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Difference schemes of high accuracy for a Sobolev-type pseudoparabolic equation

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In this work, numerical algorithms of higher-order accuracy are constructed and studied for a pseudoparabolic equation that describes the filtration process in fractured-porous media. The increase in the order of accuracy is achieved in various ways. First, only the spatial variables are approximated, as in the method of lines. Then, to solve the resulting system of linear ordinary differential equations, the finite difference method and the finite element method are applied. The application of these methods makes it possible to achieve a higher order of approximation for the difference schemes. Schemes of fourth-order accuracy in the spatial variables and second-order in time are presented, as well as schemes of fourth-order accuracy in all variables. Based on the stability theory of three-level difference schemes, stability conditions for the proposed algorithms are obtained. Using a special technique for solving the difference schemes, a priori estimates are derived, and based on them, theorems on convergence and accuracy are proven in the class of smooth solutions to the differential problem. An implementation algorithm is proposed for the difference scheme constructed using the finite element method. Test examples for one-dimensional and two-dimensional equations are also provided, demonstrating the higher-order accuracy of the proposed schemes.

Keywords: pseudoparabolic equation, filtration equation, finite difference method, finite element method, higher-order accuracy schemes, stability, convergence, accuracy estimates.

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Introduction

In the general case, pseudoparabolic equations are written in the following form:

$$\frac{\partial}{\partial t}[A(u)] + B(u) = 0,$$

these equations belong to composite-type equations. Here $A(u)$, $B(u)$ are elliptic operators [1]. Problems in semiconductor physics, plasma physics, and hydrodynamics of stratified and filtered liquids are examples of such equations. Let us present some of them. Mathematical models of Rossby waves in oceanology [2] are given as

$$\frac{\partial}{\partial t}Lu + \beta {u'}_2 = g(x,t), \;\; (x,t) \in Q_T,$$

$Lu = \sum\limits_{m=1}^{3} L_m$, $L_m = \partial^2 u/\partial x_m^2$, ${u'}_2 = \partial u/\partial x_2$, β is a constant, and the equation

$$\frac{\partial}{\partial t}(Lu + \theta u) + \mu^2 Lu + \lambda u = g(x,t), \;\; (x,t) \in Q_T \tag{1}$$

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describes the process of filtration in a fractured porous liquid [1]. Here θ, μ, λ are constants. Besides, we can mention the equation of moisture transfer in soils [3]:

$$\frac{\partial u}{\partial t} = Lu + g(x,t), \;\; (x,t) \in Q_T,$$

where $Lu = \sum_{m=1}^{p} L_m u, \;\; L_m u = (k_\alpha(x){u'}_m)'_m + \frac{\partial}{\partial t}(k_\alpha(x){u'}_m)'_m$. Here $Q_T = \{(x,t) : x \in \Omega, \; t \in (0,T]\}$, $\Omega = \{x = (x_1, x_2, x_3) : 0 < x_k < l_k, \; k = 1,2,3\}$.

Such problems were studied by analytical methods in [4–6]. Numerical methods for solving problems of this type were considered in [1,2], where difference schemes with second-order accuracy in both variables were constructed under the assumption of sufficient smoothness of the solution to the differential problem. In [7–9] for Sobolev-type equations, high-order accuracy schemes were constructed and studied in classes of nonsmooth solutions.

Initial high-order accuracy difference schemes for multidimensional parabolic equations were developed and analyzed in [10–12], where it was demonstrated that fourth-order accuracy in spatial variables and second-order accuracy in time could be achieved. In [13–15], compact difference schemes for various parabolic equations were constructed and investigated. In particular, monotone difference schemes for linear non-homogeneous parabolic equations and Fisher (Kolmogorov–Petrovskii–Piskunov) equations were constructed in [13]. The convergence of the proposed methods in the uniform metric C is proved. The results obtained are generalized to arbitrary semilinear parabolic equations with a nonlinear sink of arbitrary type and to quasilinear equations. Note also the paper [14], which studies compact and monotone difference schemes: first- and second-order in time and fourth-order in space, developed for linear, semilinear and quasilinear parabolic equations. Similar results were obtained in [15] for one-dimensional and multidimensional quasilinear stationary equations; where conservative compact and monotone difference schemes were constructed. Compact and monotone difference schemes of the fourth-order accuracy in spatial variables (and first-order in time) that maintain the conservatism properties were constructed and investigated for the first time in [16]. High-order accuracy difference schemes for convection-diffusion problems are constructed in paper [17, 18].

This paper examines the issues of constructing and studying high accuracy difference schemes for equation (1) with first kind boundary conditions. In this case, the main attention is paid to obtaining an estimate of the accuracy of difference schemes in classes of smooth solutions. The approximation error was studied, stability conditions were obtained, and theorems on the convergence and accuracy of the considered schemes were proved. In addition, test calculations are performed to confirm the high accuracy of difference schemes.

1 Statement of the problem

Let the following initial and boundary conditions be specified for (1):

$$u|_{t=0} = u_0(x), \;\; x \in \overline{\Omega} = \Omega + \Gamma, \tag{2}$$

$$u|_{x \in \Gamma = \partial\bar{\Omega}} = \mu(t), \;\; t \in (0,T]. \tag{3}$$

Let $u(x,t) \in H = \overset{\circ}{W}{}_2^1(\Omega)$, $\frac{\partial u}{\partial t} \in L_2[0,T]$. Let us put the following problem in correspondence to (1)–(3):

$$a_3\left(\frac{du(t)}{dt}, \vartheta\right) + a_2(u(t),\vartheta) + a_1(u(t),\vartheta) = (g(t),\vartheta), \;\; u(0) = u_0, \; \forall \vartheta(x) \in H, \tag{4}$$

where

$$a_3(u,\vartheta) = \iint\limits_{\Omega} \left(\sum_{k=1}^{3} u_{x_k}\vartheta_{x_k} + \theta u \vartheta\right) dx, \;\; a_2(u,\vartheta) = \mu^2 \iint\limits_{\Omega} \sum_{k=1}^{3} u_{x_k}\vartheta_{x_k} dx, \;\; a_1(u,\vartheta) = \lambda \iint\limits_{\Omega} u\vartheta dx,$$

$u = u(t) \in H$, $\forall t \in [0, T]$, i.e. $u(t)$ is a function of abstract argument t with values in Hilbert space H. In $\overset{\circ}{W}{}_2^1(\Omega)$ we define the scalar product

$$(u(x), \vartheta(x)) = \iint\limits_{\Omega} \left(u\vartheta + \sum_{m=1}^{3} \frac{\partial u}{\partial x_m} \cdot \frac{\partial \vartheta}{\partial x_m} \right) dx$$

and the norms

$$\|u(x_1, x_2, x_3)\|^2_{W_2^1(\Omega)} = \iint\limits_{\Omega} \left(u^2 + \sum_{m=1}^{3} \left(\frac{\partial u}{\partial x_m} \right)^2 \right) dx.$$

Here $c_3 \|u\|_1^2 \le a_3(u, u) \le C_3 \|u\|_1^2$, $0 \le a_2(u, u) \le C_2 \|u\|_1^2$, $0 \le a_1(u, u) \le C_1 \|u\|_1^2$, $c_3 > 0$, $C_1 = C_1(\lambda)$, $C_2 = C_2(\mu)$, $C_3 = C_3(\theta)$.

2 Approximation in space

We introduce subspace $H_h \subset H$. The scalar product and energy norm [14] in H_h are defined by $(y, \vartheta)_A = (Ay, \vartheta)$ and $\|y\|_A = \sqrt{(y, y)_A}$, respectively. Let us approximate equation (1) in space variables. We introduce a grid $\bar\omega_h = \bar\omega_{h_1} \times \bar\omega_{h_2} \times \bar\omega_{h_3}$, $\bar\omega_{h_m} = \left\{ x_m = i_m h_m,\ i_m = \overline{0, N}_m,\ h_m = l_m/N_m \right\}$, $m = 1, 2, 3$ in $\bar\Omega$. Here $\bar\omega_h = \omega_h + \gamma_h$. We define $H_h = \overset{\circ}{W}{}_2^1(\omega_h)$ with the norm defined as

$$\|\vartheta\|^2_{1h} = \sqrt{\sum_{i_1}^{N_1} \sum_{i_2}^{N_2} \sum_{i_3}^{N_3} h_1 h_2 h_3 \sum_{i=1}^{3} (\vartheta_{\bar x_i})^2} \le M,$$

where M is a positive constant, $\vartheta = \vartheta(i_1 h_1, i_2 h_2, i_3 h_3)$,

$$\vartheta_{\bar x_1} = \left[\vartheta(i_1 h_1, i_2 h_2, i_3 h_3) - \vartheta((i_1 - 1) h_1, i_2 h_2, i_3 h_3)\right] / h_1,$$

$$\vartheta_{\bar x_2} = \left[v(i_1 h_1, i_2 h_2, i_3 h_3) - \vartheta(i_1 h_1, (i_2 - 1) h_2, i_3 h_3)\right] / h_2\,,$$

$$\vartheta_{\bar x_3} = \left[v(i_1 h_1, i_2 h_2, i_3 h_3) - \vartheta(i_1 h_1, i_2 h_2, (i_3 - 1) h_3)\right] / h_3\,.$$

Approximating $a_m(u, \vartheta)$ by quadrature formulas, from (4) we come to the definition of an approximate grid solution:

$$a_{3,h}\left(\frac{du_h(t)}{dt}, \vartheta \right) + a_{2,h}(u_h(t), \vartheta) + a_{1,h}(u_h(t), \vartheta) = (g_h(t), \vartheta),\ \ \forall \vartheta(x) \in H_h,$$

$$u_h(0) = u_{0,h}.$$

This corresponds to the following problem:

$$D \frac{du_h(t)}{dt} + A u_h(t) = g_h(t), \quad u_h(0) = u_{0,h}, \tag{5}$$

where $D = \sum\limits_{m=1}^{3} \Lambda_m + \theta E$, $A = \mu^2 \sum\limits_{m=1}^{3} \Lambda_m + \lambda E$, $\Lambda_m y = y_{x_m \bar x_m}$, $u_{h,0} = P_h u_0(x)$, $P_h : H \to H_h$, $g_h(t) = P_h g(x, t)$,

$$y_{x_1 \bar x_1} = (y((i_1 + 1) h_1, i_2 h_2, i_3 h_3) - 2y(i_1 h_1, i_2 h_2, i_3 h_3) + y((i_1 - 1) h_1, i_2 h_2, i_3 h_3)) / h_1^2,$$

$$y_{x_2 \bar x_2} = (y(i_1 h_1, (i_2 + 1) h_2, i_3 h_3) - 2y(i_1 h_1, i_2 h_2, i_3 h_3) + y(i_1 h_1, (i_2 - 1) h_2, i_3 h_3)) / h_2^2,$$

$$y_{x_3\bar{x}_3} = (y(i_1h_1, i_2h_2, (i_3+1)h_3) - 2y((i_1h_1, i_2h_2, i_3h_3)) + y(i_1h_1, i_2h_2, (i_3-1)h_3))/\,h_3^2.$$

Operators $D \in H_h$ and $A \in H_h$ are approximates respectively,

$$L + \theta E, \quad \mu^2 L + \lambda E \tag{6}$$

with second-order error.

Based on the Taylor expansion formula, we obtain:

$$\bar{\Lambda} u = \sum_{m=1}^{3} \Lambda_m u + \sum_{\substack{m,l=1\\ m\neq l}}^{3} \frac{h_m^2}{12} \Lambda_m \Lambda_l + O(|h|^4), \tag{7}$$

where $|h| = \sqrt{h_1^2 + h_2^2 + h_3^2}$. Then, from (7), neglecting $O(|h|^4)$, we obtain the following operators:

$$\bar{D} = \bar{\Lambda} + \theta E, \quad \bar{A} = \mu^2 \bar{\Lambda} + \lambda E, \tag{8}$$

which approximate (6) to the fourth-order in h. Hence, instead of (5), we obtain the semi-discrete problem:

$$\bar{D}\frac{du_h}{dt} + \bar{A}u_h = \bar{g}_h, \quad t \in (0, T], \quad u_h(0) = u_{h,0}, \tag{9}$$

where $\bar{D} \in H_h$, $\bar{A} \in H_h$, $\bar{g}_h = g + \sum\limits_{m=1}^{3} \frac{h_m^2}{12} \Lambda_m g$.

It's clear that

$$D = D^* > 0, \quad \bar{D} = \bar{D}^* > 0, \quad A = A^* > 0, \quad \bar{A} = \bar{A}^* > 0. \tag{10}$$

In what follows, in (9), we use $u = u_h \in H_h$ instead of u_h, i.e. equations (9), (10) have the following form:

$$\bar{D}\dot{u} + \bar{A}u = \bar{g}, \quad u(0) = u_0, \tag{11}$$

where $\dot{u} = du/dt$.

3 Approximation in time

Let y approximate $u = u_h \in H_h$. We introduce a grid $\omega_\tau = \{t_n = n\tau,\ n = 1, 2, ..., M,\ \tau = T/M\}$ uniform in t. Here $\tau > 0$ is the time step. We replace system (11) with the following difference scheme:

$$\bar{D}y_{\overset{\circ}{t}} + \bar{A}y^{(\sigma_1,\sigma_2)} = \varphi, \quad y^0 = u_0, \quad y^1 = u_1, \tag{12}$$

where $y_{\overset{\circ}{t}} = (y^{n+1} - y^{n-1})/(2\tau)$, $y^n = y(t_n)$, $u_1 = (E - \tau\bar{D}^{-1}\bar{A})u_0 + \tau\bar{D}^{-1}g(x, 0)$, φ approximates g,

$$y^{(\sigma_1,\sigma_2)} = \sigma_1 y^{n+1} + (1 - \sigma_1 - \sigma_2)y^n + \sigma_2 y^{n-1} = y^n + \tau(\sigma_1 - \sigma_2)y_{\overset{\circ}{t}} + \frac{\tau^2}{2}(\sigma_1 + \sigma_2)y_{\bar{t}t}, \tag{13}$$

where $y_{\bar{t}t} = (y^{n+1} - 2y^n + y^{n-1})/\tau^2$. We write the difference scheme (12) using identity (13) in the following form:

$$\bar{B}y_{\overset{\circ}{t}} + \tau^2\bar{D}y_{\bar{t}t} + \bar{A}y = \varphi, \quad y^0 = u_0, \quad y^1 = u_1, \tag{14}$$

with the operators

$$\bar{D} = (\sigma_1 + \sigma_2)\bar{A}/2, \quad \bar{B} = \bar{D} + \tau(\sigma_1 - \sigma_2)\bar{A}. \tag{15}$$

We denote the errors of scheme (14) by $z = y - u$. Then, from (14) for z, we obtain:

$$\bar{B}z_{\overset{\circ}{t}} + \tau^2 \bar{D}z_{\bar{t}t} + \bar{A}z = \psi, \quad z^0 = 0, \quad z^1 = 0, \tag{16}$$

where ψ is the approximation error of scheme (14) for the solution $u(x,t)$ of the equation (1). By direct calculation we can verify that $\psi = O(\tau^2 + |h|^4)$. Now we approximate (11) by the difference scheme [8]:

$$\bar{D}y_t - \gamma\bar{A}\dot{y}_t + \bar{A}y^{(0.5)} = \varphi_1, \quad \gamma\bar{D}\dot{y}_t + \alpha\bar{A}y_t + \beta\bar{A}\dot{y}^{(0.5)} = \varphi_2, \tag{17}$$

$$y^0 = u_0, \quad \dot{y}^0 = \bar{D}^{-1}(f^0 - \bar{A}u_0), \tag{18}$$

where $y_t = (y^{n+1} - y^n)/\tau$, $\dot{y}_t = (\dot{y}^{n+1} - \dot{y}^n)/\tau$, $y^{(0.5)} = (y^{n+1} + y^n)/2$, $\dot{y}^{(0.5)} = (\dot{y}^{n+1} + \dot{y}^n)/2$, $\varphi_1 = \frac{1}{\tau}\int\limits_{t_n}^{t_{+n1}} \bar{g}(t)dt$, $\varphi_2 = \frac{1}{\gamma\tau}\int\limits_{t_n}^{t_{n+1}} \bar{g}(t)(s_1\vartheta_2^{(1)} + s_2\vartheta_2^{(3)})dt$, $s_1 = 15\gamma - 35\alpha/3$, $s_2 = 140\gamma - 350\alpha/3$, $\vartheta_2^{(1)} = 1/2$, $\vartheta_2^{(3)} = \tau\xi(1-\xi)(\xi - 1/2)$, $\xi = \tau^{-1}(t - t_n)$. Thus, consider the following algorithms:

— scheme 1^0 — a difference approximation of fourth order in space (8) and second order in time (12);

— scheme 2^0 — a difference approximation of fourth order in space (8) and fourth order in time (17), (18).

4 Stability and accuracy

To study the stability of scheme (12), we use well-known theorems on the stability of three-layer difference schemes. Since $\bar{D}, \bar{B}$ are self-adjoint positive difference operators, according to Theorem 1 from [19, p. 231], provided that the following conditions are met:

$$\bar{A} > 0, \quad \bar{D} > \frac{1}{4}\bar{A}, \tag{19}$$

$$\bar{B} + \frac{\tau}{2}\frac{\rho - 1}{\rho + 1}\bar{A} \geq 0, \quad \rho \geq 1, \tag{20}$$

the following a priori estimate holds:

$$\|y^{n+1}\|_{\mathfrak{A}} \leq \rho\|y^n\|_{\mathfrak{A}}, \quad n = 0, 1, ..., \quad \rho \geq 1, \tag{21}$$

where $\|y^n\|_{\mathfrak{A}} = \frac{1}{4}\left\|y^n + y^{n+1}\right\|_{\bar{A}}^2 + \left\|y^{n+1} - y^n\right\|_{\bar{D} - \frac{1}{4}\bar{A}}^2$.

Conditions (19), (20) considering (15) take the following form:

$$\left(\frac{\sigma_1 + \sigma_2}{2} - \frac{1}{4}\right)\bar{A} > 0, \tag{22}$$

$$\left[\frac{\sigma_1 + \sigma_2}{2} + \tau(\sigma_1 - \sigma_2) + \frac{\tau}{2}\frac{\rho - 1}{\rho + 1}\right]\bar{A} \geq 0. \tag{23}$$

Since $\rho \geq 1$, from (22), (23) we obtain that the difference scheme (12) is stable for all τ and h, if its parameters satisfy the following inequalities

$$\sigma_1 + \sigma_2 > 0.5, \quad \sigma_1 \geq \sigma_2. \tag{24}$$

Consequently, the following theorem holds.

Theorem 1. If conditions (24) are satisfied, scheme (12) is stable with respect to the initial data and estimate (21) holds for its solution $y^n \in H_h$.

Based on Theorem 1 and Theorem 3 in [19, p. 257], the following statement holds.

Theorem 2. Let conditions (24) be satisfied. Then the solution to the difference scheme (12) is stable with respect to the initial data and the right-hand side, and for its solution $y^n \in H_h$, the following a priori estimate holds:

$$\left\|y^{n+1}\right\|_{\mathfrak{M}} \leq e^{0.5ct_n}\left(\left\|y^0\right\|_{\mathfrak{M}(0)} + \sum_{k=0}^{n}\left\|\bar{B}_1^{-1}\varphi^k\right\|_{\bar{D}}\right), \tag{25}$$

where $\bar{B}_1 = \bar{B}/(2\tau) + \bar{D}$, $\left\|y^{n+1}\right\|_{\mathfrak{M}} = \frac{1}{4}\left\|y^{n+2} + y^{n+1}\right\|_{\bar{A}}^2 + \left\|y^{n+2} - y^{n+1}\right\|_{\bar{D}-\frac{1}{4}\bar{A}}^2$.

Considering (16) and (25), we obtain the following theorem.

Theorem 3. Let conditions (24) be satisfied. Then the solution to scheme (12) $y^n \in H_h$ converges to a smooth solution to the differetial problem (1)–(3), i.e.

$$\|y(x_i,t_n) - u(x_i,t_n)\|_{1h} \leq M(|h|^4 + \tau^2), \quad (x_i,t_n) \in \bar{\omega}_{\tau h} = \bar{\omega}_\tau \times \bar{\omega}_{h_\alpha}, \quad \bar{\omega}_\tau = \omega_\tau \cup \{0\}\,.$$

Let us consider the accuracy of scheme (17), (18). Let $z^n = y^n - u^n$, $\dot{z}^n = \dot{y}^n - \dot{u}^n$. Substituting $y^n = z^n + u^n$ and $\dot{y}^n = \dot{z}^n + \dot{u}^n$ into (17), (18), we obtain:

$$\bar{D}z_t - \bar{\gamma}\dot{z}_t + \bar{z}^{(0.5)} = \psi_1, \quad \gamma\bar{D}\dot{z}_t + \alpha\bar{z}_t + \beta\dot{z}^{(0.5)} = \psi_2, \quad z^0 = 0, \quad \dot{z}^0 = 0,$$

$$\psi_1 = O(\tau^4), \ \psi_2 = (\alpha+\beta-\gamma)\bar{A}\dot{\bar{u}} + \frac{\tau^2}{24}\left[(\alpha+3\beta-\gamma)\bar{A}\dddot{\bar{u}} - (3\gamma - 2\alpha\dddot{\bar{g}})\right] + O\left(\tau^4\right),$$

where $\bar{u} = u(\bar{t}_n)$, $\bar{t}_n = t_n + \theta\tau$, $0 < \theta < 1$. Hence, if the following conditions are met

$$\gamma = \alpha + \beta, \quad \alpha, \ \beta, \ \gamma = O(\tau^2), \tag{26}$$

then $\psi_1 = \psi_2 = O\left(\tau^4\right)$.

For vector scheme (17), (18) with commuting operators $\bar{D}$ and $\bar{A}$, i.e. $\bar{A}\bar{D} = \bar{D}\bar{A}$, the following estimate was obtained in [8]:

$$\|u_h(t) - u(t)\|_{\bar{A}} + \|u_{h,t}(t) - u_t(t)\|_{\bar{D}} \leq M\tau^4.$$

Condition $\bar{A}\bar{D} = \bar{D}\bar{A}$ is overloaded. To avoid it, we introduce $w = \bar{D}^{1/2}y$, $\dot{w} = \bar{D}^{1/2}\dot{y}$ instead of $y, \dot{y}$. Note that $\left(\bar{D}^{1/2}\right)^* = \bar{D}^{1/2} > 0$ and there is an inverse operator $\bar{D}^{-1/2} = \left(\bar{D}^{-1/2}\right)^* > 0$.

After obvious transformations, from (17), (18) we obtain:

$$\tilde{D}w_t - \gamma\tilde{A}\dot{w}_t + \tilde{A}w^{(0.5)} = \tilde{\varphi}_1, \quad \gamma\tilde{D}\dot{w}_t + \alpha\tilde{A}w_t + \beta\tilde{A}\dot{w}^{(0.5)} = \tilde{\varphi}_2, \tag{27}$$

$$w^0 = \bar{D}^{1/2}u_0, \quad \dot{w}^0 = \bar{D}^{1/2}(\bar{g}^0 - \bar{A}u_0),$$

where $\tilde{\varphi}_j = \bar{D}^{-1/2}\varphi_j$, $j = 1,2$, $\tilde{D} = E$, $\tilde{A} = \bar{D}^{-1/2}\bar{A}\bar{D}^{-1/2}$. Here $\tilde{D}$, $\tilde{A}$ are self-adjoint, positive, and commutating operators. Eliminating $\dot{w}$ from (27) we obtain:

$$B_1 w^{n+1} + B_2 w^n + B_3 w^{n-1} = \tau F_n, \quad n = 1, 2, ..., \tag{28}$$

where w^0, w^1 are given

$$B_1 = \gamma\tilde{D}^2 + \frac{\tau}{2}(\gamma+\beta)\tilde{A}\tilde{D} + \frac{\tau^2}{12}\left(3\beta+\alpha\right)\tilde{A}^2,$$

$$B_2 = -2\gamma\tilde{D}^2 + \frac{\tau^2}{6}\left(3\beta-\alpha\right)\tilde{A}^2,$$

$$B_3 = \gamma \tilde{D}^2 - \frac{\tau}{2}(\gamma+\beta)\tilde{A}\tilde{D} + \frac{\tau^2}{12}(3\beta+\alpha)\tilde{A}^2,$$

$$F_n = \left(\gamma\tilde{D} + \frac{\tau}{2}\beta\tilde{A}\right)\tilde{\varphi}_1^n + \frac{\tau^2}{12}\tilde{A}\tilde{\varphi}_2^n - \left(\gamma\tilde{D} - \frac{\tau}{2}\beta\tilde{A}\right)\tilde{\varphi}_1^{n-1} - \frac{\tau^2}{12}\tilde{A}\tilde{\varphi}_2^{n-1}.$$

We rewrite equation (28) in canonical form:

$$\overline{B}w_{\overset{\circ}{t}} + \tau^2\overline{R}w_{\bar{t}t} + \overline{A}w = \overline{F}. \tag{29}$$

The operators in (29) have the following form:

$$\bar{B} = \tau(B_1 - B_3) = \tau(\gamma+\beta)\tilde{A}\tilde{D},$$

$$\bar{R} = \frac{1}{2\tau}(B_1 + B_3) = \frac{1}{\tau}\left(\gamma\tilde{D}^2 + \frac{\tau^2}{12}(3\beta+\alpha)\tilde{A}^2\right),$$

$$\bar{A} = \frac{1}{\tau}(B_1 + B_2 + B_3) = \tau\beta\tilde{A}^2,$$

$$\bar{F} = \tau\gamma\tilde{D}\tilde{\varphi}_{1,\bar{t}}^n + \tau\beta\tilde{A}\frac{\tilde{\varphi}_1^n + \tilde{\varphi}_1^{n-1}}{2} + \frac{\tau^2}{12}\tilde{A}\tilde{\varphi}_{2,\bar{t}}^n. \tag{30}$$

Here $\bar{B}$, $\bar{A}$ are self-adjoint positive operators, $\bar{R}^* = \bar{R}$.

The scheme stability condition (29) $\bar{R} > \bar{A}/4$ is satisfied if,

$$\alpha > 0, \quad \gamma > 0, \tag{31}$$

β is a free parameter. Therefore, based on the methodology given in [19, 20], for solving scheme (29), we obtain the following estimate:

$$\|w^n\|_{\bar{A}}^2 \le \|w^0\|_{\bar{A}}^2 + \frac{1}{2}\sum_{k=0}^{n}\tau\left\|\bar{F}_k\right\|_{\bar{B}^{-1}}^2. \tag{32}$$

From (32) considering (30), we obtain:

$$\|y^n\|_{\tilde{A}^2} \le \|y^0\|_{\tilde{A}^2} +$$

$$M\max_k\left(\frac{\gamma}{\sqrt{\beta(\gamma+\beta)}}\left\|\tilde{\varphi}_{1,\bar{t}}^k\right\|_{\tilde{A}^{-1}\tilde{D}} + \sqrt{\frac{\beta}{\gamma+\beta}}\left\|\frac{\tilde{\varphi}_1^k + \tilde{\varphi}_1^{k-1}}{2}\right\|_{\tilde{A}\tilde{D}^{-1}} + \frac{\tau^2}{12\sqrt{\beta(\gamma+\beta)}}\left\|\tilde{\varphi}_{2,\bar{t}}^k\right\|_{\tilde{A}\tilde{D}^{-1}}\right), \tag{33}$$

where M is a positive constant.

Let us apply (33) to estimate the error $z = y - u$ of scheme (29), which satisfies equation $\bar{B}z_{\overset{\circ}{t}} + \tau^2\bar{R}z_{\bar{t}t} + \bar{A}z = \psi$, where $\psi = \bar{F} - (\bar{B}u_{\overset{\circ}{t}} + \tau^2\bar{R}u_{\bar{t}t} + \bar{A}u)$. Hance, we get the following estimate for z:

$$\|z^n\|_{\tilde{A}^2} \le$$

$$M\max_k\left(\frac{\gamma}{\sqrt{\beta(\gamma+\beta)}}\left\|\psi_{1,\bar{t}}^k\right\|_{\tilde{A}^{-1}\tilde{D}} + \sqrt{\frac{\beta}{\gamma+\beta}}\left\|\frac{\psi_1^k + \psi_1^{k-1}}{2}\right\|_{\tilde{A}\tilde{D}^{-1}} + \frac{\tau^2}{12\sqrt{\beta(\gamma+\beta)}}\left\|\psi_{2,\bar{t}}^k\right\|_{\tilde{A}\tilde{D}^{-1}}\right).$$

Here $\psi_1,\ \psi_2$ are the approximation errors of the vector scheme (17).

Similarly we obtain results for $\dot{z} = \dot{y} - \dot{u}(t_n)$. Therefore, $\|z^n\|_{\tilde{A}^2} = \|u^n - y^n\|_{\tilde{A}^2} = O(\tau^4)$ and $\|\dot{z}^n\|_{\tilde{A}^2} = \|\dot{u}^n - \dot{y}^n\|_{\tilde{A}^2} = O(\tau^4)$ at time point t_n, $n = 1, 2, \ldots$ Based on (26), (31), (33), we obtain the following result.

Theorem 4. Let conditions (26), (31) be satisfied. Then, for $u(x,t) \in C^6[0,T]$, scheme (17), (18) converges to the solution to problem (11), i.e. the following accuracy estimates hold:

$$\|z(t)\|_{\tilde{A}^2} \leq M\tau^4, \quad \|\dot{z}(t))\|_{\tilde{A}^2} \leq M\tau^4, \quad \forall t \in [0,T].$$

Similarly, we obtain accuracy estimates for scheme 2^0.

Theorem 5. Let the approximation conditions (26) be satisfied. Then, if condition (31) is satisfied, the solution to scheme 2^0 converges to a sufficiently smooth solution to problem (1)–(3), i.e.

$$\|z(t)\|_{1h} + \|\dot{z}(t)\|_{1h} \leq M(|h|^4 + \tau^4), \quad z, \dot{z} \in H_h.$$

5 *Algorithm for implementing the scheme*

To implement (27) we rewrite it in the following form:

$$m_{11}w^{n+1} + m_{12}\dot{w}^{n+1} = \phi_1, \quad m_{21}w^{n+1} + m_{22}\dot{w}^{n+1} = \phi_2, \tag{34}$$

where

$$m_{11} = \tilde{D} + \tau\tilde{A}/2, \;\; m_{12} = -\tau^2\tilde{A}/12, \;\; m_{21} = \alpha\tilde{A}, \;\; m_{22} = \gamma\tilde{D} + \tau\beta\tilde{A}/2,$$

$$\phi_1 = \tau\tilde{\varphi}_1 + \left(\tilde{D} + \tau\tilde{A}/2\right)w^n - \tau^2\tilde{A}\dot{w}^n/12, \quad \phi_2 = \tau\tilde{\varphi}_2 + \alpha\tilde{A}w^n + \left(\gamma\tilde{D} + \tau\beta\tilde{A}/2\right)\dot{w}^n.$$

The integrals $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ can be calculated, for example, using Simpson's formula.

Considering the commutability of $\tilde{A}$, $\tilde{D}$, we eliminate $\dot{w}^{n+1}$ from (34):

$$Cy^{n+1} = F, \tag{35}$$

where $C = \gamma\tilde{D}^2 + \tau(\beta+\gamma)\tilde{A}\tilde{D}/2 + \tau^2(3\beta+\alpha)\tilde{A}^2/12$, $F = m_{22}\phi_1 - m_{12}\phi_2$.

To solve (35), we factorize the operator C:

$$C = \gamma C_1C_2 = \gamma[\tilde{D}^2 - (x_1+x_2)\tau\tilde{A}\tilde{D} + x_1x_2\tau^2\tilde{A}^2], \;\; C_k = (\tilde{D} - x_k\tau\tilde{A}), \;\; k = 1,2.$$

Therefore, the algorithm for solving (35) has the following form:

$$\gamma C_1\bar{w} = F, \quad C_2w^{n+1} = \bar{w}.$$

The value of $\dot{w}^{n+1}$ is determined from

$$\left(\gamma\tilde{D} + \tau\beta\tilde{A}/2\right)\dot{w}^{n+1} = \phi_2 - \alpha\tilde{A}w^{n+1}.$$

The implementation of scheme (12) is not difficult, for example, for $\sigma_1 = \sigma_2 = \sigma$, it is implemented as follows:

$$(\bar{D} - \sigma\tau\bar{A})y^{n+1} = (1-2\sigma)\tau\bar{A}y^n + (\bar{D} + \sigma\tau\bar{A})y^{n-1} + \tau\phi, \quad n = 1,2,...,$$

$$y^0 = u_{h,0}, \quad y^1 = u_{h,1}.$$

6 Numerical experiments

6.1 One-dimensional case

Let us choose the parameters of problem (1)–(3): $l = \pi$, $T = 1$, $\mu = \theta = 1$, $\lambda = -1$. Then, instead of (1)–(3), we obtain:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2} - u\right) + \frac{\partial^2 u}{\partial x^2} - u = 0,$$

$$(x,t) \in Q_T = \{x: \ 0 < x < \pi, \ t \in (0,1]\},$$

$$u(0,t) = u(\pi,t) = 0, \quad t \in (0,1],$$

$$u(x,0) = \sin x, \quad x \in [0,\pi].$$

The exact solution is $u(x,t) = e^{-t}\sin x$. The parameters of scheme (17), (18) are given by the values of $\gamma = \tau^2$, $\alpha = 9\tau^2/7$, $\beta = -2\tau^2/7$.

The order of the convergence rate is determined by the following formulas: $p^h = \log_2(\|z\| / \|z_{1/2}\|)$, $p^\tau = \log_2(\|z\| / \|z_{1/2}\|)$, where $z_{1/2} = y_{h/2,\tau/2} - u_{h/2,\tau/2}$.

Table 1

Convergence rates in spatial and temporal variables

h	τ	Error	Order
0.01	0.01	0.00038	–
0.005	0.005	$1.93E-05$	4.26
0.0025	0.0025	$1.27E-06$	3.93
0.00125	0.00125	$8.09E-08$	3.98

6.2 Two-dimensional case

We choose the parameters of problem (1)–(3) in the following form: $l_1 = l_2 = \pi$, $T = 1$, $\mu = \theta = 1$, $\lambda = -1$. Then, instead of (1)–(3), we obtain:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u\right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u = 0,$$

$$(x,y,t) \in Q_T = \{(x,y): \ 0 < x < \pi, \ 0 < y < \pi, \ t \in (0,1]\},$$

$$u(x,y,t) = 0, \quad (x,y) \in \partial\bar{\Omega}, \quad t \in (0,1],$$

$$u(x,y,0) = \sin x \sin y, \quad x \in [0,\pi], \quad y \in [0,\pi].$$

The exact solutions is $u(x,y,t) = e^{-t}\sin x \sin y$. The parameters of scheme (17), (18) are given by the values of $\gamma = \tau^2$, $\alpha = 9\tau^2/7$, $\beta = -2\tau^2/7$.

Table 2

Convergence rates in spatial and temporal variables

h_1	h_2	τ	Error	Order
1/10	1/10	0.05	$3.78E-02$	–
1/20	1/20	0.05	$2.49E-03$	3.97
1/40	1/40	0.05	$1.61E-04$	3.98
1/80	1/80	0.05	$1.01E-05$	3.97

Tables 1 and 2 show the rate of convergence of the approximate solution to the exact solution when conditions (26), (31) are satisfied.

Conclusion

A high-accuracy numerical method was developed and investigated for solving the first boundary value problem for a pseudoparabolic equation. Based on the stability theory results for difference schemes, it was possible to obtain a priori estimates and, on their basis, prove the convergence of the constructed algorithm with a fourth-order rate in both variables. An algorithm for implementing the methods is given. Based on a computational experiment, test calculations were verified to confirm the theoretical data.

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Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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