof of Lemma 4* taking into account (25), (27) It is carried out by direct calculation using the well-known Cauchy-Bunyakovsky inequality. From (29) we have

$$\|\tau'(x)\|_{L_2(0,1)} \leq C \|Q(x)\|_{L_2(0,1)} \leq C \|f(x, y)\|_{L_2(\Omega)}. \quad (33)$$

From (17) by virtue (33) by direct calculation it is not difficult to establish that

$$\|u(x, y)\|_{W_2^1(\Omega_1)} \leq C \|f(x, y)\|_{L_2(\Omega)}, \quad (34)$$

where  $W_2^1(\Omega)$  is S.L. Sobolev's space. From (18) and (34) we have

$$\begin{aligned} & D_{0x}^{\alpha-1} \|u(x, y)\|_{L_2(0,1)}^2 + \int_0^x \|u_y(t, y)\|_{L_2(0,1)}^2 dt + \|u(x, y)\|_{W_2^1(\Omega_2)}^2 \leq \\ & \leq C \left[ \int_0^x \|f(t, y)\|_{L_2(0,1)}^2 + \int_0^x d\xi \int_{\xi}^1 |f(\xi, x)|^2 dt + \|f(x, y)\|_{L_2(\Omega)}^2 \right]. \end{aligned} \quad (35)$$

Thus, summarizing the above statements, the following theorem is proved.

*Theorem 1.* For any function  $f(x, y) \in C^1(\bar{\Omega})$ ,  $f(A) = 0$  there is a unique regular solution to the problem  $M_1A$  (1), (3)-(5) and it is represented in the form (31) and satisfies the inequality (35). From (35) or (19) and (34) it is followed the the validity of the estimate

$$\|u(x, y)\|_{L_2(\Omega_0)} + \|u_y(x, y)\|_{L_2(\Omega_0)} + \|u(x, y)\|_{W_2^1(\Omega_1)} \leq C \|f(x, y)\|_{L_2(\Omega)}. \quad (36)$$

*Definition 2.* The function  $u(x, y) \in L_2(\Omega)$  is called a strong solution to problem  $M_1A$ , if there is a sequence of functions  $\{u_n(x, y)\}$ ,  $u_n(x, y) \in V$ , satisfying conditions (3)–(5), such that

$$\|u_n(x, y) - u(x, y)\|_{L_2(\Omega)} \rightarrow 0, \quad \|Lu_n(x, y) - f(x, y)\|_{L_2(\Omega)} \rightarrow 0 \text{ for } n \rightarrow \infty.$$

*Theorem 2.* For any function  $f(x, y) \in L_2(\Omega)$  there is a unique strong solution  $u(x, y)$  to the problem  $M_1A$ . This solution can be represented as (31) and satisfies the estimate (36).

*The proof of Theorem 2* in the presence of a representation of the solution (31) and the estimate (36) is proved in the same way as in [22–24].

By  $B_1$  we denote a closure in space  $L_2(\Omega)$ , of fractional differential operator given at the set of functions  $V$ , satisfying conditions (3)–(5), with expression (2).

According to the definition of a strong solution to the problem  $M_1A$ ,  $u(x, y)$  is a strong solution to the problem  $M_1A$  only and only then, when  $u(x, y) \in D(B_1)$ , where  $D(B_1)$  is a definition domain of operator  $B_1$ .

From theorem 2 it follows that operator  $B_1$  is closed and its definition domain is dense in  $L_2(\Omega)$ ; there exists an inverse operator  $B_1^{-1}$ , it is defined in all  $L_2(\Omega)$  and quite continuous.

In this regard, a natural question arises: is there an eigenvalue of the operator  $B_1^{-1}$ , and therefore to the problem? The main result is the theorem on the absence of eigenvalues of the operator  $B_1^{-1}$ .

*Theorem 3.* Integral operator

$$B_1^{-1} f(x, y) = \iint_{\Omega} M_1(x, y, x_1, y_1) f(x_1, y_1) dx_1 dy_1, \quad (37)$$

where  $M_1(x, y, x_1, y_1) \in L_2(\Omega \times \Omega)$  is Volterra in  $L_2(\Omega)$ .

*Proof.* To prove Theorem 3, we need to show that the operator  $B_1^{-1}$  defined by formula (37) is completely continuous and quasinilpotent. Since the complete continuity of this operator follows from the fact that  $M_1(x, y, x_1, y_1) \in L_2(\Omega \times \Omega)$ , show that  $B_1^{-1}$  is quasinilpotent, i.e.

$$\lim_{n \rightarrow \infty} \|B_1^{-1}\|_{L_2(\Omega) \rightarrow L_2(\Omega)}^{\frac{1}{n}} = 0, \quad (38)$$

where  $B_1^{-n} = B_1^{-1} [B_1^{-(n-1)}]$ ,  $n = 1, 2, \dots$

From (37) by direct calculation, taking into account (32) is not difficult to obtain that

$$B_1^{-n} f(x, y) = \iint_{\Omega} M_n(x, y, x_1, y_1) f(x_1, y_1) dx_1 dy_1, \quad (39)$$

where

$$M_n(x, y, x_1, y_1) = \iiint_{\Omega} M_1(x, y, x_2, y_2) M_{(n-1)}(x_2, y_2, x_1, y_1) dx_2 dy_2, \quad n = 2, 3, \dots$$

*Lemma 5.* For iterated kernels  $M_n(x, y, x_1, y_1)$  there is an assessment

$$|M_n(x, y, x_1, y_1)| \leq \left(\frac{3}{2}\right)^{n-1} N^n \frac{\Gamma^n(\gamma)}{\Gamma(n\gamma)} (x - x_1)^{n\gamma-1}, \quad (40)$$

where  $\gamma = (2 + \theta)\beta$ ,  $N = Cd$ ,  $C$  is coefficient from the assessment (26),

$$d = \max_{\substack{(x,y) \in \Omega \\ (x_1,y_1) \in \Omega}} |(x - x_1)^{1-\gamma} M_1(x, y, x_1, y_1)|, \quad \text{if } \gamma < 1.$$

$$d = \max_{\substack{(x,y) \in \Omega \\ (x_1,y_1) \in \Omega}} |M_1(x, y, x_1, y_1)|, \quad \text{if } \gamma \geq 1.$$

The proof of Lemma 5 we carry out by induction method over  $n$ .

For  $n = 1$  the inequality

$$|M_1(x, y, x_1, y_1)| \leq N(x - x_1)^{\gamma-1}$$

follows from representation (32) taking into account estimate (26).

Let be (40) valid for  $n = k - 1$ . Let's prove the validity of this formula for  $n = k$ .

Using inequality (40) for  $n = 1$  and  $n = k - 1$  we have

$$\begin{aligned} |M_k(x, y, x_1, y_1)| &= \left| \iint_{\Omega} M_1(x, y, x_2, y_2) \cdot M_{(k-1)}(x_2, y_2, x_1, y_1) dx_2 dy_2 \right| \leq \\ &\leq \iint_{\Omega} |M_1(x, y, x_2, y_2)| \cdot |M_{(k-1)}(x_2, y_2, x_1, y_1)| dx_2 dy_2 \leq \\ &\leq \iint_{\Omega} \theta(x - x_2) N(x - x_2)^{\gamma-1} \theta(x_2 - x_1) \left(\frac{3}{2}\right)^{k-2} N^{k-1} \frac{\Gamma^{k-1}(\gamma)}{\Gamma[(k-1)\gamma]} (x_2 - x_1)^{(k-1)\gamma-1} dx_2 dy_2 \leq \\ &\leq \left(\frac{3}{2}\right)^{k-1} N^k \frac{\Gamma^{k-1}(\gamma)}{\Gamma[(k-1)\gamma]} \int_{x_1}^x (x - x_2)^{\gamma-1} (x_2 - x_1)^{(k-1)\gamma-1} dx_2 = \end{aligned}$$

$$= \left(\frac{3}{2}\right)^{k-1} N^k \frac{\Gamma^{k-1}(\gamma)}{\Gamma[(k-1)\gamma]} (x-x_1)^{k\gamma-1} \int_0^1 \sigma^{\gamma-1} (1-\sigma)^{(k-1)\gamma-1} d\sigma = \left(\frac{3}{2}\right)^{k-1} N^k \frac{\Gamma^k(\gamma)}{\Gamma(k\gamma)} (x-x_1)^{k\gamma-1},$$

which proves Lemma 5.

Using the consistently known Schwarz inequality and Lemma 5 from the representation (39) we have

$$\begin{aligned} \|B_1^{-n} f(x, y)\|_{L_2(\Omega)}^2 &= \iint_{\Omega} |B_1^{-n} f(x, y)|^2 dx dy = \iint_{\Omega} \left[ \iint_{\Omega} M_n(x, y, x_1, y_1) f(x_1, y_1) dx_1 dy_1 \right]^2 dx dy \leq \\ &\leq \iint_{\Omega} \left[ \left( \iint_{\Omega} |M_n(x, y, x_1, y_1)|^2 dx_1 dy_1 \right) \left( \iint_{\Omega} |f(x_1, y_1)|^2 dx_1 dy_1 \right) \right] dx dy \leq \\ &\leq \left(\frac{3}{2}N\right)^{2n} \frac{\Gamma^{2n}(\gamma)}{[(2n\gamma-1)](2n\gamma)\Gamma^2(n\gamma)} \|f(x, y)\|_{L_2(\Omega)}^2. \end{aligned}$$

From here we get

$$\|B_1^{-n}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \left(\frac{3N}{2}\right)^n \left(4 - \frac{2}{n\gamma}\right)^{-\frac{1}{2}} \frac{\Gamma^n(\gamma)}{\Gamma(1+n\gamma)}.$$

From the latter it is not difficult to establish equality (38). *Theorem 3 is proved.*

*Corollary 1.* Problem  $M_1A$  is Volterra nature problem.

*Corollary 2.* For any complex number  $\lambda$  the equation  $B_1 u(x, y) - \lambda u(x, y) = f(x, y)$  is unambiguously solvable at all  $f(x, y) \in L_2(\Omega)$ .

Let now  $\Omega_1$  is a domain bounded by segments  $AB$  and characteristics  $AC : x + y = 0$ ,  $BC : x - y = 1$  of equation (1) and smooth curve  $AD : y = -\gamma(x)$ ,  $0 < x < l$ , where  $0,5 < l \leq 1$ ;  $\gamma(0) = 0$ ,  $l + \gamma(l) = 1$ , if  $l < 1$  and  $\gamma(l) = 0$ , if  $l = 1$  is located inside the characteristic triangle  $0 < x + y \leq x - y < 1$ .

A generalization of the problem in the domain  $\Omega$  is the following non-local problem for equation (1), where in the hyperbolic part of the mixed domain, the non-local condition pointwise connects the values of the tangent derivative of the desired solution on the characteristic  $AC$  with the derivatives in the direction of the characteristic of the desired function on an arbitrary curve  $AD$  lying inside the characteristic triangle, with the ends at the origin and on the characteristic  $BC$  (at a point  $B$ ).

**Problem  $M_1B$ .** Find a solution of equation (1) satisfying the conditions (3), (4) and

$$[u_x - u_y] [\theta_0(t)] + \mu(t) [u_x - u_y] [\theta^*(t)] = 0, \quad 0 < t < 1, \tag{41}$$

where  $\theta_0(t)$ ,  $(\theta^*(t))$  is an affix of the intersection point of the characteristic  $AC$  (curve  $AD$ ) with the characteristic coming out of the point  $(t, 0)$ ,  $0 < t < 1$ ,  $\mu(t)$  is a given function.

In the case when  $\alpha = 1$ , the problem  $M_1B$  coincides with nonlocal problem for mixed parabolic and hyperbolic equation with non-characteristic line of type change. In this case, regular and strong solvability issues and Volterra property of problem  $M_1B$  are investigated in [21–24]. Note that the problem  $M_1B$ , when  $\mu(x) = 0$  coincides with the problem of Tricomi for diffusion and wave equation, and in the case when  $\mu(t) = \infty$  coincides with the problem  $M_1A$ .

Similarly, as in the case of the problem  $M_1A$ , the concept of a regular and strong solution to the problem is introduced. Applying the methodology of proofs of theorems 1–3, the following theorem is proved.

*Theorem 4.* Let be  $\mu(t) \in C^1[0, 1]$  and  $\mu(x) \neq -1$ ,  $0 \leq x \leq 1$ . Then :

- a) for any function  $f(x, y) \in C^1(\bar{\Omega})$ ,  $f(A) = 0$  there is a unique regular solution to the problem  $M_1B$  (1), (3), (4), (41) and it is represented in the form (31) and satisfies the inequality (35);
- b) for any function  $f(x, y) \in L_2(\Omega)$  there exists a unique strong solution  $u(x, y)$  to problem  $M_1B$ . This solution can be presented in the form (31) and satisfies estimate (36);
- c) the problem  $M_1B$  is Volterra nature problem.

### 3 Solvability and existence of eigenvalues of local and nonlocal problems for the diffusion-wave equation

In domain  $\Omega$  of considered section 2 we investigate the following problem: **Problem  $M_2A$** . Find a solution of equation (1) satisfying the conditions

$$u|_{AA_0 \cup A_0B_0} = 0, \quad (42)$$

$$u_x + u_y|_{AD \cup BD} = 0. \quad (43)$$

*Definition 3.* The regular solution to the problem  $M_2A$  in the domain  $\Omega$  will be called the function  $u(x, y) \in W$ , where  $W = \{(x, y) : u(x, y) \in C(\bar{\Omega}) \cap C^{1,1}(\Omega \cup AD \cup BD), D_{0x}^\alpha u(x, y), u_{yy}(x, y) \in C(\Omega_0), u(x, y) \in C^{2,2}(\Omega_1)\}$ , satisfying equation (1) in  $\Omega_0 \cup \Omega_1$  and conditions (42)–(43).

*Definition 4.* The function  $u(x, y) \in L_2(\Omega)$  is called a strong solution to the problem  $M_2A$ , if there exists  $\{u_n(x, y)\}$ ,  $u_n(x, y) \in W$ , satisfying conditions (42)–(43), such that  $\|u_n(x, y) - u(x, y)\|_{L_2(\Omega)} \rightarrow 0$ ,  $\|Lu_n(x, y) - f(x, y)\|_0 \rightarrow 0$ , for  $n \rightarrow \infty$ .

Similarly, as in section 2, the regular solvability of the problem  $M_2A$ .

*Theorem 5.* For any function  $f \in L_2(\Omega)$  there is a unique strong solution  $u(x, y)$  to problem  $M_2A$ . This solution can be presented in the form

$$u(x, y) = \iint_{\Omega} K(x, y; x_1, y_1) f(x_1, y_1) dx_1 dy_1, \quad (44)$$

where  $K(x, y; x_1, y_1) \in L_2(\Omega \times \Omega)$ , and satisfies estimate (36).

Similarly as in the problem  $M_1A$ , the solution to problem  $M_2A$  in domain  $\Omega_1$  we seek in the form (15). Based on (43) from (15) we find

$$v(\xi) = -\tau'(\xi) - 2 \int_{\xi}^{\varphi(\xi)} f_1(\xi, \eta_1) d\eta_1, \quad 0 \leq \xi \leq 1, \quad (45)$$

where  $\eta = \varphi(\xi)$ ,  $0 \leq \xi \leq \xi_0$ ,  $\varphi(\xi_0) = 1$  is an equation of the curve  $AD$  in characteristic variables  $\xi, \eta$  and  $\varphi(\xi) \equiv 1$ ,  $\xi_0 \leq \xi \leq 1$  in the case when  $D \neq B$  and  $\eta = \varphi(\xi)$ ,  $0 \leq \xi \leq 1$  when  $D = B$ .

Substituting the resulting expression  $v(\xi)$  into (15), we obtain

$$u(x, y) = \tau(\eta) + \int_{\xi}^{\eta} d\xi_1 \int_{\eta}^{\varphi(\xi_1)} f_1(\xi_1, \eta_1) d\eta_1. \quad (46)$$

The formula (45) gives an integro-differential relation between  $\tau(x)$  and  $\nu(x)$ , brought to the segment  $AB$  from hyperbolic part  $\Omega_1$ .

Taking into account (22) and (45), it is not difficult to establish that the problem  $M_2A$  is equivalent to the following Volterra integral equation of the second kind

$$\tau'(x) - \int_0^x m(x-t)\tau'(t)dt = \Phi(x), \quad 0 \leq x \leq 1, \quad (47)$$

where  $\Phi(x) = -2 \int_x^{\varphi(x)} f_1(x, \eta_1) d\eta_1 - \int_0^x dx_1 \int_0^1 E_y(x-x_1, 0, y_1) f(x_1, y_1) dy_1$ .

Since  $m(x-t)$  is a kernel with a weak feature, then there is a unique strong solution to equation (47), and it is representable as

$$\tau'(x) = \Phi(x) + \int_0^x \Gamma(x-t)\Phi(t)dt, \quad (48)$$

where  $\Gamma(x)$  is a resolvent of equation (48):

$$\Gamma(x) = \sum_{j=1}^{\infty} m_j(x), \quad m_1(x) = m(x), \quad m_{j+1}(x) = \int_0^x m_1(x-t)m_j(t)dt.$$

From (48), taking into account  $\tau(0) = 0$ , we have

$$\tau(x) = -2 \int_0^x d\xi_1 \int_{\xi_1}^{\varphi(\xi_1)} \Gamma_1(x-\xi_1) f(\xi_1, \eta_1) d\eta_1 - \int_0^x dx_1 \int_0^1 E_1(x-x_1, y_1) f(x_1, y_1) dy_1, \quad (49)$$

where

$$\Gamma_1(x) = 1 + \int_0^x \Gamma(t)dt, \quad E_1(x, y_1) = \int_0^x E_y(t, 0, y_1)\Gamma_1(x-t)dt. \quad (50)$$

Substituting (49) into (20) and (46), we obtain

$$u(x, y) = \int_0^x dx_1 \int_0^1 E_2(x-x_1, y, y_1) f(x_1, y_1) dy_1 - 2 \int_0^x d\xi_1 \int_{\xi_1}^{\varphi(\xi_1)} E_1(x-\xi_1, y) f_1(\xi_1, \eta_1) d\eta_1, \quad y > 0, \quad (51)$$

$$u(x, y) = \int_{\xi}^{\eta} d\xi_1 \int_{\eta}^{\varphi(\xi_1)} f_1(\xi_1, \eta_1) d\eta_1 - 2 \int_0^{\eta} d\xi_1 \int_{\xi_1}^{\varphi(\xi_1)} \Gamma_1(\eta-\xi_1) f_1(\xi_1, \eta_1) d\eta_1 - \int_0^{\eta} dx_1 \int_0^1 E_1(\eta-x_1, y_1) f(x_1, y_1) dy_1, \quad y < 0, \quad (52)$$

where

$$E_2(x, y, y_1) = E(x, y, y_1) - \int_0^x E_y(x, 0, y_1)E_1(x-t, y_1)dt.$$

From (51) and (52) we get (44), where the kernel has the form

$$\begin{aligned}
 K(x, y; x_1, y_1) = & \theta(y) \{ \theta(y_1) \theta(x - x_1) E_2(x - x_1, y, y_1) - \\
 & - \theta(-y_1) \theta(x - \xi_1) E_1(x - \xi_1, y) \} + \theta(-y) \{ -\theta(y_1) \theta(\eta - x_1) E_1(\eta - x_1, y_1) + \\
 & + \theta(-y_1) \left[ \frac{1}{2} \theta(\xi_1 - \xi) \theta(\eta - \xi_1) \theta(\eta_1 - \eta) - \theta(\eta - \xi_1) \Gamma_1(\eta - \xi_1) \right] \}.
 \end{aligned} \tag{53}$$

From (44), (51), (52) and properties of the solution to the first initial boundary value problem for the diffusion equation [8], as in Theorem 2 it follows all statements of Theorem 5.

By  $B_2$  we denote a closure in  $L_2(\Omega)$  of the operator given on a set of functions from  $W$ , satisfying conditions (42), (43), with expression (2).

*Theorem 6.* Let be  $\gamma(x) \neq 0$ . Then there exists  $\lambda \in C$  such that equation  $B_2 u(x, y) = \lambda u(x, y)$  has non-trivial solution  $u(x, y) \in W$ .

*Proof.* From theorem 5 it is followed, that  $B_2$  is invertible and  $B_2^{-1}$  is an operator of Hilbert-Schmidt, defined by the formula (44). Then  $B_2^{-2} \equiv (B_2^{-1})^2$  kernel operator in  $L_2(\Omega)$ .

Therefore, for the operator  $B_2^{-2}$  we apply the result of V.B. Lidskii [25] on the coincidence of matrix and spectral traces. It is also known that for the kernel operator, represented as the product of two Hilbert-Schmidt operators, the Gaal formula [26] trace calculation takes place. Using formula of Gaal, we calculate the matrix trace  $B_2^{-2}$ .

$$Sp B_2^{-2} = \iint_{\Omega} dx dy \iint_{\Omega} K(x, y; x_1, y_1) K(x_1, y_1; x, y) dx_1 dy_1. \tag{54}$$

Taking into account the representation (53) from (54), after simple transformations, we obtain

$$\begin{aligned}
 Sp B_2^{-2} = & \int_0^1 dx \int_0^1 dy \int_0^x E_1(\xi_2, y) d\xi_2 \theta[\varphi(x - \xi_2) - x] \int_0^{\varphi(x - \xi_2) - x} E_1(\eta_2, y) d\eta_2 + \\
 & + \frac{1}{4} \int_0^1 d\xi \int_{\xi}^{\varphi(\xi)} d\eta \int_{\xi}^{\eta} d\xi_1 \int_{\eta}^{\varphi(\xi_1)} \theta(\eta_1 - \xi) \Gamma_1(\eta_1 - \xi) [\Gamma_1(\eta - \xi_1) - \theta(\eta_1 - \eta)] d\eta_1 = A + B.
 \end{aligned}$$

We will show that  $A + B > 0$ . Indeed, taking into account (50) and  $\varphi(t) \neq t$  will take place  $A \geq 0$ , if

$$E_y(t, 0, y_1) > 0. \tag{55}$$

We represent the function  $E_y(t, 0, y_1)$  in the form

$$E_y(t, 0, y_1) = \frac{1}{2t} \sum_{n=0}^{+\infty} \left[ e_{1,\beta}^{1,0} \left( -\frac{|2n + y_1|}{t} \right) + e_{1,\beta}^{1,0} \left( -\frac{|2(n + 1) - y_1|}{t} \right) \right].$$

Due to the properties of the Wright function [8; 46]  $e_{1,\beta}^{1,0}(-z) > 0$ ,  $z > 0$ , therefore, from the latter we get the justice of inequality (55). Also, from (50) it easily follows that

$$\Gamma_1(\eta - \xi_1) - \theta(\eta_1 - \eta) \geq 0,$$

therefore  $B \geq 0$ .

Thus,  $A + B > 0$ , as an integral in the positive direction of a non-negative and identically non-zero function. From here we get that  $Sp B_2^{-2} > 0$ . Further, applying the results of [25], we have

$$\sum_{k=1}^{\infty} \lambda_k (B_2^{-2}) = \sum_{k=1}^{\infty} \lambda_k^2 (B_2^{-1}) > 0,$$

where  $\lambda_k (B_2^{-2})$  are eigenvalues of operator  $B_2^{-2}$ . It means that  $\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} > 0$ , where  $\lambda_k$  are eigenvalues of the problem (1), (42) and (43). This implies the existence of the eigenvalues of the problem  $M_2A$  for the diffusion-wave equation of fractional order. *Theorem 6 is proved.*

In conclusion, we note that the most interesting is the fact that in problems  $M_1A$  and  $M_2A$ , in the case when point  $D$  coincides with point  $B$ , the Volterra property or existence of the problems eigenvalues depend on the derivative directions of the desired function given in the non-characteristic curve of the hyperbolic part of the boundary.

#### Acknowledgments

This research is supported by Ministry of Education and Science of the Republic of Kazakhstan Grant AP09058677.

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## Бөлшек ретті диффузиялық-толқындық теңдеу үшін локальді және локальді емес есептердің спектрлік қасиеттері

Мақалада бөлшек ретті диффузиялық-толқындық теңдеу үшін локальді және локальді емес есептердің шешімділік мәселелері мен спектрлік қасиеттері зерттелген. Сипаттауыш және сипаттауыш емес шекаралары бар облыстарда қойылған есептердің регуляр және күшті шешімділігі дәлелденді. Есептердің бірегей шешімділігі дәлелденіп, меншікті мәндердің бар екендігі немесе Вольтерра типіндегі есеп екендігі туралы теоремалар дәлелденген.

*Кілт сөздер:* диффузиялық-толқындық теңдеу, бөлшек ретті теңдеулер, шекаралық есептер, күшті шешім, Вольтерра қасиеті, меншікті мән.

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## Спектральные свойства локальных и нелокальных задач для диффузионно-волнового уравнения дробного порядка

В статье исследованы вопросы разрешимости и спектральные свойства локальных и нелокальных задач для диффузионно-волнового уравнения дробного порядка. Доказаны регулярная и сильная разрешимости поставленных задач в областях, как с характеристической, так и с нехарактеристической границей области. Установлена однозначная разрешимость задач, и доказаны теоремы о существовании собственных значений либо вольтерровости рассматриваемых задач.

*Ключевые слова:* диффузионно-волновое уравнение, уравнения дробного порядка, краевые задачи, сильное решение, вольтерровость, собственное значение.

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