There are actual the issues related to the mathematical modeling of thermophysical processes in the electric arc of high-current disconnecting apparatuses. The heat equation is a tool for describing the physics of processes in an arc. The equation takes into account the influence of thermal sources in the arc and the effect of shrinking the arc axial section in the cathode region into a contact spot. The diameter of the contact spot is several orders of magnitude smaller than the diameter of the section of the developed arc column. This fact makes it possible to consider the spot as a mathematical point. The domain of solution changes over time according to the law determined by the conditions of contact opening. At the initial moment of time, the contacts are in a closed state and there is no domain of solution to the problem. From the mathematical point of view, the problematical character of the problem under consideration is exactly in the presence of a mobile boundary and degeneration of the solution domain at the initial moment [19].

Problems in evolutionary domains similar to considered problems are very relevant not only for modeling the processes of electrical contact vehicles, but also in the adjacent field of plasma torch design. Similar problems arise when creating new technologies in metallurgy, crystal production, laser technology and other industries. Mathematical modeling of such processes allows to carry out an optimum choice of parameters and modes of operation of technological equipment and to achieve maximum economic and environmental benefits.

It should be emphasized that for parabolic equations in domains with a moving boundary the boundary value problems are fundamentally different from classical problems. Due to the dependence of the domain size on time, the methods of separation of variables and integral transformations are not applicable to this type of problems: remaining within the framework of classical methods of mathematical physics it is not possible to coordinate a solution of the heat equation with the movement of the boundary for a heat transfer domain.

The application of the method of thermal potentials allows

MINISTRY OF EDUCATION AND SCIENCE OF THE REPUBLIC OF KAZAKHSTAN
NAO KARAGANDA UNIVERSITY NAMED AFTER ACADEMICIAN E.A. BUKETOVA

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SINGULAR INTEGRAL EQUATIONS FOR HEAT CONDUCTION PROBLEMS

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Currently, the use of contact equipment is constantly increasing. The experimental study of thermal processes is often difficult due to their transience. Therefore, in some cases, only a mathematical model can serve as the basis for obtaining additional information about dynamics of thermal processes. The monograph is devoted to formulation and study of boundary value problems for the heat conduction equation in non-cylindrical domain, the domain degenerates to a point at the initial time. The study is based on reducing the formulated problems to the second-order Volterra singular integral equation, and solving this integral equation. The methods for solving equations (when the upper and lower limits of integration coincide, the operator is not equal to zero) are not specific to ordinary Volterra equations, so they were called singular Volterra integral equations of the second kind. The monograph will be interesting to scientists, doctoral students, undergraduates, students, university teachers and everyone involved in the field of heat equations and integral equations.

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Singularity of equation (1) is incompressibility of the kernel and is expressed in the fact that the corresponding nonhomogeneous equation cannot be solved by the method of successive approximations. Equations of this type are first considered in the paper of S.N. Kharin [4] when studying the thermal field of liquid contact bridges. In this paper, he constructed the asymptotic of integrals of the double layer potential type was studied and approximate solutions of some applied problems. Also, S.N. Kharin proposed and justified the method in which the solution of the integral equation is represented in the form of asymptotic expansion in the half-integer powers of the time variable.

He also proved the theorem on the permutability (in the sense of Dirihile formula) of singular integral operators included in the reduced equations. This allowed to reduce the system of integral equations to one equation relative to one of a unknown function. Assuming that the solution exists and is unique, S.N. Kharin found the asymptotic solution of the obtained integral equation at small values of time. The asymptotic solution is acceptable to obtaine engineering calculation formulas.

Furthermore, integral equation (1) is the subject in papers by Kavokin A.A., Otelbaev M.O., Omarov T.E., Djenaliyev M.T., Ramazanov M.I., Shpadi Yu.R., Gorodnichev S.P., Koilyshov U.K. and other authors. Note that the issues of unique solvability for such integral equations in certain weight spaces are studied in [5]–[8]: it is shown that the weight function of solutions depend on the weight functions of the right parts of the equation. The study of similar integral equations, in particular, the study of spectral issues of the corresponding integral operators was also carried out in [9]–[15].

It should also be noted that when the load line moves according to the law $x = t$ [16]–[18] boundary value problems for the spectral loaded parabolic equation are reduced to such singular integral equations as (1).

Identification of peculiarities of thermal fields in the intercontact space has not only a theoretical value, but also an applied value.

mentally impossible to determine the temperature field of the contact system and the dynamics of its change over time. Therefore, it is necessary to study the processes of heat and mass exchange between electrodes [2].

At the moment of opening the contacts, the power lines are not interrupted immediately, the contacts are heated to a melting point, and a liquid metal bridge is created between them. When the contacts are opened, the bridge is divided into two parts, i.e. the contact material is transferred from one electrode to another, this leads to bridge erosion. Eventually, the smooth surface of the contacts is destroyed, this means that their normal operation is disturbed. This phenomenon can lead to disaster.

There is a need to study the temperature field of the bursting liquid bridge.

Note that solving the boundary value problems for heat equation in degenerating domain leads to the need to study of a singular Volterra integral equation of the second kind:

$$\varphi(t) - \int\limits_0^t K(t,\,\tau)\,\varphi(\tau)\,d\tau = f(t), \qquad (1)$$

where

$$K(t,\,\tau) = \frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right) + \right.$$

$$\left. + \frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\}.$$

If the domain degenerates, the integral operators become singular, i.e. operator does not tends to zero as the upper limit tends to the lower limit.

E.I. Kim [3] developed constructive methods of solving the considered thermal problems for parabolic equations. These methods are based on the use of thermal potentials and reduction of initial boundary value problems to integral equations.

CONTENTS

DESIGNATIONS AND ABBREVIATIONS

$\mathbb{R}^n$ — n-dimensional space,

$\mathbb{R}^n = \{-\infty < x_k < \infty;\ k = 1,\ 2,\ \ldots,\ n\}$;

$\mathbb{C}$ – set of complex numbers;

G — Green's function;

t — time variable;

$u(x,\ t)$ — required function, solution of the equation (problem) of mathematical physics;

$\delta(x)$ — Dirac delta function;

$\operatorname{erf} z = \dfrac{2}{\sqrt{\pi}} \int\limits_0^z \exp\left(-\zeta^2\right) d\zeta$ — error function integral;

$\operatorname{erfc} z = \dfrac{2}{\sqrt{\pi}} \int\limits_z^\infty \exp\left(-\zeta^2\right) d\zeta$ — complementary error function integral;

$\mathbb{L}$ — differential operator;

$L_1(\Omega)$ — space (of classes) of functions summable in a domain Ω;

$L_\infty(\Omega)$ — space (of classes) of essentially bounded functions in a domain Ω;

$\operatorname{Ker}\{L\}$ — kernel of an operator L;

$\operatorname{Coker}\{L\}$ — cokernel of an operator L;

$\underset{z=z_0}{res}\ f(z)$ — residue of a function $f(z)$ at the point z_0.

PREFACE

This monograph paper is devoted to the formulation and study of boundary value problems for heat equation in non-cylindrical domain degenerating to a point at the initial time of the time. The paper also considers and studies a singular Volterra integral equations that arise in solving boundary value problems.

Current state of the topic and its relevance

There is necessity of studying boundary value problems for the equations of unsteady transfer in the domains with a mobile boundary. These problems have numerous practical applications in the theoretical study of the processes of energy or mass transfer. The processes associated with changes in aggregate state of a substance, in the theory of brittle fracture in study of surface crack growth, in the theory of dams, in soil mechanics, in the thermography of oil reservoirs, etc.

In [1] S.N. Kharin obtains approximate solutions to heat equation for a domain with a moving boundary; and the approximation error is estimated based on the principle of maximum. The most important result of [1] is to obtain an analytical solution to the two-phase Stefan problem with a flow boundary condition in the form of a series of Hartrean functions.

At present, the use of contact technique is constantly increasing. Therefore, the study of thermophysical processes occurring in contacts is a necessary condition for new advances in automation and instrumentation, welding technology, electrical equipment and in various devices, where the contact elements serve as one of the main segments. In technology the current use of ultra-high and ultra-low currents in many electrical devices leads to necessitates the study of new phenomena that have remained in the shadows for the conventional current range. The experimental study of thermal processes is often difficult due to their transience. Therefore, in some cases, only a mathematical model can serve as the basis for obtaining additional information about dynamics of thermal processes.

Because of the short duration of the thermal process, it is experi-

The classes of uniqueness are

$$u(x,t) \le C \cdot \gamma_\varepsilon(x,t),\ \ \gamma_\varepsilon(x,t) \ge 2,\ \ \varepsilon_i \ge 0,\ \ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \ne 0,$$

where

$$\gamma_\varepsilon = \exp\left\{\frac{(1-\varepsilon_1)(t-x)}{a^2}\right\} \times$$

$$\times \max\left[\left(\frac{\sqrt{t}}{t-x}\right)^{1-\varepsilon_2} \exp\left\{-\left(\frac{2t-x}{2a}\right)^2 \cdot \frac{1}{t} - \varepsilon_3 t\right\};\right.$$

$$\left. 1 + \exp\left\{-\frac{(1-\varepsilon_1)(t-x)}{a^2}\right\}\right], \qquad \{x,\, t\} \in G.$$

Direct verification of the obtained solution (10) of homogeneous singular Volterra integral equation (6) is given in [30].

Remark 1.3

The solution of the second boundary value problem for the heat equation in a degenerating domain is given in [32].

Theorem 1.3

In the domain of $G = \{(x;\ t):\quad t > 0,\quad 0 < x < t\}$ *the second homogeneous boundary value problem for the heat equation*

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=t} = 0$$

has a solution

$$u(x,t) = \frac{C_1}{2a\sqrt{\pi}} \int\limits_0^t \frac{1}{\sqrt{t-\tau}}\ \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\ \nu(\tau)\, d\tau +$$

$$+\frac{C_1}{2a\sqrt{\pi}} \int\limits_0^t \frac{1}{\sqrt{t-\tau}}\ \exp\left[-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right] \times$$

to reduce boundary value problem to a Volterra equation of the second kind. It is established that if in the boundary value problem the variable domain does not degenerate to a point at the initial moment of time, the equivalent integral equation is solved by the method of successive approximations. Otherwise, the integral equation of the boundary value problem can have additional solutions, and the implementation of the Picard method is associated with serious mathematical difficulties.

Note that the study of boundary value problems for the heat equation in non-cylindrical domains was considered in [20]—-[24]. In these papers solutions of the boundary value problems are built in a domain with a uniformly moving boundary: $G = \{x,\, t|\, t > 0, 0 < x < a + kt, a \ne 0\}$.

Therefore, the issue about the study of boundary value problems in a domain with degeneracy at the initial time is not fully theoretically studied and, accordingly, is relevant.

The "starting points"for our studies are: in degenerating domains boundary value problems of heat conduction are reducing to the integral equations with a variable upper limit of integration, and the dependence of the solution to integral equations on the peculiarities of integral operators. The methods of solving such equations (when the upper and lower limits of integration coincide, the operator is not equal to zero) are not specific for the usual Volterra equations, so they were called *singular integral equations of the Volterra type of the second kind.*

The peculiarity of the problems under study in this Monograph is the degeneration of the domain at the initial time. As a result, homogeneous boundary value problems have a nontrivial solution in certain classes.

The main purpose of the study is to formulate and solve boundary value problems for the heat equation in domains degenerating at the initial time; to solve singular Volterra integral equations of the second kind; to study issues of their solvability.

Objectives of the study:

- give a statement of direct and adjoint boundary value problems for the heat equation in degenerating angular domain and describe the spaces of solution and of given functions;

- solve singular Volterra integral equations of the second kind by the Karleman-Vecua regularization method;
- establish an integral representation of the solution to the boundary value problem and show that the boundary value problem is Noether;
- find the multiplicity of eigenvalues and eigenfunctions for an integral operator, show that the multiplicity depends on the value of the spectral parameter;
- determine the classes of uniqueness solution for the boundary value problems under study.

The authors express the deep gratitude and appreciation to Doctors of Physical and Mathematical Sciences, Professors M.T. Djenaliyev, M.I. Ramazanov for attention and valuable comments.

Summary of the work

The first section is devoted to the study of the first boundary value problem for the heat equation in a degenerating angular domain: statement of the problem, its reduction to a singular Volterra integral equation of the second kind by using of thermal potentials, solving the integral equation by method of regularization, determination of the solution uniqueness classes and formulation of the main result of the study (the boundary value problem is Noether). Also, in the section, the second boundary value problem for the heat equation in the degenerating domain is set and the result of the study is formulated.

It turned out that the issues under consideration are closely related to the problem of establishing the solution uniqueness classes from [52]–[55]. This problem is actively continued, for example, in [56]–[63].

This section provides a brief overview of some papers on the solution uniqueness classes for parabolic equations.

Statement of the problem

Thus, we consider the first boundary value problem of heat conduction in a degenerating domain (a domain with a moving

has a non-zero solution that is determined by the formula:

$$u(x,t) = \frac{1}{4a^3\sqrt{\pi}}\int_0^t \frac{x}{(t-\tau)^{\frac{3}{2}}}\ \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\nu(\tau)\,d\tau + $$

$$+\frac{1}{4a^3\sqrt{\pi}}\int_0^t \frac{x-\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau, \quad (12)$$

where

$$\nu(t) = \frac{1}{2a\sqrt{\pi}}\int_0^t \frac{\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau,$$

and the function $\varphi(t)$ is determined by formula (11).

To set the class of non-trivial solution $u(x,t)$ (12) the following estimate has been established:

$$u(x,t) \le C\,\gamma(x,t)$$

here

$$\gamma(x,t) = \max\left[\frac{\sqrt{t}}{t-x}\exp\left\{-\frac{x^2}{4a^2t}\right\};\ 1+\exp\left\{\frac{t-x}{a^2}\right\}\right],$$

$$\gamma(x,t)\ge 2,\ \{x,t\}\in G. \quad (14)$$

The following Proposition is true.

Proposition 1.3

For the problem L (2) – (3) in class (4)

$$\dim\{\mathrm{Ker}\,\{L\}\} = 1.$$

The classes of uniqueness for boundary value problem (2) – (3) are defined by the following Proposition.

Proposition 1.4

to be equal to 1):

$$\varphi(t)=\frac{1}{\sqrt{t}}\,\exp\left(-\frac{t}{4a^2}\right)+\frac{\sqrt{\pi}}{2a}\mathrm{erf}\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{\pi}}{2a}. \qquad (11)$$

So, the theorem is established:

Theorem 1.1

Function (11) is a solution of a singular integral equation

$$\varphi(t)-\int\limits_0^t K(t,\,\tau)\,\varphi(\tau)\,d\tau=0,$$

where

$$K(t,\,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right)+\right.$$

$$\left.+\frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\},$$

in the weight class of functions $\sqrt{t}\,\exp\left(-\dfrac{t}{4a^2}\right)\,\varphi(t)\in L_\infty(0,\,\infty)$.

Taking into account that the required function $u(x,t)$ is represented as a sum of potentials of a double layer with densities $\nu\,(t)$ and $\varphi(t)$, *solution to boundary value problem (2) – (3)* is obtained in an explicit form according to the following theorem.

Theorem 1.2

In the domain $G=\{(x;\,t):\quad t>0,\quad 0<x<t\}$ *the homogeneous boundary value problem*

$$\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}=0,$$

$$u(x,\,t)|_{x=0}=0,\quad u(x,\,t)|_{x=t}=0,$$

boundary):

In the domain of $G=\{(x;\,t):\quad t>0,\quad 0<x<t\}$ *to find a solution to the heat conduction equation*

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}, \qquad (2)$$

the following boundary conditions must be satisfied:

$$u(x,\,t)|_{x=0}=0,\quad u(x,\,t)|_{x=t}=0, \qquad (3)$$

where $u(x,t)$ belongs to the class:

$$u(x,t)\le C\,\exp\left\{\frac{t-x}{a^2}\right\}\max\left[\frac{\sqrt{t}}{t-x}\exp\left\{-\left(\frac{2t-x}{2a}\right)^2\frac{1}{t}\right\};\right.$$

$$\left.1+\exp\left\{-\frac{t-x}{a^2}\right\}\right],\qquad (x,t)\in G, \quad (4)$$

It is required to show that boundary value problem (2) – (3) has only one non-trivial solution in class (4) up to constant factor .

Reducing the problem to the integral equation

A solution of problem (2) – (3) is looking for as a sum of thermal potentials of the double layer with densities $\nu\,(t)$ and $\varphi(t)$ [64].

As a result, problem (2) – (3) is reduced to the study of the following integral equation:

$$\varphi(t)-\int\limits_0^t K(t,\,\tau)\,\varphi(\tau)\,d\tau=0, \qquad (5)$$

where

$$K(t,\,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right)+\right.$$

$$+\frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\Bigg\}.$$

A singularity of the equation under study is a property of the kernel $K(t,\tau)$:

$$\lim_{t\to 0}\int_0^t K(t,\tau)\,d\tau = 1, \qquad \lim_{t\to\infty}\int_0^t K(t,\tau)\,d\tau = 1,$$

and the singularity is expressed in the fact that the corresponding homogeneous equation cannot be solved by the method of successive approximations.

Due to [65], to study equation (5) it is enough to find a solution of the "simplified"equation:

$$\varphi(t) - \int_0^t k(t,\tau)\,\varphi(\tau)\,d\tau = 0, \qquad (6)$$

where

$$k(t,\tau) = \frac{1}{2a\sqrt{\pi}}\left[\frac{2t}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{t\tau}{a^2(t-\tau)}\right) + \right.$$

$$\left. +\frac{1}{(t-\tau)^{\frac{1}{2}}}\left(1-\exp\left(-\frac{t\tau}{a^2(t-\tau)}\right)\right)\right].$$

Equation (6) is studied by the Karleman-Vekua regularization method [66], [67]. For this purpose, its characteristic part is distinguished:

$$\varphi(t) - \int_0^t k_o(t,\tau)\,\varphi(\tau)\,d\tau = f_1(t), \qquad (7)$$

where

$$k_o(t,\tau) = \frac{t}{a\sqrt{\pi}\,(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\},$$

$$f_1(t) = \int_0^t k_h(t,\tau)\;\varphi(\tau)\,d\tau, \qquad (8)$$

where

$$k_h(t,\tau) = \frac{1}{2a\sqrt{\pi}\,(t-\tau)^{\frac{1}{2}}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right).$$

The following proposition is valid.

Proposition 1.1

The following limit relations hold

$$\lim_{t\to 0}\int_0^t k_o(t,\tau)\,d\tau = 1; \quad \lim_{t\to 0}\int_0^t k_h(t,\tau)\,d\tau = 0.$$

By virtue of the previous Proposition we have:

Proposition 1.2

Equation (7) is a characteristic equation for (6).

Using the Carleman-Vecua regularization method, the lemma was proved

Lemma 1.1

Integral equation (6) is equivalent to the Abel equation

$$\varphi(t) - \frac{1}{2a\sqrt{\pi}}\int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}}d\tau = \frac{C}{\sqrt{t}}. \qquad (9)$$

The solution of the Abel equation (9), i.e. the solution of the "simplified" equations (6), has the form

$$\varphi(t) = C\left[\frac{1}{\sqrt{t}} + \frac{\sqrt{\pi}}{2a}\exp\left(\frac{t}{4a^2}\right)\operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right) + \frac{\sqrt{\pi}}{2a}\exp\left(\frac{t}{4a^2}\right)\right]. \qquad (10)$$

The solution of initial equation (5) is obtained after multiplying equality (10) by $\exp\left(-\frac{t}{4a^2}\right)$ (for convenience, *const* C is assumed

Along with the operator L consider the operator

$$M(\upsilon) = a^2 \upsilon_{xx} + \upsilon_t. \qquad (1.1.3)$$

We introduce *a fundamental solution to the heat equation*

$$G_0(x,\, t,\, \xi,\, \tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}} \exp\left(-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right). \qquad (1.1.4)$$

It is known that *any solution to the heat equation can be presented as* [64]:

$$u(x,\, t) = \int\limits_{AB} u\, G_0\, d\xi - \int\limits_{AP} u\, G_0\, d\xi + \int\limits_{BQ} u\, G_0\, d\xi + \qquad (1.1.5)$$

$$+a^2 \int\limits_{BQ+PA} \left(G_0 \frac{\partial u}{\partial \xi} - u \frac{\partial G_0}{\partial \xi} \right) d\tau.$$

The following integrals for interior points of the domain $PABQ$ satisfy heat equation

$$V(x,\, t) = a^2 \int\limits_0^t G_0\, \nu(\tau)\, d\tau =$$

$$= \frac{a}{2\sqrt{\pi}} \int\limits_0^t \frac{1}{\sqrt{t-\tau}} \exp\left(-\frac{(x-\chi_1(\tau))^2}{4a^2(t-\tau)}\right) \nu(\tau)\, d\tau, \qquad (1.1.6)$$

$$W(x,\, t) = 2a^2 \int\limits_0^t \frac{\partial G_0}{\partial \xi}\, \mu(\tau)\, d\tau =$$

$$= \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{x-\chi_1(\tau)}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{(x-\chi_1(\tau))^2}{4a^2(t-\tau)}\right) \mu(\tau)\, d\tau \qquad (1.1.7)$$

$$\times \left[\frac{1}{\sqrt{\tau}} \exp\left(-\frac{\tau}{4a^2}\right) + \frac{\sqrt{\pi}}{2a} \mathrm{erf}\left(\frac{\sqrt{\tau}}{2a}\right) + \frac{\sqrt{\pi}}{2a} \right] d\tau + C_2,$$

where

$$\nu(t) = \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{\tau}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\} \times$$

$$\times \left\{ \frac{1}{\sqrt{\tau}} \exp\left(-\frac{\tau}{4a^2}\right) + \frac{\sqrt{\pi}}{2a} \mathrm{erf}\left(\frac{\sqrt{\tau}}{2a}\right) + \frac{\sqrt{\pi}}{2a} \right\} d\tau.$$

Next, we consider *an adjoint boundary value problem of heat conduction* L^*.

In the domain of $G = \{(x;\, t):\quad t > 0,\quad 0 < x < t\}$ *to find a solution to the adjoint boundary value problem for the equation*

$$-\frac{\partial u^*}{\partial t} = a^2 \frac{\partial^2 u^*}{\partial x^2}, \qquad (15)$$

with boundary conditions:

$$u^*(x,\, t)|_{t=\infty} = 0,\;\; u^*(x,\, t)|_{x=0} = 0,\;\; u^*(x,\, t)|_{x=t} = 0. \qquad (16)$$

Boundary value problem (15)–(16) is reduced to the study of the Volterra equation of the second kind:

$$\varphi^*(t) - \int\limits_t^\infty K^*(t,\tau) \varphi^*(\tau)\, d\tau = 0,\;\; t > 0, \qquad (17)$$

where

$$K^*(t,\, \tau) = \frac{1}{2\, a\sqrt{\pi}} \left[\frac{\tau + t}{(\tau - t)^{\frac{3}{2}}} \exp\left\{-\frac{(\tau+t)^2}{4a^2(\tau-t)}\right\} + \right.$$

$$\left. + \frac{1}{(\tau-t)^{\frac{1}{2}}} \exp\left\{-\frac{\tau - t}{4a^2}\right\} \right], \qquad (18)$$

and singularity of the obtained equation is the properties of the kernel:

$$\lim_{t\to+\infty}\int\limits_t^\infty K^*(t,\,\tau)\,d\tau=1,\qquad \lim_{t\to 0}\int\limits_t^\infty K^*(t,\,\tau)\,d\tau=3$$

.

The study of equation (17) is reduced to the study of the integral equation:

$$\psi^*(t)-\int\limits_t^\infty k^*(t,\tau)\psi^*(\tau)\,d\tau=0,\ \ t>0, \tag{19}$$

where

$$k^*(t,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{2\tau}{(\tau-t)^{3/2}}\exp\Big\{-\frac{\tau t}{a^2(\tau-t)}\Big\}+\right.$$

$$\left.+\frac{1}{\sqrt{\tau-t}}\Big(1-\exp\Big\{-\frac{\tau t}{a^2(\tau-t)}\Big\}\Big)\right\},$$

$$\psi^*(t)=\exp\Big\{-\frac{\tau}{4a^2}\Big\}\varphi^*(t),\quad K^*(t,\tau)=k^*(t,\tau)\exp\Big\{-\frac{\tau-t}{4a^2}\Big\}.$$

The solution of homogeneous equation (19) is the function

$$\psi^*(t)=\frac{C}{a\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\Big(-\frac{(2n+1)^2}{4a^2}t\Big). \tag{20}$$

Direct verification of the obtained solution (20) of homogeneous singular Volterra equation of the second kind (19) is done in [30].

The solution $u^*(x,t)$ of adjoint boundary value problem (15)–(16) is determined by the function $\psi^*(t)$ (20) However, this solution $u^*(x,t)$ does not belong to the class which is conjugate to class (4) of solutions to the direct boundary value problem:

$$\exp\left\{-\frac{t-x}{a^2}\right\}\cdot[\gamma(x,t)]^{-1}\cdot u(x,t)\in L_\infty(G),$$

1 Boundary value problems of heat conduction in degenerating domain

1.1 Method of thermal potentials and uniqueness classes of solutions to boundary value problems

1.1.1 Application of thermal potentials to solving boundary value problems

The Fourier method and the method of integral transforms allow us one to obtain an explicit representation of the solution to boundary value problems if domains are domains of the simplest form. The thermal potential method allows us reduce boundary value problems to integral equations. Therefore, the determination of thermal potentials and investigation of their properties are important prerequisites for solving the boundary value problems.

Consider a boundary value problem for the heat equation with one spatial variable

$$L(M)=a^2u_{xx}-u_t=0 \tag{1.1.1}$$

Following [64], consider the domain $BAEF$ (Fig. 1.1), limited

Figure 1.1 – Domain with moving boundaries

by characteristics AB и EF $(t=const)$ and curves defined by the equations

$$x=\chi_1(t)\qquad \text{for } AE.$$

and

$$x=\chi_2(t)\qquad \text{for } BF.$$

The first boundary problem in this domain is to determine the solution to heat equation (1.1.1), satisfying initial and boundary conditions

$$\begin{cases} u(x,\,t)|_{t=AB}=\varphi(x),\\ u(x,\,t)|_{x=\chi_1(t)}=\mu_1(t),\qquad u(x,\,t)|_{x=\chi_2(t)}=\mu_2(t).\end{cases} \tag{1.1.2}$$

We introduce a notation.

$$\tilde{f}_2(t) = \exp\left\{t/(4a^2)\right\} f(t) + \lambda \int\limits_0^t r(t,\tau) \exp\left\{\tau/(4a^2)\right\} f(\tau)\, d\tau. \tag{24}$$

Theorem 2.1
Homogeneous integral equation (21) *is solvable in the class*

$$\sqrt{t}\,\varphi(t) \in L_\infty(0,\infty)$$

for each right part

$$\sqrt{t}\, f(t) \in L_\infty(0;\infty)$$

and for each

$$|\lambda| > \exp(|\arg\lambda|), \quad \arg\lambda \in [-\pi;\,\pi]$$

Corresponding homogeneous equation has (N_1+N_2+1) *eigenfunctions*

$$\varphi_k(t) = \frac{1}{\sqrt{t}} \exp\left(\frac{p_k}{t} - \frac{t}{4a^2}\right) +$$
$$+\frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\lambda^2-1}{4a^2}t - \frac{\lambda\sqrt{-p_k}}{a}\right) \cdot \operatorname{erfc}\left(\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}\right),$$

where the numbers p_k *are determined by equality (23) and the general solution of integral equation (21) can be written as*

$$\varphi(t) = F(t) + \frac{\lambda^2}{4a^2}\int\limits_0^t \exp\left(\frac{\lambda^2(t-\tau)}{4a^2}\right) F(t)\, d\tau + \sum_{k=-N_1}^{N_2} C_k\varphi_k(t),$$

where

$$F(t) = \tilde{f}_2(t) - \frac{\lambda}{2a\sqrt{\pi}}\int\limits_0^t \frac{\tilde{f}_2(\tau)}{\sqrt{t-\tau}}\, d\tau,$$

a function $\sqrt{t}\cdot\exp\{-t/(4a^2)\}\cdot\tilde{f}_2(t) \in L_\infty(0,\infty)$ *defined by formula* (24).

where

$$\gamma(x,t) = \max\left[\frac{\sqrt{t}}{t-x}\exp\left\{-\left(\frac{2t-x}{2a}\right)^2\cdot\frac{1}{t}\right\};\ 1+\exp\left\{-\frac{t-x}{a^2}\right\}\right],$$

$$\{x,\,t\} \in G,$$

i.e.

$$\exp\left\{\frac{t-x}{a^2}\right\}\cdot\gamma(x,t)\cdot u^*(x,t) \notin L_1(G).$$

The following Proposition has been established.

Proposition 1.5
For the problem L^* (2) – (3) *in a class conjugated with* (4),

$$\dim\{\mathrm{Ker}\,\{L^*\}\} = 0.$$

Thus, from Propositions 1.3 and 1.5 it follows **main result** of the section:

Theorem 1.4
The problem L (2) – (3) *is Noetherian, i.e.*

$$\mathrm{ind}\,\{L\} = \dim\{\mathrm{Ker}\,\{L\}\} - \dim\{\mathrm{Coker}\,\{L\}\} = 1.$$

In **the second section** integral equation (1) is investigated with spectral parameter λ, $|\lambda| > 1$.

It is shown that the corresponding homogeneous equation at

$$|\lambda| > \exp(|\arg\lambda|, \quad \arg\lambda \in [-\pi;\,\pi])$$

has continuous spectrum, and the multiplicity of characteristic numbers increases with the increase of $|\lambda|$.

The eigenfunctions of the equation are found in an explicit form.

Solving model problems for parabolic equations in domains with a moving boundary the following singular integral equations arise

$$\varphi(t) - \lambda \int_0^t K(t,\tau)\varphi(\tau)\,d\tau = f(t), \quad t > 0, \tag{21}$$

where

$$K(t,\tau) = \frac{1}{2a\sqrt{\pi}}\left\{ \frac{t+\tau}{(t-\tau)^{3/2}} \exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right) + \right.$$

$$\left. + \frac{1}{(t-\tau)^{1/2}} \exp\left(-\frac{t-\tau}{4a^2}\right) \right\}.$$

From the property

$$\lim_{t\to 0} \int_0^t K(t,\,\tau)\,d\tau = 1, \;\; \lim_{t\to+\infty} \int_0^t K(t,\,\tau)\,d\tau = 1$$

it follows that in (21) the norm of the integral operator, acting in the class of essentially limited functions, is equal to unity.

Problem

Find in the class $\sqrt{t}\,\varphi(t) \in L_\infty(0,\infty)$ a solution $\varphi(t)$ of integral equation (21) for any given function $\sqrt{t}\,f(t) \in L_\infty(0,\infty)$ and any given complex parameter

$$\lambda \in \mathbb{C} \diagdown \{|\lambda| \le \exp(|\arg\lambda|)\}, \arg\lambda \in [-\pi;\pi].$$

In the study of integral equation (21), the method of regularization of Karleman-Vekua is used [66], [67].

The solution of the homogeneous equation corresponding to initial equation (21) is given by

$$\varphi(t) = \sum_{k=-N_1}^{N_2} C_k \left\{ \frac{1}{\sqrt{t}} \exp\left(\frac{p_k}{t} - \frac{t}{4a^2}\right) + \right.$$

$$\left. + \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\lambda^2-1}{4a^2}t - \frac{\lambda\sqrt{-p_k}}{a}\right) \cdot \operatorname{erfc}\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}\right) \right\}, \tag{22}$$

where for $k = 0, 1, 2, \ldots$

$$-p_k = \frac{a^2}{4}\left(\ln^2|\lambda| - (\arg\lambda + 2k\pi)^2\right) + i\frac{a^2}{4}\ln|\lambda|^2\,(\arg\lambda + 2k\pi)\,; \tag{23}$$

$$N_1 = \left[\frac{\ln|\lambda| + \arg\lambda}{2\pi}\right], \quad N_2 = \left[\frac{\ln|\lambda| - \arg\lambda}{2\pi}\right],$$

$N_1 + N_2 + 1$ is the number of eigenfunctions and $[a]$ is an integral part of the a. Obviously, the more $|\lambda|$ the more of eigenfunctions.

The function $\sqrt{t}\cdot\varphi(t)$ is belong to $L_\infty(0,\infty)$. Indeed, the first summands of sum (22)

$$\exp\left(\frac{p_k}{t} - \frac{t}{4a^2}\right) \in L_\infty(0,\infty).$$

For the second term of sum (22), the following containment is valid:

$$\sqrt{t}\cdot\frac{\lambda\sqrt{\pi}}{2a}\exp\left(\frac{\lambda^2-1}{4a^2}t - \frac{\lambda\sqrt{-p_k}}{a}\right)\cdot\operatorname{erfc}\left(\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}\right) \in L_\infty(0,\infty).$$

It is enough to take into account that the numbers p_k, $k \in [-N_1, N_2]$, are roots of an equation

$$1 - \lambda\exp\left\{-\frac{2}{a}\sqrt{-p}\right\} = 0,$$

for each fixed complex spectral parameter $\lambda \in \mathcal{C}$, and use an asymptotics of the function $\operatorname{erfc}(z)$ for large values z ([68], p.890, 8.254^8; [64], p.708). Obviously, there is a limit relation

$$z = \frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}} \to \{\mathrm{IP}\} \quad \text{by} \quad t\to\infty \quad \text{and for any} \quad |\lambda| > 1.$$

$$+\frac{1}{4a^3\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\exp\left(-\frac{t-\tau}{4a^2}\right)\times$$

$$\times\varphi(\tau)\,d\tau=$$

$$-\frac{\varphi(t)}{2a^2}+\frac{1}{4a^3\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\exp\left(-\frac{t-\tau}{4a^2}\right)\varphi(\tau)\,d\tau.$$

As a result, we obtain the following system of integral equations relatively unknown densities $\nu(t)$ and $\varphi(t)$ [29]:

$$\begin{cases} 0=\dfrac{\nu(t)}{2a^2}-\dfrac{1}{4\,a^3\sqrt{\pi}}\int\limits_0^t \dfrac{\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\dfrac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau, \\ 0=-\dfrac{\varphi(t)}{2a^2}+\dfrac{1}{4\,a^3\sqrt{\pi}}\int\limits_0^t \dfrac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left\{-\dfrac{t-\tau}{4a^2}\right\}\varphi(\tau)\,d\tau+ \\ \quad +\dfrac{1}{4a^3\sqrt{\pi}}\int\limits_0^t \dfrac{t}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\dfrac{t^2}{4a^2(t-\tau)}\right)\nu(\tau)\,d\tau. \end{cases} \tag{5}$$

Excluding from the system (5) $\nu(t)$, we find:

$$\nu(t)=\frac{1}{2a\sqrt{\pi}}\int_0^t \frac{\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau, \tag{6}$$

$$0=-\frac{\varphi(t)}{2a^2}+$$

$$+\frac{1}{4\,a^3\sqrt{\pi}}\int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\varphi(\tau)\,d\tau+$$

(Here: $\xi=\chi_1(\tau)$ is a moving lateral boundary.)

Functions (1.1.6) and (1.1.7) are called *thermal potentials of the simple and double layer, respectively* ([64]).

We consider the first boundary-value problem for a semi-bounded domain $x\ge\chi_1(t)$: *find a solution to the equation*

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}\qquad \text{at}\quad x\ge\chi_1(t),\quad t\ge t_0$$

satisfying conditions:

$$\begin{cases} u(x,\,t)|_{t=t_0}=\varphi(x), & x\ge\chi_1(t_0); \\ u(x,\,t)|_{x=\chi_1(t)}=\mu(t), & t\ge t_0. \end{cases}$$

Without loss of generality, we assume that $\varphi(x)=0$.

We present the solution in the form

$$u(x,\,t)=\frac{1}{2a^2}W(x,\,t)=\int_{t_0}^t \frac{\partial G_0}{\partial\xi}(x,\,t,\,\chi_1(\tau),\,\tau)\tilde{\mu}(\tau)\,d\tau=$$

$$=\frac{1}{4a^3\sqrt{\pi}}\int_{t_0}^t \frac{x-\chi_1(\tau)}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(x_0-\chi_1(\tau))^2}{4a^2(t-\tau)}\right)\tilde{\mu}(\tau)\,d\tau.$$

This function satisfies the equation for $x>\chi_1(t)$, is bounded at infinity and has zero initial value for any choice $\tilde{\mu}(t)$. When $x=\chi_1(t)$ $u(x,\,t)$ is discontinuous and her ultimate value at $x=\chi_1(t)+0$ should be equal $\mu(t)$

$$u(x,\,t)\,|_{x=\chi_1(t)+0}=\frac{\tilde{\mu}(t)}{2a^2}+$$

$$+\frac{1}{4a^3\sqrt{\pi}}\int_{t_0}^t \frac{\chi_1(t)-\chi_1(\tau)}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(\chi_1(t)-\chi_1(\tau))^2}{4a^2(t-\tau)}\right)\tilde{\mu}(\tau)\,d\tau=\mu(t),$$

$$W\,|_{x=\chi_1(t)+0}=W\,|_{x=\chi_1(t)}+\tilde{\mu}(t)$$

and

$$u(x,\,t)=\frac{1}{2a^2}W(x,\,t).$$

The relation

$$\frac{\tilde{\mu}(t)}{2a^2}+\frac{1}{4a^3\sqrt{\pi}}\int\limits_{t_0}^{t}\frac{\chi_1(t)-\chi_1(\tau)}{(t-\tau)^{\frac{3}{2}}}$$

$$\exp\left(-\frac{\left(\chi_1(t)-\chi_1(\tau)\right)^2}{4a^2(t-\tau)}\right)\tilde{\mu}(\tau)\,d\tau=\mu(t)$$

is an integral equation of Volterra type of the second kind for finding a function $\tilde{\mu}(t)$ that determines the required solution $u(x,\,t)$.

In the case of a fixed boundary of the domain: $\chi_1(t)=x_0$ integral is equal to zero and

$$u(x,\,t)=\frac{1}{2a\sqrt{\pi}}\int\limits_{t_0}^{t}\frac{x-x_0}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{\left(x-x_0\right)^2}{4a^2(t-\tau)}\right)\mu(\tau)\,d\tau.$$

1.1.2 About uniqueness classes

Cosidered issues are closely related to the problem of establishing uniqueness classes in [52]–[55]. This problem has an active continuation, for example, in works[56]–[63].

Let us give a brief overview of some works on uniqueness classes for parabolic equations.

In the domain of $Q=\{\mathbb{R}^n\times(0,T)\}$ for the boundary value problem

$$u_t(x,t)-\Delta u(x,t)=0,\;\;u(x,t)_{|t=0}=0$$

the following classes of uniqueness are established:

$u(x,t)\leq C\cdot\exp\{k|x|^2(\ln|x|)^\alpha\},\;\;\alpha\in[0,1],$ (Holmgren E. [52]).

$$=\frac{\nu\left(t\right)}{a^2\sqrt{\pi}}\int\limits_{0}^{+\infty}\exp\left(-z^2\right)\,dz=\frac{\nu\left(t\right)}{2a^2}.$$

When we use the second boundary condition then in the second integral operator in representation (4) the kernel has such singularity.

After the introduction of the replacement

$$z=\frac{x-t}{2a\sqrt{t-\tau}};\quad t-\tau=\frac{(x-t)^2}{4a^2z^2};$$

$$\tau=t-\frac{(x-t)^2}{4a^2z^2};\quad d\tau=\frac{(x-t)^2}{2a^2z^3}dz$$

and representation $x-\tau=(x-t)-(t-\tau)$ for the second term in (4) we have:

$$u_2(x,\,t)|_{x\to t-0}=$$

$$=\frac{1}{a^2\sqrt{\pi}}\lim_{x\to t-0}\int\limits_{\frac{x-t}{2a\sqrt{t}}}^{-\infty}\exp\left(-z^2-\frac{x-t}{2a^2}-\frac{(x-t)^2}{16a^4z^2}\right)\times$$

$$\times\varphi\left(t-\frac{(x-t)^2}{4a^2z^2}\right)\,dz$$

$$+\frac{1}{4a^3\sqrt{\pi}}\int\limits_{0}^{t}\frac{1}{a^2\sqrt{t-\tau}}\exp\left(-\frac{t-\tau}{4a^2}\right)\varphi(\tau)\,d\tau=$$

$$=-\frac{\varphi\left(t\right)}{a^2\sqrt{\pi}}\int\limits_{-\infty}^{0}\exp\left(-z^2\right)\,dz$$

form of a sum of thermal double layer potentials [64]:

$$u(x,t)=\frac{1}{4a^3\sqrt{\pi}}\int\limits_0^t \frac{x}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\,\nu(\tau)\,d\tau+$$
$$+\frac{1}{4a^3\sqrt{\pi}}\int\limits_0^t \frac{x-\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\,\varphi(\tau)\,d\tau. \tag{4}$$

Is known [64, p. 471] that function (4) satisfies equation (1) at any $\nu(t)$ and $\varphi(t)$. Note that every solution to heat equation (1) can be represented by (4). This assertion is justified, for example, in [64, p. 476–480].

We use conditions (2).

Note that as x tend to 0 on the right the kernel in the first integral operator in representation (4) has a Cauchy type singularity $\dfrac{1}{t-\tau}$, since $\sqrt{z}e^{-z}\le const,\quad z\ge 0$, in particular when $z=\dfrac{x^2}{4a^2(t-\tau)},\quad 0<x<t.$

To apply the first condition, we introduce the replacement:

$$z=\frac{x}{2a\sqrt{t-\tau}};\;\; t-\tau=\frac{x^2}{4a^2z^2};\;\; \tau=t-\frac{x^2}{4a^2z^2};\;\; d\tau=\frac{x^2}{2a^2z^3}dz.$$

Then for the first term in (4) we get:

$$u_1(x,\;t)|_{x\to 0+0}=$$
$$=\frac{1}{a^2\sqrt{\pi}}\lim_{x\to 0+0}\int\limits_{\frac{x}{2a\sqrt{t}}}^{+\infty}\exp\left(-z^2\right)\;\nu\left(t-\frac{x^2}{4a^2z^2}\right)\,dz=$$

$u(x,t)\le C\cdot\exp\{a_T|x|^2\}$, (Tikhonov A.N. [53]).

$u(x,t)\;\le\;C\cdot\exp\{|x|h(|x|)\},\;\displaystyle\int\limits_1^{\infty}\frac{dr}{h(r)}=\infty,$ (Täcklind S. [54]).

For the boundary value problem in $Q=\{\mathbb{R}^n\times(0,T)\}$:

$$u_t(x,t)+Au(x,t)=0,\;\; u_{|t=0}=0,$$

where A is a linear elliptic operator of orders $2p$, the following classes of uniqueness are established:

$u(x,t)\le C\cdot\exp\left\{k|x|^{\frac{2p}{2p-1}}\right\}$, (Ladyzhenskaya O.A. [56] for one equation with coefficients, depending only on t).

$u(x,t)\le C\cdot\exp\left\{k|x|^{\frac{2p}{2p-1}}\right\}$, (Oleynik O.A. [57] for systems of parabolic equations with coefficients depending on x and t; [58] for the Cauchy-Neumann problem in an unbounded domain, arbitrarily "tapering" at infinity).

We also note works of Oleinik O.A. and Radkevich E.V.[59], Gagnidze A.G. [60], Kozhevnikova L.M. [61]–[63], and others, devoted to the establishment of classes of uniqueness for parabolic equations and systems.

V.P. Mikhailov's example on the existence of a nontrivial solution for the homogeneous Dirichlet problem in the degenerating domain.

Let $Q\subset\mathbb{R}^2$ be domain bounded by a closed curve Γ: $x^2=-2t\,\ln t$, passing through the origin of coordinates and the point $\{x=0,\;t=1\}$ and symmetrical with respect to the axis $0\,t$.

The boundary value problem [55]

$$u_t(x,t)-u_{xx}(x,t)=0,\;\; u_{|\Gamma}=0,$$

has a non-trivial solution

$$u(x,t) = t^{-1/2} \exp\left\{-\frac{x^2}{4t}\right\} - 1 \in L_2(Q).$$

Note that although $u_t,\ u_{xx} \notin L_2(Q)$, the following inclusions hold:

$$t^2 u_t(x,t),\ t^2 u_{xx}(x,t) \in L_2(Q).$$

1.2 Reducing the boundary value problems to a singular Volterra integral equation

In this section, we consider the problems of heat conduction in an infinite angle, namely in the domain $G = \{(x,\, t) |\, t > 0,\, 0 < x < t\}$ that degenerates to a point at the initial moment of time. The main goal: find classes of solutions to direct and conjugate boundary value problems, show the existence of a nontrivial solution to the homogeneous direct boundary value problem, and also define classes uniqueness.

After the statement of the direct boundary value problem with the introduction of the class of solutions, we reduce this problem to an integral Volterra equation of the second kind. After transforming this integral equation it is reduced to the Abel equation. The solution of the Abel equation allows us obtain an explicit integral representation of the solution to the boundary value problem. Next, for the obtained solution we establish estimates and weight class. It is proved that the solution of the boundary value problem belongs to the required class. Then we give the conjugate boundary problem with introducing the corresponding class solutions, and we reduce the conjugate boundary problem to an integral equation. Next we find a solution to the integral equation. It is shown that the solution of the conjugate boundary value problem does not belongs to the required class. The main result of the section is formulated as a theorem.

1.2.1 Dirichlet problem for the heat equation in an infinite angular domain

In the domain $G = \{(x;\ t):\quad t > 0,\quad 0 < x < t\}$ (Fig. 1.2) *find a solution to the heat equation*

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

satisfying the boundary conditions:

$$u(x,\ t)|_{x=0} = 0, \quad u(x,\ t)|_{x=t} = 0, \tag{2}$$

where $u(x,t), \quad (x,t) \in G$ belongs to the class :

Figure 1.2 – Domain G

$$|u(x,t)| \le C \exp\left\{\frac{t-x}{a^2}\right\} \times$$
$$\times \max\left[\frac{\sqrt{t}}{t-x} \exp\left\{-\left(\frac{2t-x}{2a}\right)^2 \frac{1}{t}\right\};\ 1 + \exp\left\{-\frac{t-x}{a^2}\right\}\right]. \tag{3}$$

It is required to show that in class (3) problem (1) – (2) has only one non-trivial solution up to a constant factor .

Note that, for example, the functions

$$u_1(x,t) = x^2 + 2a^2 t,\ u_2(x,t) = t^{-1/2} \exp\left\{-\frac{x^2}{4a^2 t}\right\}, (x,t) \in G,$$

satisfying equation (1), belong to class (3).

Reducing the problem to an integral equation

We are looking for a solution of problem (1) – (2) in the namely:

$$\varphi(t) - \int_0^t k_o(t,\tau)\,\varphi(\tau)\,d\tau \;= f_1(t), \qquad (15)$$

where

$$k_o(t,\tau) = \frac{t}{a\,\sqrt{\pi}\,(t-\tau)^{\frac{3}{2}}}\;\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\},$$

$$f_1(t) = \int_0^t k_h(t,\tau)\;\varphi(\tau)\,d\tau, \qquad (16)$$

where

$$k_h(t,\tau) = \frac{1}{2a\,\sqrt{\pi}\,(t-\tau)^{\frac{1}{2}}}\;\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right).$$

Proposition 1.1
There are relations

$$\lim_{t\to 0}\int_0^t k_o(t,\,\tau)\,d\tau = 1;\quad \lim_{t\to 0}\int_0^t k_h(t,\,\tau)\,d\tau = 0.$$

Indeed:

$$\lim_{t\to 0}\int_0^t k_o(t,\,\tau)\,d\tau =$$

$$= \frac{1}{a\,\sqrt{\pi}}\lim_{t\to 0}\int_0^t \frac{t}{(t-\tau)^{\frac{3}{2}}}\;\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\,d\tau =$$

$$+\frac{1}{8a^4\,\pi}\int_0^t \frac{t}{(t-\tau)^{\frac{3}{2}}}\;\exp\left(-\frac{t^2}{4a^2(t-\tau)}\right)\times$$

$$\times\left(\int_0^\tau \frac{\tau_1}{(\tau-\tau_1)^{\frac{3}{2}}}\;\exp\left[-\frac{\tau_1^2}{4a^2(\tau-\tau_1)}\right]\;\varphi\,(\tau_1)\,d\tau_1\right)\,d\tau. \quad (7)$$

We introduce the following notation:

$$J(t) = \frac{1}{8a^4\,\pi}\int_0^t \frac{t}{(t-\tau)^{\frac{3}{2}}}\;\exp\left\{-\frac{t^2}{4a^2(t-\tau)}\right\}\times$$

$$\times\left(\int_0^\tau \frac{\tau_1}{(\tau-\tau_1)^{\frac{3}{2}}}\;\exp\left\{-\frac{\tau_1^2}{4a^2(\tau-\tau_1)}\right\}\;\varphi\,(\tau_1)\,d\tau_1\right)\,d\tau.$$

Changing the order of integration, we obtain:

$$J(t) = \frac{1}{8a^4\,\pi}\int_0^t t\,\tau_1\varphi\,(\tau_1)\left(\int_{\tau_1}^t \frac{1}{(t-\tau)^{\frac{3}{2}}\,(\tau-\tau_1)^{\frac{3}{2}}}\times\right.$$

$$\left.\times\exp\left\{-\left(\frac{t^2}{4a^2(t-\tau)}+\frac{\tau_1^2}{4a^2(\tau-\tau_1)}\right)\right\}\;d\tau\right)\,d\tau_1. \quad (8)$$

For the inner integral:

$$I(t,\tau_1) = \int_{\tau_1}^t \frac{1}{(t-\tau)^{\frac{3}{2}}\,(\tau-\tau_1)^{\frac{3}{2}}}\times$$

$$\times\exp\left\{-\left(\frac{t^2}{4a^2(t-\tau)}+\frac{\tau_1^2}{4a^2(\tau-\tau_1)}\right)\right\}\;d\tau$$

the replacement

$$\left\| \begin{array}{l} z = \sqrt{\dfrac{t-\tau}{\tau-\tau_1}};\ \tau = \dfrac{z^2\tau_1 + t}{z^2+1};\ d\tau = \dfrac{2z(\tau_1 - t)dz}{(z^2+1)^2}; \\ t-\tau = \dfrac{z^2(t-\tau_1)}{z^2+1};\ \tau-\tau_1 = \dfrac{t-\tau_1}{z^2+1}; \end{array} \right\|$$

and the table integral

$$\int\limits_0^\infty \exp\{-\mu z^2 - \eta/z^2\}dz =$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{\mu}}\exp\{-2\sqrt{\mu\eta}\},\ \ \mu>0,\ \ \eta>0;$$

lead to the result:

$$I(t,\tau_1) =$$

$$= \int\limits_0^\infty \frac{2(z^2+1)}{z^2(t-\tau_1)^2}\exp\left\{-\frac{t^2(z^2+1)}{4a^2z^2(t-\tau_1)} - \frac{\tau_1^2(z^2+1)}{4a^2(t-\tau_1)}\right\}dz =$$

$$= \left\| \begin{array}{l} A(t,\tau_1) = \dfrac{2}{(t-\tau_1)^2}\exp\left\{-\dfrac{\tau_1^2+t^2}{4a^2(t-\tau_1)}\right\}; \\ \alpha(t,\tau_1) = \dfrac{\tau_1^2}{4a^2(t-\tau_1)};\ \beta(t,\tau_1) = \dfrac{t^2}{4a^2(t-\tau_1)} \end{array} \right\| =$$

is given by the formula

$$y(x) = f(x) + \int\limits_a^x R\,(x,t)\ f(t)\,dt,$$

where $R\,(x,t)$ is the resolvent (resolving kernel) of equations (13), then solution of the equation

$$y(x) + \int\limits_a^x K\,(x,t)\ \ e^{\alpha(x-t)}y(t)\,dt = f(x)$$

has the form:

$$y(x) = f(x) + \int\limits_a^x R\,(x,t)\ \ e^{\alpha(x-t)}f(t)\,dt.$$

This fact also holds for the corresponding homogeneous equation.

Therefore, it is enough to find a solution to the "simplified" equations:

$$\varphi(t) - \int\limits_0^t k(t,\tau)\,\varphi(\tau)\,d\tau\ = 0, \qquad (14)$$

where

$$k(t,\tau) = \frac{1}{2a\sqrt{\pi}}\left\{\frac{2t}{(t-\tau)^{\frac{3}{2}}}\ \exp\left(-\frac{t\tau}{a^2(t-\tau)}\right) + \right.$$

$$\left. + \frac{1}{(t-\tau)^{\frac{1}{2}}}\left(1 - \exp\left(-\frac{t\tau}{a^2(t-\tau)}\right)\right)\right\}.$$

Solving a characteristic equation

According to the Carleman-Vekua regularization method to study equation (14) we distinguish its characteristic part, according to the law $x = t$ [16]–[18] boundary value problems for the spectral loaded parabolic equation are reduced to such singular integral equations as (10).

We consider the homogeneous equation (10):

$$\varphi(t) - \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \left[\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}} \; \exp\left\{ -\frac{(t+\tau)^2}{4a^2(t-\tau)} \right\} + \right.$$

$$\left. + \frac{1}{(t-\tau)^{\frac{1}{2}}} \; \exp\left\{ -\frac{t-\tau}{4a^2} \right\} \right] \varphi(\tau)\, d\tau = 0, \quad (t > 0) \quad (11)$$

Using the ratios:

$$t + \tau = 2t - (t - \tau), \quad \frac{(t+\tau)^2}{4a^2\,(t-\tau)} = \frac{t\tau}{a^2\,(t-\tau)} + \frac{t-\tau}{4a^2},$$

we get:

$$\varphi(t) - \int\limits_0^t \frac{1}{2a\sqrt{\pi}} \left\{ \frac{2t}{(t-\tau)^{\frac{3}{2}}} \; \exp\left(-\frac{t\tau}{a^2(t-\tau)} \right) - \right.$$

$$- \frac{1}{(t-\tau)^{\frac{1}{2}}} \exp\left(-\frac{t\tau}{a^2(t-\tau)} \right) +$$

$$\left. + \frac{1}{(t-\tau)^{\frac{1}{2}}} \right\} \exp\left(-\frac{t-\tau}{4a^2} \right) \; \varphi(\tau)\, d\tau = 0. \quad (12)$$

Remark 1.1 [65, ?]
If the solution of the integral equation

$$y(x) + \int\limits_a^x K\,(x,t)\; y(t)\, dt = f(x) \quad (13)$$

$$= A(t,\tau_1) \int\limits_0^\infty \frac{z^2+1}{z^2} \exp\{-\alpha(t,\tau_1)z^2 - \beta(t,\tau_1)/z^2\} dz =$$

$$= A(t,\tau_1) \int\limits_0^\infty \exp\{-\alpha(t,\tau_1)z^2 - \beta(t,\tau_1)/z^2\} dz +$$

$$+ A(t,\tau_1) \int\limits_0^\infty \exp\{-\beta(t,\tau_1)z^2 - \alpha(t,\tau_1)/z^2\} dz =$$

$$= 2a\sqrt{\pi} \frac{t+\tau_1}{t\tau_1\,(t-\tau_1)^{\frac{3}{2}}} \exp\left\{ -\frac{(t+\tau_1)^2}{4a^2(t-\tau_1)} \right\}.$$

Substituting $I(t,\tau_1)$ into (8), we have:

$$J(t) = \frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{t+\tau_1}{(t-\tau_1)^{\frac{3}{2}}} \exp\left\{ -\frac{(t+\tau_1)^2}{4a^2(t-\tau_1)} \right\} \varphi\,(\tau_1)\; d\tau_1. \quad (9)$$

Taking into account (9) we rewrite (7) in the form:

$$\varphi(t) - \int\limits_0^t K(t,\,\tau)\; \varphi(\tau)\, d\tau = 0, \quad (10)$$

where

$$K(t,\,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)} \right) + \right.$$

$$\left. + \frac{1}{(t-\tau)^{\frac{1}{2}}} \exp\left(-\frac{t-\tau}{4a^2} \right) \right\}.$$

The kernel $K(t,\,\tau)$ has properties:
1) $K(t,\,\tau) > 0$ and continuously as $0 < \tau < t < \infty$;

2) $\lim\limits_{t\to t_0}\int\limits_{t_0}^{t} K(t,\,\tau)\,d\tau = 0,\ t_0 \geq \varepsilon > 0;$

3) $\lim\limits_{t\to 0}\int\limits_{0}^{t} K(t,\,\tau)\,d\tau = 1.$

Property 1) is obvious. Let us prove the validity of property 3), i.e. show that

$$\lim_{t\to 0}\int\limits_0^t \frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right)+\right.$$

$$\left.+\frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\}d\tau = 1.$$

We introduce a replacement:

$$x = \sqrt{t-\tau}.$$

Then we get:

$$\int\limits_0^t K(t,\,\tau)\,d\tau = \frac{2}{\sqrt{\pi}}\exp\left(\frac{2t}{a^2}\right)\times$$

$$\times\int\limits_0^{\sqrt{t}}\exp\left\{-\left(\frac{t}{ax}+\frac{x}{2a}\right)^2\right\}\left(\frac{t}{a\,x^2}-\frac{1}{2a}\right)dx+$$

$$+\frac{1}{a\sqrt{\pi}}\int\limits_0^{\sqrt{t}}\exp\left(-\frac{x^2}{4a^2}\right)dx =$$

$$= \exp\left(\frac{2t}{a^2}\right)\operatorname{erfc}\left(\frac{3\sqrt{t}}{2a}\right)+\operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right).$$

Here, calculating the first integral, we used the replacement

$$z = \frac{t}{ax}+\frac{x}{2a};\quad dz = -\left(\frac{t}{a\,x^2}-\frac{1}{2a}\right)dx,$$

and, calculating the first integral, we used the replacement

$$\xi = \frac{x}{2a}.$$

Besides, we have

$$\lim_{t\to\infty}\int\limits_0^t K(t,\,\tau)\,d\tau = 1.$$

The last limit relation follows from the asymptotic formula for $\operatorname{erfc}(z)$ as large z [64, ?].

Similarly, we obtain

$$\int\limits_{t_0}^t K(t,\,\tau)\,d\tau = \exp\left(\frac{2t}{a^2}\right)\operatorname{erfc}\left(\frac{3t-t_0}{2a\sqrt{t-t_0}}\right)+\operatorname{erf}\left(\frac{\sqrt{t-t_0}}{2a}\right).$$

This implies the validity of property 2).

Study of the integral equation

Singularity of the equation under study is the property 3) of the kernel $K(t,\,\tau)$, and this singularity is expressed in the fact that the corresponding homogeneous equation cannot be solved by the method of successive approximations.

Issues of unique solvability for such integral equations in certain weight spaces are studied in [5]–[8]: it is shown that the weight function of solutions depend on the weight functions of the right parts of the equation. The study of similar integral equations, in particular, the study of spectral issues of the corresponding integral operators was also carried out in [9]–[15].

It should also be noted that when the load line moves

Finally, we get:

$$\varphi(t) - \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} d\tau = \frac{C}{\sqrt{t}}. \qquad (24)$$

Thus, "simplified" integral equation (14) is reduced to equation (24), i.e., to the Abel integral equation of the second kind.

So, the lemma is proved.

Lemma 1.1

The integral equation

$$\varphi(t) - \int\limits_0^t k(t,\tau)\,\varphi(\tau)\,d\tau = 0,$$

where

$$k(t,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ \frac{2t}{(t-\tau)^{\frac{3}{2}}} \exp\left\{ -\frac{t\tau}{a^2(t-\tau)} \right\} + \right.$$

$$\left. + \frac{1}{(t-\tau)^{\frac{1}{2}}} \left(1 - \exp\left\{ -\frac{t\tau}{a^2(t-\tau)} \right\} \right) \right\},$$

is equivalently reduced to the Abel equation

$$\varphi(t) - \frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} d\tau = \frac{C}{\sqrt{t}}.$$

Solving the Abel equation

Remark 1.2 [65, ?]

$$= \left\| z = \frac{t}{a\sqrt{t-\tau}} \right\| = \frac{2}{\sqrt{\pi}} \lim_{t\to 0} \int\limits_{\frac{\sqrt{t}}{a}}^{\infty} \exp\left\{ -\left(z^2 - \frac{t}{a^2} \right) \right\} dz =$$

$$= \lim_{t\ 0} \left\{ \exp\left(\frac{t}{a^2} \right) \cdot erfs \left(\frac{\sqrt{t}}{a} \right) \right\} = 1.$$

The validity of the equality:

$$\lim_{t\to 0} \int\limits_0^t k_h(t,\tau)\,d\tau = 0$$

follows from an estimate:

$$k_h(t,\tau) = \frac{1}{2a\sqrt{\pi}\,(t-\tau)^{\frac{1}{2}}} \left(1 - \exp\left\{ -\frac{t\tau}{a^2(t-\tau)} \right\} \right) \leq$$

$$\leq \frac{1}{2a\sqrt{\pi}\,(t-\tau)^{\frac{1}{2}}},$$

i.e. function $k_h(t,\tau)$ has a weak singularity. By virtue of Proposition 1.1, it follows

Proposition 1.2

Equation (15) is a characteristic equation for (14).

Furthermore, we consider that the right side of equation (15) is known; we find its solution, i.e., we find solution to the characteristic equation for (14).

Integral equation (15) we reduce to equation with a difference kernel. To do this, we make replacements:

$$t = \frac{1}{y}, \quad \tau = \frac{1}{x}; \quad \psi(y) = \frac{1}{\sqrt{y}}\varphi\left(\frac{1}{y}\right); \quad f_2(y) = \frac{1}{\sqrt{y}} f_1\left(\frac{1}{y}\right). \qquad (17)$$

Then we obtain an equation of the form:

$$\psi(y) - $$

$$-\int_y^\infty \frac{1}{a\sqrt{\pi}(x-y)^{\frac{3}{2}}}\,\exp\left\{-\frac{1}{a^2(x-y)}\right\}\psi(x)\,dx = f_2(y),$$

$$(y > 0) \quad (18)$$

The solution to equation (18) can be found either by the operational method [29] or by reducing it to the Riemann boundary value problem [16]. In this case the index of the boundary value problem is equal to 1 [16], and the function $\psi(y) = C \quad (C - const)$ is a solution of the homogeneous equation:

$$\psi(y) - \int_y^\infty \frac{1}{a\sqrt{\pi}(x-y)^{\frac{3}{2}}}\,\exp\left\{-\frac{1}{a^2(x-y)}\right\}\psi(x)\,dx = 0,$$

corresponding to equation(18).

A solution of nonhomogeneous equation (18) has the form [16]:

$$\psi(y) = f_2(y) + \int_y^\infty r_-(y-x)\,f_2(x)\,dx + C, \quad (C - const), \quad (19)$$

where

$$r_-(y) = \frac{1}{a\sqrt{\pi}\,(-y)^{\frac{3}{2}}}\sum_{n=1}^\infty n\cdot\exp\left\{-\frac{n^2}{a^2(-y)}\right\}.$$

Taking into account replacements (17) from (19) we get a

For calculating the second integral, we use the known result:

$$\int_0^\infty \exp\left\{-\mu x^2 - \frac{\eta}{x^2}\right\}\,dx = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\mu}}\exp\left\{-2\sqrt{\mu\eta}\right\}.$$

Then:

$$I_n^{(2)}(t;\,\tau) =$$

$$= \frac{a\sqrt{\pi}}{n\,t\,\sqrt{t-\tau}}\exp\left\{-\frac{(n^2+1)t\,\tau}{a^2(t-\tau)}\right\}\exp\left\{-2\cdot\frac{n\,t\tau}{a^2(t-\tau)}\right\} =$$

$$= \frac{a\sqrt{\pi}}{n\,t\,\sqrt{t-\tau}}\exp\left\{-\frac{(n+1)^2 t\,\tau}{a^2(t-\tau)}\right\}.$$

So,

$$I_n(t;\,\tau) = I_n^{(1)}(t;\,\tau) - I_n^{(2)}(t;\,\tau) =$$

$$= \frac{a\sqrt{\pi}}{n\,t\,\sqrt{t-\tau}}\left(\exp\left\{-\frac{n^2 t\,\tau}{a^2(t-\tau)}\right\} - \exp\left\{-\frac{(n+1)^2 t\,\tau}{a^2(t-\tau)}\right\}\right).$$

Substituting into (23), we get:

$$J(t,\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}}\sum_{n=1}^\infty\left(\exp\left\{-\frac{n^2 t\,\tau}{a^2(t-\tau)}\right\} - \right.$$

$$\left. - \exp\left\{-\frac{(n+1)^2 t\,\tau}{a^2(t-\tau)}\right\}\right) = \frac{1}{2a\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{t\,\tau}{a^2(t-\tau)}\right\}.$$

Then equation (22) takes the form:

$$\varphi(t) = \int_0^t \left\{\frac{1}{2a\sqrt{\pi\,(t-\tau)}}\left(1 - \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right) + \right.$$

$$\left. + \frac{1}{2a\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{t\,\tau}{a^2(t-\tau)}\right\}\right\}\varphi(\tau)\,d\tau + \frac{C}{\sqrt{t}}.$$

$$\tau_1-\tau=\frac{(t-\tau)\ z^2}{1+z^2};\quad t-\tau_1=\frac{t-\tau}{1+z^2};\quad \frac{\tau_1}{t-\tau_1}=\frac{t\,z^2+\tau}{t-\tau};$$

$$\frac{\tau_1}{\tau_1-\tau}=\frac{t\,z^2+\tau}{(t-\tau)z^2};\quad d\tau_1=\frac{(t-\tau)\cdot 2z}{\left(1+z^2\right)^2}dz.$$

After substitution, the last integrals take the form:

$$I_n^{(1)}(t;\,\tau)=\frac{2}{t-\tau}\exp\left\{-\frac{n^2t\,\tau}{a^2(t-\tau)}\right\}\int\limits_0^{\infty}\exp\left\{-\frac{n^2t^2\,z^2}{a^2(t-\tau)}\right\}\,dz,$$

$$I_n^{(2)}(t;\,\tau)=\frac{2}{t-\tau}\exp\left\{-\frac{(n^2+1)t\,\tau}{a^2(t-\tau)}\right\}\times$$

$$\times\int\limits_0^{\infty}\exp\left\{-\frac{n^2t^2z^2}{a^2(t-\tau)}-\frac{\tau^2}{a^2(t-\tau)z^2}\right\}dz.$$

Now, using replacement

$$\xi=\frac{n\,t\,z}{a\sqrt{t-\tau}}$$

for the first integral, we get:

$$I_n^{(1)}(t;\,\tau)=$$

$$=\frac{2}{t-\tau}\exp\left\{-\frac{n^2t\,\tau}{a^2(t-\tau)}\right\}\frac{a\sqrt{t-\tau}}{n\,t}\int\limits_0^{\infty}\exp\left\{-\xi^2\right\}\,d\xi=$$

$$=\frac{a\sqrt{\pi}}{n\,t\,\sqrt{t-\tau}}\exp\left\{-\frac{n^2t\,\tau}{a^2(t-\tau)}\right\}.$$

solution of nonhomogeneous equation (15):

$$\varphi(t)=f_1(t)+\int\limits_0^t r\,(t,\tau)\ f_1(\tau)\,d\tau+\frac{C}{\sqrt{t}},\qquad(20)$$

where the resolvent $r(t,\tau)$ is a function

$$r(t,\tau)=\frac{t}{a\sqrt{\pi}\ (t-\tau)^{\frac{3}{2}}}\sum_{n=1}^{\infty}n\cdot\exp\left\{-n^2\frac{t\tau}{a^2(t-\tau)}\right\}.\qquad(21)$$

Reducing a "simplified" equation to Abel equation

Now, we solve equation (14), i.e., a "simplified" variant of equation (11).

Using the formula for the solution of characteristic equation (20), Taking into account relations (16) for the function $f_1(t)$, we obtain:

$$\varphi(t)=$$

$$=\int\limits_0^t\frac{1}{2a\,\sqrt{\pi\,(t-\tau)}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)\,\varphi(\tau)\,d\tau+$$

$$+\int\limits_0^t r\,(t,\tau)\,\left(\int\limits_0^{\tau}\frac{1}{2a\,\sqrt{\pi\,(\tau-\tau_1)}}\left(1-\exp\left\{-\frac{\tau\tau_1}{a^2(\tau-\tau_1)}\right\}\right)\times\right.$$

$$\left.\times\varphi(\tau_1)\,d\tau_1\right)$$

$$d\tau+\frac{C}{\sqrt{t}}.$$

Changing the integration order on the right side of the obtained equation and changing roles τ ang τ_1, we have:

$$\varphi(t) = \int\limits_0^t \left\{ \frac{1}{2a\sqrt{\pi\,(t-\tau)}} \left(1 - \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right) + \right.$$

$$\left. + \int\limits_\tau^t r\,(t,\tau_1)\,\frac{1}{2a\sqrt{\pi\,(\tau_1-\tau)}}\cdot \left(1 - \exp\left\{-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}\right) d\tau_1 \right\}\times$$

$$\times\,\varphi(\tau)\,d\tau + \frac{C}{\sqrt{t}}.. \quad (22)$$

We calculate the inner integral in (22):

$$J\,(t,\tau) =$$

$$= \int\limits_\tau^t r\,(t,\tau_1)\,\frac{1}{2a\sqrt{\pi\,(\tau_1-\tau)}} \left(1 - \exp\left\{-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}\right) d\tau_1 =$$

$$= \frac{t}{2a^2\pi}\sum_{n=1}^{\infty}\int\limits_\tau^t \frac{n}{(t-\tau_1)^{\frac{3}{2}}\sqrt{\tau_1-\tau}}\exp\left\{-n^2\frac{t\tau_1}{a^2(t-\tau_1)}\right\}\times$$

$$\times\left(1 - \exp\left\{-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}\right) d\tau_1\ \cdot$$

or:

$$J\,(t,\tau) = \frac{t}{2a^2\pi}\sum_{n=1}^{\infty} n\cdot$$

$$\int\limits_\tau^t \frac{1}{(t-\tau_1)^{\frac{3}{2}}\sqrt{\tau_1-\tau}}\exp\left\{-n^2\frac{t\tau_1}{a^2(t-\tau_1)}\right\}\times$$

$$\times\left(1 - \exp\left\{-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}\right) d\tau_1 \ = \frac{t}{2a^2\pi}\sum_{n=1}^{\infty} n\cdot I_n(t,\tau)\,. \quad (23)$$

Consider the integral:

$$I_n(t;\,\tau) = \int\limits_\tau^t \frac{1}{(t-\tau_1)^{\frac{3}{2}}\,(\tau_1-\tau)^{\frac{1}{2}}}\exp\left\{-\frac{n^2t\,\tau_1}{a^2(t-\tau_1)}\right\}\times$$

$$\times\left(1 - \exp\left\{-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}\right) d\tau_1 \ = I_n^{(1)}(t;\,\tau) - I_n^{(2)}(t;\,\tau),$$

where

$$I_n^{(1)}(t;\,\tau) = \int\limits_\tau^t \frac{1}{(t-\tau_1)^{\frac{3}{2}}\,(\tau_1-\tau)^{\frac{1}{2}}}\exp\left\{-\frac{n^2t\,\tau_1}{a^2(t-\tau_1)}\right\} d\tau_1\,,$$

$$I_n^{(2)}(t;\,\tau) = \int\limits_\tau^t \frac{1}{(t-\tau_1)^{\frac{3}{2}}\,(\tau_1-\tau)^{\frac{1}{2}}}\times$$

$$\times\exp\left\{-\left(\frac{n^2t\,\tau_1}{a^2(t-\tau_1)} + \frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right)\right\} d\tau_1\,.$$

We calculate the integrals $I_n^{(1)}(t;\,\tau)$ and $I_n^{(2)}(t;\,\tau)$.
We make a replacement:

$$z = \sqrt{\frac{\tau_1-\tau}{t-\tau_1}};\quad \tau_1 = \frac{t\,z^2+\tau}{1+z^2};$$

$$= \left\| z = (1+y^2)^{1/2} \right\| =$$

$$= \frac{\sqrt{t}}{a^2} \int\limits_1^{\infty} \exp\left\{ -\frac{t(z^2-1)^2}{4a^2z^2} \right\} dz -$$

$$- \frac{\sqrt{t}}{a^2} \int\limits_0^{1} \exp\left\{ -\frac{t(z^2-1)^2}{4a^2z^2} \right\} dz \leq$$

$$\leq \frac{\sqrt{t}}{a^2} \exp\{t/(2a^2)\} \int\limits_0^{\infty} \exp\left\{ -\frac{t}{4a^2} \left(z^2 + 1/z^2 \right) \right\} dz =$$

$$= \frac{\sqrt{t}}{a^2} \exp\{t/(2a^2)\} \cdot \frac{1}{2}\sqrt{\pi} \frac{2a}{\sqrt{t}} \exp\{-2t/(4a^2)\} = \frac{\sqrt{\pi}}{a}.$$

Thus, we obtain:

$$\nu_1(t) \leq \frac{C_7}{\sqrt{t}}, \quad \nu_2(t) \leq C_8. \tag{1.2.32}$$

Using (1.2.32), we estimate terms $u_{11}(x,t)$ and $u_{12}(x,t)$ of the solution

$$u_{11}(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int\limits_0^{t} \frac{x}{(t-\tau)^{3/2}} \exp\left\{ -\frac{x^2}{4a^2(t-\tau)} \right\} \nu_1(\tau) d\tau \leq$$

$$\leq \frac{C_7}{4a^3\sqrt{\pi}} \int\limits_0^{t} \frac{x}{\sqrt{\tau}(t-\tau)^{3/2}} \exp\left\{ -\frac{x^2}{4a^2(t-\tau)} \right\} d\tau =$$

$$= \left\| y = \sqrt{\frac{x}{t-\tau}} \right\| = \frac{C_7\sqrt{x}}{2a^3\sqrt{\pi}} \int\limits_{\sqrt{x/t}}^{\infty} \frac{y}{\sqrt{ty^2 - x}} \exp\left\{ -\frac{xy^2}{4a^2} \right\} dy =$$

The solution of the Abel equation of the second kind:

$$y(x) + \lambda \int\limits_a^{x} \frac{y(t)}{\sqrt{x-t}} dt = f(x)$$

has the form:

$$y(x) = F(x) + \pi\lambda^2 \int\limits_a^{x} \exp\left[\pi\lambda^2 (x-t) \right] F(t)\, dt, \tag{25}$$

where

$$F(x) = f(x) - \lambda \int\limits_a^{x} \frac{f(t)}{\sqrt{x-t}} dt.$$

The solution of equation (24) according to formula (25) can be written as:

$$\varphi(t) = F(t) + \frac{1}{4a^2} \int\limits_0^{t} \exp\left(\frac{t-\tau}{4a^2} \right) F(\tau)\, d\tau,$$

where

$$F(t) = C \cdot \left\{ \frac{1}{\sqrt{t}} + \frac{1}{2a\sqrt{\pi}} \int\limits_0^{t} \frac{d\tau}{\sqrt{\tau(t-\tau)}} \right\} = C \cdot \left\{ \frac{1}{\sqrt{t}} + \frac{\sqrt{\pi}}{2a} \right\}.$$

Then

$$\varphi(t) =$$

$$= C \cdot \left\{ \frac{1}{\sqrt{t}} + \frac{\sqrt{\pi}}{2a} + \frac{1}{4a^2} \int\limits_0^{t} \exp\left(\frac{t-\tau}{4a^2} \right) \cdot \left(\frac{1}{\sqrt{\tau}} + \frac{\sqrt{\pi}}{2a} \right) d\tau \right\} =$$

$$= C\cdot\left\{\frac{1}{\sqrt{t}}+\frac{\sqrt{\pi}}{2a}\right\}+\frac{\mathrm{C}}{4a^{2}}\exp\left(\frac{t}{4a^{2}}\right)\times$$

$$\times\left\{\int\limits_{0}^{t}\exp\left(-\frac{\tau}{4a^{2}}\right)\frac{d\tau}{\sqrt{\tau}}+\frac{\sqrt{\pi}}{2a}\int\limits_{0}^{t}\exp\left(-\frac{\tau}{4a^{2}}\right)d\tau\right\}.$$

After simplifications we get:

$$\varphi(t)=C\cdot\left\{\frac{1}{\sqrt{t}}+\frac{\sqrt{\pi}}{2a}\exp\left(\frac{t}{4a^{2}}\right)\operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{\pi}}{2a}\exp\left(\frac{t}{4a^{2}}\right)\right\} \tag{26}$$

the solution of Abel equation (24), i.e. the solution of "simplified" equations (14).

Note that after multiplying equality (26) by $\exp\left(-\frac{t}{4a^{2}}\right)$, we get (Remark 1.1) a solution of initial equation (11) (for convenience, *const* C is equal to 1):

$$\varphi(t)=\frac{1}{\sqrt{t}}\exp\left(-\frac{t}{4a^{2}}\right)+\frac{\sqrt{\pi}}{2a}\operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right)+\frac{\sqrt{\pi}}{2a}. \tag{27}$$

So, the theorem is proved:

Theorem 1.1

In the weight class of functions $\sqrt{t}\exp\left(-\frac{t}{4a^{2}}\right)\varphi(t)\in L_{\infty}(0,\infty)$ *function (27) is a solution to the singular integral equation*

$$\varphi(t)-\int\limits_{0}^{t}K(t,\tau)\,\varphi(\tau)\,d\tau=0,$$

where

$$K(t,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left(-\frac{(t+\tau)^{2}}{4a^{2}(t-\tau)}\right)+\right.$$

$$\nu_{2}(t)=\frac{1}{2a\sqrt{\pi}}\int\limits_{0}^{t}\frac{\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left\{-\frac{\tau^{2}}{4a^{2}(t-\tau)}\right\}\varphi_{2}(\tau)d\tau.$$

We have:

$$\nu_{1}(t)=\frac{1}{2a\sqrt{\pi}}\int\limits_{0}^{t}\frac{\sqrt{\tau}}{(t-\tau)^{3/2}}\exp\left\{-\frac{t\tau}{4a^{2}(t-\tau)}\right\}d\tau=$$

$$=\left\|y=\frac{\tau}{t-\tau}\right\|=\frac{1}{2a\sqrt{\pi t}}\int\limits_{0}^{\infty}\frac{y}{1+y}\exp\left\{-\frac{ty}{4a^{2}}\right\}d\sqrt{ty}=$$

$$=\frac{2}{\sqrt{\pi t}}\int\limits_{0}^{\infty}\exp\left\{-\frac{ty}{4a^{2}}\right\}d\left(\frac{\sqrt{ty}}{2a}\right)-$$

$$-\frac{2}{\sqrt{\pi t}}\int\limits_{0}^{\infty}\frac{1}{1+z^{2}}\exp\left\{-\frac{t\,z^{2}}{4a^{2}}\right\}d\left(\frac{\sqrt{t}\,z}{2a}\right)\leq\frac{C_{7}}{\sqrt{t}}.$$

Furthermore,

$$\nu_{2}(t)=\frac{1}{2a\sqrt{\pi}}\int\limits_{0}^{t}\frac{\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left\{-\frac{\tau^{2}}{4a^{2}(t-\tau)}\right\}\varphi_{2}(\tau)d\tau\leq$$

$$\leq\left\|\tau=t-(t-\tau),\ y=\sqrt{\frac{\tau}{t-\tau}}\right\|\leq$$

$$\leq\frac{\sqrt{t}}{2a^{2}}\int\limits_{0}^{\infty}\frac{2y}{(1+y^{2})^{1/2}}\exp\left\{-\frac{ty^{4}}{4a^{2}(1+y^{2})}\right\}dy-$$

$$-\frac{\sqrt{t}}{2a^{2}}\int\limits_{0}^{\infty}\frac{2y}{(1+y^{2})^{3/2}}\exp\left\{-\frac{ty^{4}}{4a^{2}(1+y^{2})}\right\}dy=$$

$$\overline{u}_{22}(x,t)|_{x=t} = -\frac{1}{2a^2}\,\mathrm{erfc}\left(-\frac{\sqrt{t}}{2a}\right).$$

Thus, we have:

$$|u_2^{(1)}(x,t)| \le C_3\frac{1}{\sqrt{t}}\exp\left\{-\frac{x^2}{4a^2t}\right\},$$

$$|u_2^{(2)}(x,t)| \le C_4\frac{\sqrt{t}}{t-x}\exp\left\{-\frac{x^2}{4a^2t}\right\},$$

$$|u_{22}(x,t)| \le C_5\exp\left\{\frac{t-x}{a^2}\right\}. \quad (1.2.30)$$

The first two inequalities can be replaced by the following:

$$|u_{21}(x,t)| \le C_6\frac{\sqrt{t}}{t-x}\cdot\exp\left\{-\frac{x^2}{4a^2t}\right\}, \quad (1.2.31)$$

$$\text{since}\quad \frac{1}{\sqrt{t}} < \frac{\sqrt{t}}{t-x},\ \forall t > x > 0.$$

Furthermore, for the first term of function $u(x,t)$ (1.2.29) we have:

$$u_1(x,t) = \frac{1}{4a^3\sqrt{\pi}}\int\limits_0^t \frac{x}{(t-\tau)^{3/2}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\nu(\tau)d\tau.$$

We represent the function $\nu(t)$ as a sum:

$$\nu(t) = \nu_1(t) + \nu_2(t),$$

where

$$\nu_1(t) = \frac{1}{2a\sqrt{\pi}}\int\limits_0^t \frac{\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi_1(\tau)d\tau,$$

$$+\frac{1}{(t-\tau)^{\frac{1}{2}}}\exp\left(-\frac{t-\tau}{4a^2}\right)\Bigg\}.$$

Solution of the initial boundary value problem

Since the desired function $u(x,t)$ is represented as a sum of potentials of double layer with densities $\nu(t)$ and $\varphi(t)$ (4), $\nu(t)$ and $\varphi(t)$ are defined by formulas (6) and (27), accordingly, we find *a solution of boundary value problem (1) – (2)* explicitly according to the following theorem.

Theorem 1.2

In the domain $G = \{(x;\ t):\quad t > 0,\quad 0 < x < t\}$ *the homogeneous boundary value problem*

$$\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = 0,$$

$$u(x,\ t)\big|_{x=0} = 0,\quad u(x,\ t)\big|_{x=t} = 0,$$

has a nonzero solution, which is determined by the formula:

$$u(x,t) = \frac{1}{4a^3\sqrt{\pi}}\int\limits_0^t \frac{x}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\nu(\tau)\,d\tau +$$

$$+\frac{1}{4a^3\sqrt{\pi}}\int\limits_0^t \frac{x-\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau, \quad (1.2.28)$$

where

$$\nu(t) = \frac{1}{2a\sqrt{\pi}}\int\limits_0^t \frac{\tau}{(t-\tau)^{\frac{3}{2}}}\exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\varphi(\tau)\,d\tau,$$

and function $\varphi(t)$ *is determined according to formula (27).*

Estimate of a non-trivial solution

To establish a class of the nontrivial solution $u(x,t)$ (1.2.28) we establish its estimate in order of growth:

$$u(x,t) = u_1(x,t) + u_2(x,t), \qquad (1.2.29)$$

where

$$\varphi(t) = \frac{1}{\sqrt{t}} \exp\left\{-\frac{t}{4a^2}\right\} + \frac{\sqrt{\pi}}{2a}\left[\operatorname{erf}\left(\frac{\sqrt{t}}{2a}\right)+1\right] = \varphi_1(t)+\varphi_2(t).$$

We estimate the second term from (1.2.29). We have for $\varphi_1(t)$:

$$u_{21}(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\} \times$$

$$\times \frac{1}{\sqrt{\tau}} \exp\left\{-\frac{\tau}{4a^2}\right\} d\tau = \left\| y = \frac{\tau}{t-\tau} \right\| = \frac{1}{4a^3\sqrt{\pi}} \exp\left\{-\frac{x^2}{4a^2 t}\right\} \times$$

$$\times \left[\int\limits_0^\infty \frac{x-t}{t} \cdot \frac{1}{\sqrt{y}} \exp\{-\alpha^2 y\} dy + \int\limits_0^\infty \frac{1}{y^{1/2}(1+y)} \exp\{-\alpha^2 y\} dy \right] =$$

$$= u_2^{(1)}(x,t) + u_2^{(2)}(x,t), \quad \alpha = \frac{x-t}{2a\sqrt{t}}.$$

Next,

$$u_2^{(1)}(x,t) = \frac{1}{2a^2\sqrt{t}} \exp\left\{-\frac{x^2}{4a^2 t}\right\},$$

$$u_2^{(2)}(x,t) = \frac{1}{2a^3\sqrt{\pi}} \exp\left\{-\frac{x^2}{4a^2 t}\right\} \int\limits_0^\infty \frac{1}{1+z^2} \exp\{-\alpha^2 z^2\}\, dz \le$$

$$\le \frac{1}{2a^3\sqrt{\pi}} \exp\left\{-\frac{x^2}{4a^2 t}\right\} \int\limits_0^\infty \exp\{-\alpha^2 z^2\}\, dz =$$

$$= \frac{1}{2a^2} \cdot \frac{\sqrt{t}}{t-x} \exp\left\{-\frac{x^2}{4a^2 t}\right\}.$$

For $\varphi_2(t)$:

$$u_{22}(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\} \frac{\sqrt{\pi}}{2a} \times$$

$$\times \left[\operatorname{erf}\left(\frac{\sqrt{\tau}}{2a}\right)+1\right] d\tau \le$$

$$\left\| \text{так как} \qquad \varphi_2(\tau) \le \frac{\sqrt{\pi}}{a}, \ \forall\, 0 < \tau < t < \infty \right\|$$

$$\le \frac{1}{4a^4} \int\limits_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\} d\tau = \frac{\sqrt{\pi}}{a} \cdot \overline{u}_{22}(x,t).$$

$$\overline{u}_{22}(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int\limits_0^t \frac{x-\tau}{(t-\tau)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\} d\tau =$$

$$\left\| y = \frac{1}{\sqrt{t-\tau}} \right\| = \frac{1}{a^2\sqrt{\pi}} \exp\left\{\frac{t-x}{a^2}\right\} \times$$

$$\times \int\limits_{\frac{1}{\sqrt{t}}}^\infty \exp\left\{-\left(\frac{t-x}{2a}y + \frac{1}{2ay}\right)^2\right\} d\left(-\frac{t-x}{2a}y - \frac{1}{2ay}\right) =$$

$$= -\frac{1}{2a^2} \exp\left\{\frac{t-x}{a^2}\right\} \operatorname{erfc}\left(-\frac{\sqrt{t}}{a} + \frac{x}{2a\sqrt{t}}\right).$$

From here it follows:

$$\overline{u}_{22}(x,t)\big|_{x=0} = -\frac{1}{2a^2} \exp\left\{\frac{t}{a^2}\right\} \operatorname{erfc}\left(-\frac{\sqrt{t}}{a}\right),$$

As

$$\frac{d}{dt}\left(\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right)=\frac{3t-20a^2x^2}{4a\left(t-4a^2x^2\right)^{3/2}}$$

and

$$\left(\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right)^2=\frac{t^2}{a^2(t-4a^2x^2)}+\frac{5t}{4a^2}-x^2,$$

then (1.2.41) can be rewritten in the form

$$\int\limits_0^{\frac{\sqrt{t}}{2a}}\frac{3t-20a^2x^2}{\left(t-4a^2x^2\right)^{3/2}}\exp\left(-\frac{t^2}{a^2(t-4a^2x^2)}\right)dx=$$

$$=\frac{5\pi}{4a}\mathrm{erfc}\left(\frac{\sqrt{t}}{a}\right)-\frac{\sqrt{\pi}}{2\sqrt{t}}\exp\left(-\frac{t}{a^2}\right). \quad (1.2.42)$$

We introduce the notation for the left side of equality (1.2.42):

$$J(t)=\int\limits_0^{\frac{\sqrt{t}}{2a}}\frac{3t-20a^2x^2}{\left(t-4a^2x^2\right)^{3/2}}\exp\left(-\frac{t^2}{a^2(t-4a^2x^2)}\right)dx.$$

We introduce a replacement $z=\sqrt{t-4a^2x^2}$. Then

$$J(t)=\int\limits_0^{\frac{\sqrt{t}}{2a}}\frac{5(t-4a^2x^2)-2t}{\left(t-4a^2x^2\right)^{3/2}}\exp\left(-\frac{t^2}{a^2(t-4a^2x^2)}\right)dx=$$

$$=\frac{5}{2a}\int\limits_0^{\sqrt{t}}\frac{1}{\sqrt{t-z^2}}\exp\left(-\frac{t^2}{a^2z^2}\right)dz-$$

$$=\frac{C_7}{a^2\sqrt{\pi t}}\exp\left\{-\frac{x^2}{4a^2t}\right\}\times$$

$$\times\int\limits_{\sqrt{x/t}}^{\infty}\exp\left\{-\frac{x}{4a^2}\left(y^2-\frac{x}{t}\right)\right\}d\left(\frac{\sqrt{x}}{2a}\sqrt{y^2-\frac{x}{t}}\right)=$$

$$=\frac{C_7}{2a^2}\cdot\frac{1}{\sqrt{t}}\exp\left\{-\frac{x^2}{4a^2t}\right\}.$$

$$u_{12}(x,t)\le\frac{C_8}{8a^3}\int\limits_0^t\frac{x}{(t-\tau)^{3/2}}\exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}d\tau=$$

$$\left\|y=\frac{1}{\sqrt{t-\tau}}\right\|=$$

$$=\frac{C_8}{2a^2}\int\limits_{\frac{1}{\sqrt{t}}}^{\infty}\exp\left\{-\frac{x^2}{4a^2}y^2\right\}d\left(\frac{x}{2a}y\right)=\frac{C_8\sqrt{\pi}}{4a^2}\cdot\mathrm{erfc}\left(\frac{x}{2a\sqrt{t}}\right).$$

Thus, we have

$$u_1(x,t)\le\left[C_9+C_{10}\cdot\frac{1}{\sqrt{t}}\exp\left\{-\frac{x^2}{4a^2t}\right\}\right]. \quad (1.2.33)$$

Estimates for functions (they are exact in order of growth) $u_1(x,t)$, $u_{21}(x,t)$ and $u_{22}(x,t)$ (1.2.30), (1.2.31) и (1.2.33) determine the following estimate:

$$|u(x,t)|\le C\,\gamma(x,t), \quad (1.2.34)$$

where

$$\gamma(x,t)=\max\left[\frac{\sqrt{t}}{t-x}\exp\left\{-\frac{x^2}{4a^2t}\right\};\ 1+\exp\left\{\frac{t-x}{a^2}\right\}\right],$$

$$\gamma(x,t)\geq 2,\ \{x,t\}\in G, \quad (1.2.35)$$

i.e.,

$$\gamma(x,t)=\begin{cases}\dfrac{\sqrt{t}}{t-x}\exp\left\{-\dfrac{x^2}{4a^2t}\right\}, & \{x,t\}\in G_1;\\ 1+\exp\left\{\dfrac{t-x}{a^2}\right\}, & \{x,t\}\in G_2\cup S;\end{cases}$$

where

$$S=\left\{\{x,t\}\in G\Big|\frac{\sqrt{t}}{t-x}\exp\left\{-\frac{x^2}{4a^2t}\right\}=1+\exp\left\{\frac{t-x}{a^2}\right\}\right\};$$

$$G_1=\left\{\{x,t\}\in G\Big|\frac{\sqrt{t}}{t-x}\exp\left\{-\frac{x^2}{4a^2t}\right\}>1+\exp\left\{\frac{t-x}{a^2}\right\}\right\};$$

$$G_2=G\setminus\{G_1\cup S\}.$$

The following proposition is established

Proposition 1.3

For the problem L (1) – (2) in the class (3)

$$\dim\{\text{Ker}\,\{L\}\}=1.$$

It follows from the foregoing that for boundary value problem (1) – (2) **the solution uniqueness classes** are defined by the following proposition.

Proposition 1.4

For boundary value problem (1) – (2) the solution uniqueness

fications we get the equality

$$\int_0^{\frac{\sqrt{t}}{2a}}\exp\left(-x^2\right)\text{erfc}\left(\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right)dx=$$

$$=\frac{\sqrt{\pi}}{2}\left(\exp\left(-\frac{5t}{4a^2}\right)\text{erfc}\left(\frac{\sqrt{t}}{a}\right)-\text{erfc}\left(\frac{3\sqrt{t}}{2a}\right)\right).$$

We differentiate the last equality with respect to t from two sides

$$\frac{1}{4a\sqrt{t}}\exp\left(-\frac{t}{4a^2}\right)\lim_{x\to\frac{\sqrt{t}}{2a}}$$

$$\left(\text{erfc}\left(\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right)\right)-$$

$$-\frac{2}{\sqrt{\pi}}\int_0^{\frac{\sqrt{t}}{2a}}\exp\left(-x^2\right)\frac{d}{dt}\left(\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right)\times$$

$$\times\exp\left(-\left(\frac{t}{a\sqrt{t-4a^2x^2}}+\frac{\sqrt{t-4a^2x^2}}{2a}\right)^2\right)dx=$$

$$=\frac{\sqrt{\pi}}{2}\left(\exp\left(-\frac{5t}{4a^2}\right)\left[-\frac{5t}{4a^2}\text{erfc}\left(\frac{\sqrt{t}}{a}\right)-\frac{1}{a\sqrt{\pi t}}\exp\left(-\frac{t}{a^2}\right)\right]+\right.$$

$$\left.+\frac{3}{2a\sqrt{\pi t}}\exp\left(-\frac{9t}{4a^2}\right)\right). \quad (1.2.41)$$

In the first term on the left side of equality (1.2.41) the limit is equal to zero.

Consider the integral $I_2(t)$. Since

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int\limits_0^t \exp\left(-x^2\right)\, dx,$$

then

$$I_2(t) = \frac{1}{a} \int\limits_0^t \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}} \times$$

$$\times \exp\left(-\frac{t\tau}{a^2(t-\tau)} + \frac{\tau}{4a^2}\right) \int\limits_0^{\frac{\sqrt{\tau}}{2a}} \exp\left(-x^2\right)\, dx\, d\tau =$$

$$= \frac{1}{a} \int\limits_0^{\frac{\sqrt{t}}{2a}} \exp\left(-x^2\right) \int\limits_{4a^2x^2}^t \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{t\tau}{a^2(t-\tau)} + \frac{\tau}{4a^2}\right) d\tau\, dx.$$

To calculate the inner integral, we introduce a replacement $z = \sqrt{t-\tau}$. After that, we change the integration order and use a replacement $\xi = \dfrac{t}{az} + \dfrac{z}{2a}$.

Then, similarly as in the integral $I_3(t)$, we get

$$I_2(t) = 2\sqrt{\pi} \exp\left(\frac{9t}{4a^2}\right) \int\limits_0^{\frac{\sqrt{t}}{2a}} \exp\left(-x^2\right) \times$$

$$\times \operatorname{erfc}\left(\frac{t}{a\sqrt{t-4a^2x^2}} + \frac{\sqrt{t-4a^2x^2}}{2a}\right) dx. \qquad (1.2.40)$$

We substitute (1.2.38)–(1.2.40) into (1.2.37). After simpli-

classes are

$$u(x,t) \le C \cdot \gamma_\varepsilon(x,t), \;\; \gamma_\varepsilon(x,t) \ge 2, \;\; \varepsilon_i \ge 0, \;\; \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \ne 0,$$

where

$$\gamma_\varepsilon = \exp\left\{\frac{(1-\varepsilon_1)(t-x)}{a^2}\right\} \times$$

$$\times \max\left[\left(\frac{\sqrt{t}}{t-x}\right)^{1-\varepsilon_2} \exp\left\{-\left(\frac{2t-x}{2a}\right)^2 \cdot \frac{1}{t} - \varepsilon_3 t\right\};\right.$$

$$\left. 1 + \exp\left\{-\frac{(1-\varepsilon_1)(t-x)}{a^2}\right\}\right], \qquad \{x,\, t\} \in G.$$

Analyzing the previous expression for a function $\gamma_\varepsilon(x,t)$, we get:

1. $\gamma_\varepsilon(x,t) \le \gamma(x,t)$, $\{x,t\} \in G$; 2. $\exists G_\varepsilon \subset G$, $\operatorname{meas}\{G_\varepsilon\} > 0 :\; \gamma_\varepsilon(x,t) < \gamma(x,t)$.

A direct verification of obtained solution (26) to singular homogeneous integral Volterra (14) equation was carried out in [30].

Indeed, after substituting the function (26) into equation (14), taking into account the fact that function (26) is a solution of Abel equations (24), it is necessary to show that function (26) satisfies the equation:

$$\frac{1}{2a\sqrt{\pi}} \int\limits_0^t \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{t\tau}{a^2(t-\tau)}\right) \varphi(\tau)\, d\tau = \frac{1}{\sqrt{t}}, \quad (t>0). \qquad (1.2.36)$$

We substitute (26) into (1.2.36):

$$\frac{1}{2a\sqrt{\pi}} \left[\int\limits_0^t \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}} \exp\left(-\frac{t\tau}{a^2(t-\tau)}\right) \frac{1}{\sqrt{\tau}}\, d\tau + \right.$$

$$+\frac{\sqrt{\pi}}{2a}\int\limits_0^t \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left(-\frac{t\tau}{a^2(t-\tau)}\right)\exp\left(\frac{\tau}{4a^2}\right)\ \mathrm{erf}\left(\frac{\sqrt{\tau}}{2a}\right)\ d\tau+$$

$$\left.+\frac{\sqrt{\pi}}{2a}\int\limits_0^t \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left(-\frac{t\tau}{a^2(t-\tau)}\right)\exp\left(\frac{\tau}{4a^2}\right)d\tau\right] = \frac{1}{\sqrt{t}}.$$

Thus, it is necessary to show correctness of the equality

$$\frac{1}{2a\sqrt{\pi}}\left(I_1(t)+I_2(t)+I_1(t)+I_3(t)\right) = \frac{1}{\sqrt{t}},\quad (t>0),\ (1.2.37)$$

where

$$I_1(t) = \int\limits_0^t \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left(-\frac{t\tau}{a^2(t-\tau)}\right)\frac{1}{\sqrt{\tau}}\,d\tau,$$

$$I_2(t) = \frac{\sqrt{\pi}}{2a}\int\limits_0^t \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left(-\frac{t\tau}{a^2(t-\tau)}+\frac{\tau}{4a^2}\right)\ \mathrm{erf}\left(\frac{\sqrt{\tau}}{2a}\right)\ d\tau,$$

$$I_3(t) = \frac{\sqrt{\pi}}{2a}\int\limits_0^t \frac{t+\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left(-\frac{t\tau}{a^2(t-\tau)}+\frac{\tau}{4a^2}\right)d\tau.$$

For the first integral, a replacement $z=\sqrt{\frac{\tau}{t-\tau}}$ is introduced. Then

$$I_1(t) = 2\int\limits_0^\infty \exp\left(-\frac{t}{a^2}z^2\right)\ dz + 2\int\limits_0^\infty \frac{z^2}{z^2+1}\exp\left(-\frac{t}{a^2}z^2\right)\ dz.$$

In $I_1(t)$ the first integral is the Euler-Poisson integral. For the second integral we use formula 3.466(2) [68]. As a result,

we obtain

$$I_1(t) = \frac{2a\sqrt{\pi}}{\sqrt{t}} - \pi\exp\left(\frac{t}{a^2}\right)\ \mathrm{erfc}\left(\frac{\sqrt{t}}{a}\right).(1.2.38)$$

For the integral $I_3(t)$ a replacement $z=\sqrt{t-\tau}$ is introduced. Then

$$I_3(t) = \frac{\sqrt{\pi}}{a}\exp\left(\frac{5t}{4a^2}\right)\int\limits_0^{\sqrt{t}} \frac{2t-z^2}{z^2}\exp\left(-\frac{t^2}{a^2z^2}-\frac{z^2}{4a^2}\right)\ dz.$$

Since

$$\frac{t^2}{a^2z^2}+\frac{z^2}{4a^2} = \left(\frac{t}{az}+\frac{z}{2a}\right)^2 - \frac{t}{a^2},$$

$$\left(\frac{2t}{z^2}-1\right)\ dz = -2a\,d\left(\frac{t}{az}+\frac{z}{2a}\right),$$

then

$$I_3(t) = -2\sqrt{\pi}\exp\left(\frac{9t}{4a^2}\right)\int\limits_0^{\sqrt{t}}\exp\left\{-\left(\frac{t}{az}+\frac{z}{2a}\right)^2\right\}$$

$$d\left(\frac{t}{az}+\frac{z}{2a}\right).$$

After a replacement $\xi=\frac{t}{az}+\frac{z}{2a}$ we get

$$I_3(t) = \pi\exp\left(\frac{9t}{4a^2}\right)\ \mathrm{erf}\left(\frac{3\sqrt{t}}{2a}\right).(1.2.39)$$

With the help of relationships:

$$\tau + t = 2\tau - (\tau - t),\ \frac{(\tau+t)^2}{4a^2(\tau-t)} = \frac{\tau t}{a^2(\tau - t)} + \frac{\tau - t}{4a^2},$$

the study of equation (1.2.52) is reduced to the study of the integral equations:

$$\psi^*(t) - \int\limits_t^\infty k^*(t,\tau)\psi^*(\tau)\,d\tau = 0,\ t > 0, \qquad (1.2.55)$$

where

$$k^*(t,\tau) = \frac{1}{2a\sqrt{\pi}}\left\{\frac{2\tau}{(\tau-t)^{3/2}}\exp\Big\{-\frac{\tau t}{a^2(\tau-t)}\Big\}+\right.$$

$$\left.+\frac{1}{\sqrt{\tau - t}}\Big(1 - \exp\Big\{-\frac{\tau t}{a^2(\tau - t)}\Big\}\Big)\right\},$$

$$\psi^*(t) = \exp\Big\{-\frac{\tau}{4a^2}\Big\}\varphi^*(t), \quad K^*(t,\tau) = k^*(t,\tau)\exp\Big\{-\frac{\tau - t}{4a^2}\Big\}.$$

With replacements $t = \dfrac{1}{t_1}$, $\tau = \dfrac{1}{\tau_1}$ and the notation $y(t_1) = \dfrac{1}{t_1^{3/2}}\psi^*\left(\dfrac{1}{t_1}\right)$ we transform integral equation (1.2.55) to the equation with a difference kernel:

$$t_1 \cdot y_1(t_1) -$$

$$-\frac{1}{2a\sqrt{\pi}}\int\limits_0^{t_1}\frac{1}{(t_1-\tau_1)^{1/2}}\Big(1 - \exp\Big\{-\frac{1}{a^2(t_1-\tau_1)}\Big\}\Big)y(\tau_1)\,d\tau_1 -$$

$$-\frac{t}{a}\int\limits_0^{\sqrt{t}}\frac{1}{z^2\sqrt{t - z^2}}\exp\left(-\frac{t^2}{a^2z^2}\right)dz.$$

After a replacement $y = z^2$ we get

$$J(t) = \frac{5}{4a}\int\limits_0^t y^{-1/2}(t-y)^{-1/2}\exp\left(-\frac{t^2}{a^2y}\right)dy -$$

$$-\frac{t}{2a}\int\limits_0^t y^{-3/2}(t-y)^{-1/2}\exp\left(-\frac{t^2}{a^2y}\right)dy.$$

For the first integral we apply formulas (3.471(2)) and (9.224), for the second integral — (3.471(3)) from [68].

Then the integral $J(t)$ takes the form

$$J(t) = \frac{5\sqrt{\pi}}{4a}\int\limits_{\frac{\sqrt{t}}{a^2}}^\infty u^{-1/2}e^{-u}\,du - \frac{\sqrt{\pi}}{2\sqrt{t}}\exp\left(-\frac{t}{a^2}\right).$$

Using sequentially formulas (3.381(3)) and (8.359(3)) from [68] we get

$$J(t) = \frac{5\pi}{4a}\operatorname{erfc}\left(\frac{\sqrt{t}}{a}\right) - \frac{\sqrt{\pi}}{2\sqrt{t}}\exp\left(-\frac{t}{a^2}\right).$$

The result coincides with the right side of equality (1.2.42).

So, from proved identity (1.2.42) it follows that function (26) satisfies equation (14) and, accordingly, function (27) is a solution to equation (11).

Remark 1.3

Solving the second boundary value problem for the heat equation in a degenerating domain is given in [32].

Theorem 1.3

In the domain $G = \{(x;\ t):\quad t>0,\quad 0<x<t\}$ *the second homogeneous boundary value problem for the heat equation*

$$\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = 0,$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=t} = 0$$

has a solution

$$u(x,t) = \frac{C_1}{2a\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\ \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\}\ \nu(\tau)\,d\tau +$$

$$+\frac{C_1}{2a\sqrt{\pi}}\int_0^t \frac{1}{\sqrt{t-\tau}}\ \exp\left\{-\frac{(x-\tau)^2}{4a^2(t-\tau)}\right\}\times$$

$$\times\left\{\frac{1}{\sqrt{\tau}}\exp\left(-\frac{\tau}{4a^2}\right) + \frac{\sqrt{\pi}}{2a}\mathrm{erf}\left(\frac{\sqrt{\tau}}{2a}\right) + \frac{\sqrt{\pi}}{2a}\right\}\,d\tau + C_2,$$

where

$$\nu(t) = \frac{1}{2a\sqrt{\pi}}\int_0^t \frac{\tau}{(t-\tau)^{\frac{3}{2}}}\ \exp\left\{-\frac{\tau^2}{4a^2(t-\tau)}\right\}\times$$

$$\times\left\{\frac{1}{\sqrt{\tau}}\exp\left(-\frac{\tau}{4a^2}\right) + \frac{\sqrt{\pi}}{2a}\mathrm{erf}\left(\frac{\sqrt{\tau}}{2a}\right) + \frac{\sqrt{\pi}}{2a}\right\}\,d\tau.$$

1.2.2 To the solution of an adjoint boundary value problem. Main result

Statement of the problem

We consider an adjoint boundary value problem of heat conduction L^*.

In the domain $G = \{(x;\ t):\quad t>0,\quad 0<x<t\}$ *to find*

Then we obtain

$$\int_t^\infty K^*(t,\,\tau)d\tau =$$

$$= -\frac{2}{\sqrt{\pi}}\int_0^\infty \exp\left\{-\left(\frac{t}{ax}-\frac{x}{2a}\right)^2 - \frac{2t}{a^2}\right\} d\left(\frac{t}{ax}-\frac{x}{2a}\right) +$$

$$+\frac{2}{\sqrt{\pi}}\int_0^\infty \exp\left\{-\frac{\tau-t}{4a^2}\right\} d\left(\frac{\sqrt{\tau-t}}{2a}\right) =$$

$$= \frac{2\exp\{-2t/a^2\}}{\sqrt{\pi}}\int_{-\infty}^\infty \exp\{-y^2\}dy + \frac{2}{\sqrt{\pi}}\int_{-\infty}^\infty \exp\{-y^2\}dy =$$

$$= 2\exp\{-2t/a^2\} + 1,\ t>0. \quad (1.2.54)$$

From (1.2.54) it follows that $K^*(t,\,\tau)$ (1.2.53) has properties:

1) $K^*(t,\,\tau) > 0$ and continuously at $0<\tau<t<t<\infty$, и $\forall t>0\ K(t,\tau)\in L_1(\mathbb{R}_+)$;

2) the integral $\int_t^\infty K^*(t,\,\tau)\,d\tau$ is a strictly decreasing function in the variable t at $(0,\infty)$;

3) $\lim\limits_{t\to+\infty}\int_t^\infty K^*(t,\,\tau)\,d\tau = 1$;

4) $\lim\limits_{t\to 0}\int_t^\infty K^*(t,\,\tau)\,d\tau = 3$.

Investigation of integral equation(1.2.52)

Singularity of the equation under study is the property 3) of the kernel $K(t,\,\tau)$ and is expressed in the fact that the corresponding nonhomogeneous equation cannot be solved by method of successive approximations.

$$= \frac{2a\sqrt{\pi}(\tau_1+t)}{\tau_1 t(\tau_1-t)^{\frac{3}{2}}} \exp\left\{-\frac{(\tau_1+t)^2}{4a^2(\tau_1-t)}\right\},$$

i.e.

$$I^*(t,\tau_1) = \frac{2a\sqrt{\pi}(\tau_1+t)}{\tau_1 t(\tau_1-t)^{\frac{3}{2}}} \exp\left\{-\frac{(\tau_1+t)^2}{4a^2(\tau_1-t)}\right\}. \qquad (1.2.50)$$

Substituting (1.2.50) into (1.2.49), we have:

$$J^*(t) = \frac{1}{2a\sqrt{\pi}} \int_t^\infty \frac{\tau_1+t}{(\tau_1-t)^{\frac{3}{2}}} \exp\left\{-\frac{(\tau_1+t)^2}{4a^2(\tau_1-t)}\right\} \varphi^*(\tau_1)\, d\tau_1. \qquad (1.2.51)$$

Taking into account (1.2.49) and (1.2.51), (1.2.48) can be rewritten in the form:

$$\varphi^*(t) - \int_t^\infty K^*(t,\tau)\varphi^*(\tau)\, d\tau = 0,\ \ t>0, \qquad (1.2.52)$$

where

$$K^*(t,\,\tau) = \frac{1}{2\,a\sqrt{\pi}}\left[\frac{\tau+t}{(\tau-t)^{\frac{3}{2}}}\ \exp\left\{-\frac{(\tau+t)^2}{4a^2(\tau-t)}\right\} + \right.$$
$$\left. + \frac{1}{(\tau-t)^{\frac{1}{2}}}\ \exp\left\{-\frac{\tau-t}{4a^2}\right\}\right]. \qquad (1.2.53)$$

Let's calculate the integral from the function $K^*(t,\tau)$ (1.2.53).

We introduce a replacement

$$\left\| \begin{array}{l} x = \sqrt{\tau - t};\ \ \dfrac{\tau+t}{4a\,(\tau-t)^{\frac{3}{2}}}d\tau = -d\left(\dfrac{t}{ax} - \dfrac{x}{2a}\right); \\ \dfrac{(\tau+t)^2}{4a^2(\tau-t)} = \left(\dfrac{t}{ax} - \dfrac{x}{2a}\right)^2 + \dfrac{2t}{a^2}. \end{array} \right\|$$

a solution to the adjoint boundary value problem for the equation

$$-\frac{\partial u^*}{\partial t} = a^2 \frac{\partial^2 u^*}{\partial x^2}, \qquad (1.2.43)$$

with the boundary conditions:

$$u^*(x,\ t)|_{t=\infty} = 0,\ \ u^*(x,\ t)|_{x=0} = 0,\ \ u^*(x,\ t)|_{x=t} = 0. \qquad (1.2.44)$$

Reducing the problem to an integral equation

Similar to direct problem a solution of problem (1.2.43)–(1.2.44) we are looking for as a sum of potentials of the double layer:

$$u^*(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int_t^\infty \frac{-x}{(\tau-t)^{\frac{3}{2}}} \exp\left\{-\frac{x^2}{4a^2(\tau-t)}\right\} \nu^*(\tau)\, d\tau +$$
$$+ \frac{1}{4a^3\sqrt{\pi}} \int_t^\infty \frac{\tau-x}{(\tau-t)^{\frac{3}{2}}} \exp\left\{-\frac{(\tau-x)^2}{4a^2(\tau-t)}\right\} \varphi^*(\tau)\, d\tau. \qquad (1.2.45)$$

Using conditions (1.2.48) and properties of thermal potentials, we have the following system of integral equations relative to unknown densities $\nu^*(t)$ and $\varphi^*(t)$ [34]:

$$\left\{ \begin{array}{l} \dfrac{\nu^*(t)}{2a^2} = -\dfrac{1}{4\,a^3\sqrt{\pi}} \displaystyle\int_t^\infty \dfrac{\tau}{(\tau-t)^{\frac{3}{2}}}\ \exp\left\{-\dfrac{\tau^2}{4a^2(\tau-t)}\right\}\varphi^*(\tau)\, d\tau, \\ \dfrac{\varphi^*(t)}{2a^2} = \dfrac{1}{4\,a^3\sqrt{\pi}} \displaystyle\int_t^\infty \dfrac{1}{(\tau-t)^{\frac{1}{2}}}\ \exp\left\{-\dfrac{\tau-t}{4a^2}\right\}\ \varphi^*(\tau)\, d\tau - \\ \quad -\dfrac{1}{4a^3\sqrt{\pi}} \displaystyle\int_t^\infty \dfrac{t}{(\tau-t)^{\frac{3}{2}}}\ \exp\left\{-\dfrac{t^2}{4a^2(\tau-t)}\right\}\ \nu^*(\tau)\, d\tau. \end{array} \right. \qquad (1.2.46)$$

Excluding from system (1.2.46) $\nu^*(t)$, we find:

$$\nu^*(t) = -\frac{1}{2a\sqrt{\pi}}\int\limits_t^\infty \frac{\tau}{(\tau-t)^{\frac{3}{2}}}\ \exp\left\{-\frac{\tau^2}{4a^2(\tau-t)}\right\}\ \varphi^*(\tau)\,d\tau, \tag{1.2.47}$$

$$0 = -\frac{\varphi^*(t)}{2a^2} + \frac{1}{4\,a^3\sqrt{\pi}}\int\limits_t^\infty \frac{1}{(\tau-t)^{\frac{1}{2}}}\ \exp\left\{-\frac{\tau-t}{4a^2}\right\}\ \varphi^*(\tau)\,d\tau +$$
$$+\frac{1}{8a^4\,\pi}\int\limits_t^\infty \frac{t}{(\tau-t)^{\frac{3}{2}}}\ \exp\left\{-\frac{t^2}{4a^2(\tau-t)}\right\}\times$$
$$\times\int\limits_t^\infty \frac{\tau_1}{(\tau_1-\tau)^{\frac{3}{2}}}\ \exp\left\{-\frac{\tau_1^2}{4a^2(\tau_1-\tau)}\right\}\ \varphi^*(\tau_1)\,d\tau_1\,d\tau. \tag{1.2.48}$$

We introduce the following notation:

$$J^*(t) = \frac{1}{4a^2\,\pi}\int\limits_t^\infty t\tau_1\varphi^*(\tau_1)\left[\int\limits_t^{\tau_1}\frac{1}{(\tau-t)^{\frac{3}{2}}\,(\tau_1-\tau)^{\frac{3}{2}}}\times\right.$$
$$\left.\times\exp\left\{-\frac{t^2}{4a^2(\tau-t)} - \frac{\tau_1^2}{4a^2(\tau_1-t)}\right\}d\tau\right]d\tau_1. \tag{1.2.49}$$

Using substitutions:

$$\left\|\begin{array}{l} z = \sqrt{\dfrac{\tau-t}{\tau_1-\tau}};\ \tau = \dfrac{z^2\tau_1+t}{z^2+1};\ d\tau = \dfrac{2z(\tau_1-t)dz}{(z^2+1)^2}; \\ \tau - t = \dfrac{z^2(\tau_1-t)}{z^2+1};\ \tau_1-\tau = \dfrac{\tau_1-t}{z^2+1}; \end{array}\right\|$$

and a tabular integral

$$\int\limits_0^\infty \exp\{-\mu z^2 - \eta/z^2\}dz = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}\exp\{-2\sqrt{\mu\eta}\},\ \ \mu>0,\ \ \eta>0,$$

for the inner integral from (1.2.49), we get

$$I^*(t,\tau_1) = \int\limits_t^{\tau_1}\frac{1}{(\tau-t)^{\frac{3}{2}}\,(\tau_1-\tau)^{\frac{3}{2}}}\times$$
$$\times\exp\left\{-\frac{t^2}{4a^2(\tau-t)} - \frac{\tau_1^2}{4a^2(\tau_1-t)}\right\}d\tau =$$
$$= \int\limits_0^\infty \frac{2(z^2+1)}{z^2(\tau_1-t)^2}\exp\left\{-\frac{t^2(z^2+1)}{4a^2z^2(\tau_1-t)} - \frac{\tau_1^2(z^2+1)}{4a^2(\tau_1-t)}\right\}dz =$$

$$= \left\|\begin{array}{l} A(t,\tau_1) = \dfrac{2}{(\tau_1-t)^2}\exp\left\{-\dfrac{\tau_1^2+t^2}{4a^2(\tau_1-t)}\right\}; \\ \alpha(\tau_1,t) = \dfrac{\tau_1^2}{4a^2(\tau_1-t)};\ \beta(\tau_1,t) = \dfrac{t^2}{4a^2(\tau_1-t)} \end{array}\right\| =$$
$$= A(\tau_1,t)\int\limits_0^\infty \frac{z^2+1}{z^2}\exp\{-\alpha(\tau_1,t)z^2 - \beta(\tau_1,t)/z^2\}dz =$$
$$= A(\tau_1,t)\int\limits_0^\infty \exp\{-\alpha(\tau_1,t)z^2 - \beta(\tau_1,t)/z^2\}dz +$$
$$+ A(\tau_1,t)\int\limits_0^\infty \exp\{-\beta(\tau_1,t)z^2 - \alpha(\tau_1,t)/z^2\}dz =$$

So, from (1.2.69), (1.2.71), and (1.2.74) we finally have:

$$u^*_{hom}(x,t) = \frac{C}{4a^3\sqrt{\pi}} \sum_{n=0}^{\infty} \int\limits_t^{\infty} \left[\frac{x-\tau}{(\tau-t)^{3/2}} \exp\left\{ -\frac{(x-\tau)^2}{4a^2(\tau-t)} \right\} + \right.$$

$$\left. + \frac{x+\tau}{(\tau-t)^{3/2}} \exp\left\{ -\frac{(x+\tau)^2}{4a^2(\tau-t)} \right\} \right] \varphi^*_n(\tau)\, d\tau =$$

$$= C \sum_{n=0}^{\infty} u^*_{hom,n}(x,t). \qquad (1.2.75)$$

Each summand $\varphi^*_n(t)$, $n = 0, 1, 2, ...,$ of series (1.2.65), and the kernel of integral expression (1.2.75) are non-negative. Then for breaking the condition

$$\exp\left\{ \frac{t-x}{a^2} \right\} \cdot \gamma(x,t) \cdot u^*(x,t) \in L_1(G),$$

i.e., for breaking the condition (1.2.66), it is sufficient a violation of this condition for a single summand $u^*_{hom,n}(x,t)$ of series (1.2.75) that represents the solution $u^*_{hom}(x,t)$ (1.2.75) to initial homogeneous boundary problem (1.2.43)–(1.2.44). Let us show this for a summand with an index 0, corresponding to the first summand of series (1.2.65). This summand is a constant. Thus, taking into account the inequality $\gamma(x,t) \geq 2$, the replacement

$$z = \frac{\sqrt{\tau-t}}{2a} - \frac{t-x}{2a\sqrt{\tau-t}}$$

and the representation

$$\frac{\tau-x}{(\tau-t)^{3/2}}\, d\tau = \frac{(\tau-t)+(t-x)}{(\tau-t)^{3/2}}\, d\tau =$$

$$-t_1 \cdot \frac{1}{2a\sqrt{\pi}} \int\limits_0^{t_1} \frac{2}{(t_1-\tau_1)^{3/2}} \exp\left\{ -\frac{1}{a^2(t_1-\tau_1)} \right\} y(\tau_1)\, d\tau_1 = 0. \qquad (1.2.56)$$

Applying the Laplace transform to (1.2.56), we get

$$-\bar{y}'(p) - \frac{1}{2a\sqrt{p}} \Big(1 - \exp\Big(-\frac{2\sqrt{p}}{a} \Big) \Big) \bar{y}(p) +$$

$$+ \frac{d}{dp} \Big\{ \exp\Big(-\frac{2\sqrt{p}}{a} \Big) \bar{y}(p) \Big\} = 0,$$

i.e., we have:

$$\bar{y}'(p) + \frac{1}{2a\sqrt{p}} \cdot \frac{\mathrm{ch}\frac{\sqrt{p}}{a}}{\mathrm{sh}\frac{\sqrt{p}}{a}} \bar{y}(p) = 0. \qquad (1.2.57)$$

The general solution of differential equation (1.2.57) is determined by the following formula:

$$\bar{y}(p) = \frac{C}{sh\frac{\sqrt{p}}{a}}, \quad C = \text{const}.$$

To find the original of this function, we rewrite it as a series:

$$\bar{y}(p) = 2C \sum_{n=0}^{\infty} \exp\Big(-\frac{(2n+1)\sqrt{p}}{a} \Big). \qquad (1.2.58)$$

Applying the inverse Laplace transform to (1.2.58), we have:

$$y(t_1) = \frac{C}{a\sqrt{\pi} t_1^{3/2}} \sum_{n=0}^{\infty} (2n+1) \exp\Big(-\frac{(2n+1)^2}{4a^2 t_1} \Big). \qquad (1.2.59)$$

By virtue of the reverse substitutions $t = \frac{1}{t_1}$, $\tau = \frac{1}{\tau_1}$ and the notation $y(t_1) = \frac{1}{t_1^{3/2}}\psi^*\left(\frac{1}{t_1}\right)$ equality (1.2.58) takes the form:

$$\psi^*(t) = \frac{C}{a\sqrt{\pi}}\sum_{n=0}^{\infty}(2n+1)\exp\Big(-\frac{(2n+1)^2}{4a^2}t\Big). \qquad (1.2.60)$$

Thus, formula (1.2.60) determines the solution of homogeneous equations (1.2.55).

Direct verification of obtained solution (1.2.60) to singular Volterra homogeneous integral equation (1.2.55) we have made in [30]. Indeed, substitute function (1.2.60) into equation (1.2.55):

$$\psi^*(t) = \frac{C}{2a^2\pi}\sum_{n=0}^{\infty}(2n+1)\ \left(I_1(n,t) + I_2(n,t) - I_3(n,t)\right), \qquad (1.2.61)$$

where

$$I_1(n,t) = 2\exp\left(-\frac{t}{a^2}\right)\int_t^{\infty}\frac{\tau}{(\tau-t)^{3/2}}\exp\left(-\frac{t^2}{a^2(\tau-t)}\right)\times$$

$$\times\exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right)\,d\tau,$$

$$I_2(n,t) =$$

$$= \int_t^{\infty}\frac{1}{(\tau-t)^{1/2}}\exp\left(-\frac{t^2}{a^2(\tau-t)}\right)\exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right)\,d\tau,$$

$$I_3(n,t) = \exp\left(-\frac{t}{a^2}\right)\int_t^{\infty}\frac{\tau}{(\tau-t)^{1/2}}\exp\left(-\frac{t^2}{a^2(\tau-t)}\right)\times$$

Furthermore, using the replacement:

$$z = \sqrt{\frac{\tau-t}{\theta-\tau}},$$

we get

$$I(x,t,\theta) = \frac{2}{(\theta-t)^2}\exp\left\{-\frac{x^2+\theta^2}{4a^2(\theta-t)}\right\}\times$$

$$\times\left[\int_0^{\infty}\exp\left(-\frac{\theta^2}{4a^2(\theta-t)}\cdot z^2 - \frac{x^2}{4a^2(\theta-t)}\cdot\frac{1}{z^2}\right)dz +\right.$$

$$\left.+\int_0^{\infty}\exp\left(-\frac{\theta^2}{4a^2(\theta-t)}\cdot z^2 - \frac{x^2}{4a^2(\theta-t)}\cdot\frac{1}{z^2}\right)\frac{dz}{z^2}\right] =$$

$$= \frac{2a\sqrt{\pi}(x+\theta)}{(\theta-t)^{3/2}x\theta}\exp\left\{-\frac{(x+\theta)^2}{4a^2(\theta-t)}\right\}. \qquad (1.2.73)$$

Here, calculating improper integrals we use formula (3.472(3)) from [68, c.355]:

$$\int_0^{\infty}\exp\left\{-\mu y^2 - \frac{\eta}{y^2}\right\} = \frac{\sqrt{\pi}}{2\sqrt{\mu}}\exp\{-2\sqrt{\mu\eta}\}.$$

Substituting expression (1.2.73) into integral (1.2.72), we get

$$u_{n1}^*(x,t) = \frac{1}{4a^3\sqrt{\pi}}\times$$

$$\times\int_t^{\infty}\frac{x+\tau}{(\tau-t)^{3/2}}\exp\left\{-\frac{(x+\tau)^2}{4a^2(\tau-t)}\right\}\varphi_n^*(\tau)\,d\tau. \qquad (1.2.74)$$

tation (1.2.45)):

$$u^*_{hom}(x,t) = C\left[\sum_{n=0}^{\infty} u^*_{n1}(x,t) + \sum_{n=0}^{\infty} u^*_{n2}(x,t)\right], \qquad (1.2.69)$$

where

$$u^*_{n1}(x,t) =$$

$$= \frac{1}{4a^3\sqrt{\pi}} \int_t^{\infty} \frac{-x}{(\tau-t)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(\tau-t)}\right\} \nu^*_n(\tau)\, d\tau, \quad (1.2.70)$$

$$u^*_{n2}(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int_t^{\infty} \frac{\tau-x}{(\tau-t)^{3/2}} \exp\left\{-\frac{(\tau-x)^2}{4a^2(\tau-t)}\right\} \varphi^*_n(\tau)\, d\tau. \qquad (1.2.71)$$

Substituting (1.2.68) into (1.2.70), we have:

$$u^*_{n1}(x,t) = \frac{1}{4a^3\sqrt{\pi}} \int_t^{\infty} \frac{x}{(\tau-t)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(\tau-t)}\right\} \times$$

$$\times \frac{1}{2a\sqrt{\pi}} \int_{\tau}^{\infty} \frac{\theta}{(\theta-\tau)^{3/2}} \exp\left\{-\frac{\theta^2}{4a^2(\tau-\theta)}\right\} \varphi^*_n(\theta) d\,\theta d\,\tau =$$

$$= \frac{1}{8a^4\pi} \int_t^{\infty} x\theta\varphi^*_n(\theta) I(x,t,\theta) d\,\theta, \qquad (1.2.72)$$

where

$$I(x,t,\theta) =$$

$$= \int_t^{\theta} \frac{1}{(\tau-t)^{3/2}(\theta-\tau)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(\tau-t)} - \frac{\theta^2}{4a^2(\theta-\tau)}\right\} d\,\tau.$$

$$\times \exp\left(-\frac{(2n+1)^2}{4a^2}\tau\right) d\tau.$$

After the replacement $\sqrt{\tau-t}$ the integral $I_2(n,t)$ takes the form

$$I_2(n,t) = \frac{2a\sqrt{\pi}}{2n+1} \exp\left(-\frac{(2n+1)^2}{4a^2}t\right). \qquad (1.2.62)$$

For the integral $I_1(n,t)$ we have

$$I_1(n,t) = 2\exp\left(-\frac{t}{a^2}\right) \exp\left(-\frac{(2n+1)^2}{4a^2}t\right) \left[\int_t^{\infty} \frac{1}{(\tau-t)^{1/2}} \times\right.$$

$$\times \exp\left(-\frac{t^2}{a^2(\tau-t)}\right) \exp\left(-\frac{(2n+1)^2}{4a^2}(\tau-t)\right) d\tau +$$

$$+t \int_t^{\infty} \frac{1}{(\tau-t)^{3/2}} \exp\left(-\frac{t^2}{a^2(\tau-t)}\right)$$

$$\left.\exp\left(-\frac{(2n+1)^2}{4a^2}(\tau-t)\right) d\tau\right].$$

After replacing $\sqrt{\tau-t}$ and applying the known ratios:

$$\int_0^{\infty} \exp\{-\mu x^2 - \eta/x^2\} dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}} \exp\{-2\sqrt{\mu\eta}\}, \quad \mu>0, \ \eta>0;$$

$$\int_0^{\infty} \exp\{-\mu x^2 - \eta/x^2\} \frac{dx}{x^2} = \frac{1}{2}\sqrt{\frac{\pi}{\eta}} \exp\{-2\sqrt{\mu\eta}\}, \quad \mu>0, \ \eta>0,$$

the integral $I_1(n,t)$ takes the form

$$I_1(n,t) = \frac{2a\sqrt{\pi}(2n+3)}{2n+1} \exp\left(-\frac{(2n+3)^2}{4a^2}t\right). \qquad (1.2.63)$$

We get in the same way:

$$I_3(n,t) = \frac{2a\sqrt{\pi}}{2n+1} \exp\left(-\frac{(2n+3)^2}{4a^2}t\right). \qquad (1.2.64)$$

After substitution (1.2.62)–(1.2.64) in (1.2.61) we'll get it:

$$\psi^*(t) = \frac{C}{a\sqrt{\pi}} \sum_{n=0}^{\infty} \left\{ (2n+2) \exp\left(-\frac{(2n+3)^2}{4a^2}t\right) + \exp\left(-\frac{(2n+1)^2}{4a^2}t\right) \right\} =$$

$$= \frac{C}{a\sqrt{\pi}} \left(\exp\left(-\frac{t}{4a^2}\right) + \sum_{n=0}^{\infty}(2n+3) \exp\left(-\frac{(2n+3)^2}{4a^2}t\right) \right) =$$

$$= \frac{C}{a\sqrt{\pi}} \sum_{n=-1}^{\infty} (2n+3) \exp\left(-\frac{(2n+3)^2}{4a^2}t\right) =$$

$$= \frac{C}{a\sqrt{\pi}} \sum_{n=0}^{\infty} (2n+1) \exp\left(-\frac{(2n+1)^2}{4a^2}t\right).$$

Thus, function $\psi^*(t)$ (1.2.60) satisfies equation (1.2.55) and, by virtue of Remark 1.1, the function

$$\varphi^*(t) = \frac{C}{a\sqrt{\pi}} \sum_{n=0}^{\infty} (2n+1) \exp\left(-\frac{n^2+n}{a^2}t\right) = C \sum_{n=0}^{\infty} \varphi_n^*(t) \qquad (1.2.65)$$

is a solution to full equation (1.2.52).

The solution $u^*(x,t)$ of adjoint boundary problem (1.2.43)–

(1.2.44) is defined by function $\varphi^*(t)$ (1.2.65). This solution $u^*(x,t)$ does not belong to the class, that is conjugate to class (3) of solutions to the direct boundary value problem:

$$\exp\left\{-\frac{t-x}{a^2}\right\} \cdot [\gamma(x,t)]^{-1} \cdot u(x,t) \in L_\infty(G),$$

where

$$\gamma(x,t) =$$

$$= \max\left[\frac{\sqrt{t}}{t-x} \exp\left\{-\left(\frac{2t-x}{2a}\right)^2 \cdot \frac{1}{t}\right\};\ 1 + \exp\left\{-\frac{t-x}{a^2}\right\}\right],$$

$$\{x,\, t\} \in G,$$

i.e.

$$\exp\left\{\frac{t-x}{a^2}\right\} \cdot \gamma(x,t) \cdot u^*(x,t) \notin L_1(G). \qquad (1.2.66)$$

Indeed, the last relation is valid. First of all, according to formula (1.2.47) we define the function $\nu^*(t)$:

$$\nu^*(t) = C \sum_{n=0}^{\infty} \nu_n^*(t), \qquad (1.2.67)$$

where

$$\nu_n^*(t) =$$

$$= -\frac{1}{2a\sqrt{\pi}} \int_t^{\infty} \frac{\tau}{(\tau-t)^{\frac{3}{2}}} \exp\left\{-\frac{\tau^2}{4a^2(\tau-t)}\right\} \varphi_n^*(\tau)\,d\tau. \qquad (1.2.68)$$

Then the solution to initial boundary value problem (1.2.43) –(1.2.44) is determined by the formula (see represen-

To study equation (2.2.2) we isolate its characteristic part, namely

$$\tilde{\varphi}(t) - \lambda \int_0^t k_0(t,\tau)\,\tilde{\varphi}(\tau)\,d\tau = f_1(t), \qquad (2.2.4)$$

where

$$k_0(t,\tau) = \frac{t}{a\sqrt{\pi}\,(t-\tau)^{3/2}} \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\},$$

$$f_1(t) = \tilde{f}(t) + \lambda \int_0^t k_h(t,\tau)\tilde{\varphi}(\tau)\,d\tau, \qquad (2.2.5)$$

where

$$k_h(t,\tau) = \frac{1}{2a\sqrt{\pi}\,(t-\tau)^{1/2}} \left(1 - \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right).$$

Equation (2.2.4) is a characteristic equation for the equation (2.2.1), because

$$\lim_{t\to 0} \int_0^t k_0(t,\tau)\,d\tau = 1; \quad \lim_{t\to 0} \int_0^t k_h(t,\tau)\,d\tau = 0.$$

Assume that the right side of equation (2.2.4) is known. We find its solution, i.e., a solution of characteristic equation (2.2.4).

Similarly [16] integral equation (2.2.4) we reduce to the equation with a difference kernel. For this aim we make replacements:

$$t = \frac{1}{y}; \quad \tau = \frac{1}{x}; \quad \psi(y) = \frac{1}{\sqrt{y}}\tilde{\varphi}\left(\frac{1}{y}\right); \quad f_2(y) = \frac{1}{\sqrt{y}} f_1\left(\frac{1}{y}\right). \qquad (2.2.6)$$

$$= \left(\frac{1}{\sqrt{\tau - t}} + \frac{t-x}{(\tau-t)^{3/2}}\right) d\tau = 2\cdot d\left(\sqrt{\tau - t} - \frac{t-x}{\sqrt{\tau - t}}\right);$$

$$\frac{(\tau - x)^2}{4a^2(\tau - t)} = \frac{((\tau - t) + (t-x))^2}{4a^2(\tau - t)} =$$

$$= \left(\frac{\sqrt{\tau - t}}{2a} - \frac{t-x}{2a\sqrt{\tau - t}}\right)^2 + \frac{t-x}{a^2}$$

for the inner integral, we get:

$$\int_0^\infty \int_0^t \gamma(x,t)\cdot u^*_{hom,0}(x,t)\,dx\,dt \geq$$

$$\geq \frac{C}{2a^4\pi} \int_0^\infty \int_0^t \int_t^\infty \left[\frac{x+\tau}{(\tau-t)^{3/2}} \exp\left\{-\frac{(x+\tau)^2}{4a^2(\tau-t)}\right\} + \right.$$

$$\left. + \frac{x-\tau}{(\tau-t)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(\tau-t)}\right\}\right] d\tau\,dx\,dt \geq$$

$$\geq \frac{C}{2a^4\pi} \int_0^\infty \int_0^t \int_t^\infty \frac{x-\tau}{(\tau-t)^{3/2}} \exp\left\{-\frac{(x-\tau)^2}{4a^2(\tau-t)}\right\} d\tau\,dx\,dt =$$

$$= \frac{2C}{a^3\sqrt{\pi}} \int_0^\infty \int_0^t \exp\left\{-\frac{x-t}{a^2}\right\} dx\,dt = +\infty.$$

So, homogeneous boundary problem L^* (1.2.43) – (1.2.44) in the class, conjugated with class (3), has only a trivial solution.

The following proposition is established

Proposition 1.5

For problem L^* (1) – (2) *in a class that is conjugated to class* (3),

$$\dim\{\mathrm{Ker}\,\{L^*\}\} = 0.$$

From Propositions 1.3 and 1.5 it follows **the main result** of the section:

Theorem 1.4

Boundary value problem L (1) – (2) *is a Noetherian problem, i.e.,*

$$\operatorname{ind}\{L\} = \dim\{\operatorname{Ker}\{L\}\} - \dim\{\operatorname{Coker}\{L\}\} = 1.$$

The results presented in this section are published in [29], [30].

Thus, by using the Carleman-Vecua regularization method for the singular Volterra integral equation of the second kind, a solution has been found in a closed form. Also in the angular domain we find a solution of the first boundary value problem (direct problem); and we show the existence of the trivial solution to the corresponding conjugate boundary problem in the given class. It is established that the problem is Noetherian. These results are developed for the second boundary value problem in a degenerating domain.

equation (2.1.1) rewrite in another form

$$\varphi(t) - \int_0^t \frac{1}{2a\sqrt{\pi}} \left\{ \frac{2t}{(t-\tau)^{3/2}} \exp\left\{ -\frac{t\tau}{a^2(t-\tau)} \right\} - \right.$$

$$- \frac{1}{(t-\tau)^{1/2}} \exp\left\{ -\frac{t\tau}{a^2(t-\tau)} \right\} +$$

$$\left. + \frac{1}{(t-\tau)^{1/2}} \right\} \cdot \exp\left\{ -\frac{t-\tau}{4a^2} \right\} \varphi(\tau)\, d\tau = f(t). \qquad (2.2.1)$$

From [65, **?**] we have that it is enough to find a solution to the "simplified" equation,

$$\tilde{\varphi}(t) - \lambda \int_0^t k(t,\tau)\, \tilde{\varphi}(\tau)\, d\tau = \tilde{f}(t), \qquad (2.2.2)$$

$$\tilde{\varphi}(t) = \exp\left\{t/(4a^2)\right\} \varphi(t),$$

$$\tilde{f}(t) = \exp\left\{t/(4a^2)\right\} f(t),$$

where

$$k(t,\tau) = \frac{1}{2a\sqrt{\pi}} \left\{ \frac{2t}{(t-\tau)^{3/2}} \exp\left\{ -\frac{t\tau}{a^2(t-\tau)} \right\} \right.$$

$$\left. + \frac{1}{(t-\tau)^{1/2}} \left(1 - \exp\left\{ -\frac{t\tau}{a^2(t-\tau)} \right\} \right) \right\},$$

Classes for "simplified"equation (2.2.2) are:

$$\begin{cases} \sqrt{t} \cdot \exp\left\{-t/(4a^2)\right\} \cdot \tilde{\varphi}(t) \in L_\infty(0,\infty), \\ \sqrt{t} \cdot \exp\left\{-t/(4a^2)\right\} \cdot \tilde{f}(t) \in L_\infty(0,\infty), \\ \sqrt{t} \cdot \exp\left\{-(t-\tau)/(4a^2)\right\} \cdot k(t,\tau) \in L_1(0,\infty). \end{cases} \qquad (2.2.3)$$

integral:

$$\frac{1}{a\sqrt{\pi}}\int\limits_0^{\sqrt{t}}\frac{dx}{\sqrt{t-x^2}}=\frac{1}{2a\sqrt{\pi}}\int\limits_0^{\sqrt{t}}\frac{dy}{\sqrt{y(t-y)}}=\frac{\sqrt{\pi}}{2a}.$$

Problem. *Find in a class* $\sqrt{t}\varphi(t)\in L_\infty(0,\infty)$ *a solution* $\varphi(t)$ *of integral equation (21) for any given function* $\sqrt{t}f(t)\in L_\infty(0,\infty)$ *and any given complex parameter* $\lambda\in\mathbb{C}\diagdown\{|\lambda|<\exp(|\arg\lambda|)\}$, $\arg\lambda\in[-\pi;\,\pi]$.

Note that integral equations of the form (2.1.1) arise when studying boundary value problems of the heat equation in an infinite angular domain degenerating at the initial moment of time. Such equations are called by us Volterra integral equations with "incompressible" kernel. The singularity of the studied equation is in property 3) for the kernel $K(t,\tau)$ and is expressed in the fact that the corresponding nonhomogeneous equation cannot be solved by the method of successive approximations at $|\lambda|>1$. Obviously, if $|\lambda|<1$, then equation (2.1.1) has a unique solution that is found by the method of successive approximations. The case $|\lambda|=1$ is considered in papers [29], [30], [32], [34] and in previous section, where it is shown that equation (2.1.1) has one non-trivial solution for $f(t)\equiv 0$ (up to a constant factor). Therefore, further we assume that $|\lambda|>1$.

2.2 Characteristic equation

We use the Carleman-Vekua regularization method [66], [67]. To do this, we transform equation (2.1.1). Using relations:

$$t+\tau=2t-(t-\tau),\qquad \frac{(t+\tau)^2}{4a^2\,(t-\tau)}=\frac{t\tau}{a^2\,(t-\tau)}+\frac{t-\tau}{4a^2},$$

2 Singular Volterra integral equation of the second kind with spectral parameter

The mathematical description of the thermal processes accompanying bridge erosion leads to solving the boundary value problems for the heat equation in domains with a moving boundary, namely, in domains degenerating to a point at the initial moment of time. Using the apparatus of thermal potentials, the solution of the problems under consideration is reduced to the study singular Volterra integral equations of the second kind, when the norm integral operator is equal to one. Singularity of these equations lies in the "incompressibility"of the kernel, and this singularity indicates inapplicability of classic methods for solving. Furthermore, we assume that $|\lambda|>1$. It is shown that the corresponding homogeneous equation at

$$|\lambda|>\exp(|\arg\lambda|,\quad \arg\lambda\in[-\pi;\,\pi])$$

has a continuous spectrum, and the multiplicity of characteristic numbers grows with increasing $|\lambda|$. Using the Carleman-Vekua method the singular Volterra integral equation is reduced to the Abel equation. Eigenfunctions of the equation are found in the explicit form.

Similar integral equations also arise in the study of spectrally loaded heat equations.

2.1 Statement of the problem

When solving model problems for parabolic equations in domains with a moving boundary, singular integral equations of the following form arise

$$\varphi(t)-\lambda\int\limits_0^t K(t,\tau)\varphi(\tau)\,d\tau=f(t),\quad t>0, \tag{2.1.1}$$

where

$$K(t,\tau)=\frac{1}{2a\sqrt{\pi}}\left\{\frac{t+\tau}{(t-\tau)^{3/2}}\exp\left(-\frac{(t+\tau)^2}{4a^2(t-\tau)}\right)+\right.$$

$$\left.+\frac{1}{(t-\tau)^{1/2}}\exp\left(-\frac{t-\tau}{4a^2}\right)\right\}.$$

The kernel K$(t,\,\tau)$ has properties:
1) $K(t,\,\tau)>0$ and continuously at $0<\tau<t<+\infty$;
2) $\lim\limits_{t\to t_0}\int\limits_{t_0}^{t}K(t,\,\tau)\,d\tau=0,\ t_0\geq\varepsilon>0;$
3) $\lim\limits_{t\to 0}\int\limits_{0}^{t}K(t,\,\tau)\,d\tau=1,\ \lim\limits_{t\to+\infty}\int\limits_{0}^{t}K(t,\,\tau)\,d\tau=1.$

To prove the property 3) we introduce a replacement $x=\sqrt{t-\tau}$. We have

$$\int\limits_0^t K(t,\tau)\,d\tau=-\frac{2}{\sqrt{\pi}}\exp\left\{\frac{2t}{a^2}\right\}\times$$

$$\times\int\limits_0^{\sqrt{t}}\exp\left\{-\left(\frac{t}{ax}+\frac{x}{2a}\right)^2\right\}d\left(\frac{t}{ax}+\frac{x}{2a}\right)+$$

$$+\frac{2}{\sqrt{\pi}}\int\limits_0^{\sqrt{t}}\exp\left\{-\frac{x^2}{4a^2}\right\}d\left(\frac{x}{2a}\right)=$$

$$=\exp\left\{\frac{2t}{a^2}\right\}\mathrm{erfc}\left(\frac{3\sqrt{t}}{2a}\right)+\mathrm{erf}\left(\frac{\sqrt{t}}{2a}\right).\qquad(2.1.2)$$

Then correctness of property 3) directly follows from relation (2.1.2). In addition, it also follows that the norm of the integral operator in (2.1.1), acting in the class of essentially

bounded functions is equal to unity.

The kernel $K(t,\tau)$ is integrable with weight function $t^{-1/2}$ [25], [27]. Really,

$$\int\limits_0^t\frac{K(t,\tau)}{\sqrt{\tau}}\,d\tau=$$

$$=\frac{\exp\{2t/a^2\}}{a\sqrt{\pi}}\int\limits_0^{\sqrt{t}}\frac{2\sqrt{t-x^2}}{x^2}\exp\left\{-\left(\frac{t}{ax}+\frac{x}{2a}\right)^2\right\}dx+$$

$$+\frac{1}{a\sqrt{\pi}}\int\limits_0^{\sqrt{t}}\frac{1}{\sqrt{t-x^2}}\exp\left\{-\left(\frac{t}{ax}-\frac{x}{2a}\right)^2\right\}dx+$$

$$+\frac{1}{a\sqrt{\pi}}\int\limits_0^{\sqrt{t}}\frac{1}{\sqrt{t-x^2}}\exp\left\{-\frac{x^2}{4a^2}\right\}dx=I_1(t)+I_2(t)+I_3(t).$$

After a replacement $y=1/x$ for the first integral we get an estimate:

$$I_1(t)\leq\frac{2\sqrt{t}\exp\{t/a^2\}}{a\sqrt{\pi}}\int\limits_{t^{-1/2}}^{+\infty}\exp\left\{-\frac{t^2y^2}{a^2}-\frac{1}{4a^2y^2}\right\}dy.$$

For small values t the last integral is bounded. For large values $t>>0$ we have the following estimate:

$$\frac{2\sqrt{t}\exp\{t/a^2\}}{a\sqrt{\pi}}\int\limits_0^{+\infty}\exp\left\{-\frac{t^2y^2}{a^2}-\frac{1}{4a^2y^2}\right\}dy=\frac{1}{\sqrt{t}}\leq\mathrm{const}.$$

So, the boundedness of the first integral $I_1(t)$ is established.

The integrals $I_2(t)$ and $I_3(t)$ we estimate using the following

We calculate an inner integral in (2.3.1)

$$J(t,\tau;\lambda)=$$

$$=\int_\tau^t r(t,\tau_1)\frac{1}{2a\sqrt{\pi(\tau_1-\tau)}}\left(1-\exp\left\{-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}\right)d\tau_1=$$

$$=\frac{t}{2a^2\pi}\sum_{n=1}^{\infty}\int_\tau^t\frac{n}{\lambda^n(t-\tau_1)^{3/2}\sqrt{(\tau_1-\tau)}}\exp\left\{-n^2\frac{t\tau_1}{a^2(t-\tau_1)}\right\}\times$$

$$\times\left(1-\exp\left\{-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}\right)d\tau_1=$$

$$=\frac{t}{2a^2\pi}\sum_{n=1}^{\infty}\frac{n}{\lambda^n}\left[I_n^{(1)}(t,\tau)-I_n^{(2)}(t,\tau)\right], \qquad (2.3.2)$$

where

$$I_n^{(1)}(t,\tau)=\int_\tau^t\frac{1}{(t-\tau_1)^{3/2}\sqrt{(\tau_1-\tau)}}\exp\left\{-n^2\frac{t\tau_1}{a^2(t-\tau_1)}\right\}d\tau_1,$$

$$I_n^{(2)}(t,\tau)=\int_\tau^t\frac{1}{(t-\tau_1)^{3/2}\sqrt{(\tau_1-\tau)}}\times$$

$$\times\exp\left\{-n^2\frac{t\tau_1}{a^2(t-\tau_1)}-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}d\tau_1.$$

Using a replacement $z=\sqrt{(\tau_1-\tau)/(t-\tau_1)}$, we calculate the integrals $I_n^{(1)}(t,\tau)$ and $I_n^{(2)}(t,\tau)$. We have

$$I_n^{(1)}(t,\tau)=\frac{2}{t-\tau}\exp\left\{-\frac{n^2t\tau}{a^2(t-\tau)}\right\}\int_0^\infty\exp\left\{-\frac{n^2t^2z^2}{a^2(t-\tau)}\right\}dz=$$

Then we obtain an equation of the form

$$\psi(y)-\lambda\int_y^\infty\frac{1}{a\sqrt{\pi}\,(x-y)^{3/2}}\exp\left\{-\frac{1}{a^2(x-y)}\right\}\times$$

$$\times\psi(x)\,dx=f_2(y),\quad y>0, \qquad (2.2.7)$$

where

$$\begin{cases}\exp\{-1/(4a^2y)\}\cdot\psi(y)\in L_\infty(0,\infty),\\ \exp\{-1/(4a^2y)\}\cdot f_2(y)\in L_\infty(0,\infty).\end{cases} \qquad (2.2.8)$$

A solution to equation (2.2.7) can be found either by operational method [29], [30], [32], [34] or by reducing it to the Riemann boundary value problem [16].

We introduce notion for Laplace transform of the function $\psi(y)$ as $L[\psi(y)]=\overline{\psi}(p)$. Then the formula for convolution of functions is valid [72]

$$L\left[\int_y^\infty K(y-x)\psi(x)\,dx\right]=\overline{K}(-p)\overline{\psi}(p), \qquad (2.2.9)$$

where

$$\overline{K}(-p)=\int_0^\infty K(-t)\exp\{p\,t\}\,dt.$$

Since

$$L\left[\frac{b}{2\sqrt{\pi}t^{3/2}}\exp\left\{-\frac{b^2}{4t}\right\}\right]=\exp\{-b\sqrt{p}\},\ b=\text{const},$$

then by virtue (2.2.9) equation (2.2.7) is converted to the form

$$\overline{\psi}(p)\cdot\left(1-\lambda\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right)=\overline{f}_2(p).$$

The corresponding homogeneous equation is:

$$\overline{\psi}(p)\cdot\left(1-\lambda\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}\right)=0. \qquad (2.2.10)$$

In case of

$$1-\lambda\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}=0, \qquad (2.2.11)$$

nonzero solutions of equation (2.2.10) are

$$\overline{\psi}_k(p)=C_k\cdot\delta(p-p_k),$$

where $\delta(x)$ is the delta function, $C_k=\text{const}$, $p_k(k=0,\pm1,\pm2,...)$ are roots of equation (2.2.11). Applying to last equality the inverse Laplace transform, we obtain:

$$\psi(y)=\frac{1}{2\pi i}\int\limits_{\sigma+i\infty}^{\sigma+i\infty}\delta(p-p_k)\exp\{py\}\,dp=\exp\{p_ky\}$$

(the integral is taken along a straight line $\operatorname{Re}p=\sigma$, and is understood in the sense of the principal value).

Therefore, if $p=p_k$ are roots of equation (2.2.11), then the eigenfunctions of equation (2.2.8) have the form [16, 85-89**?**]

$$\psi_k(y)=C_k\exp\{p_ky\},\quad C_k=\text{const}. \qquad (2.2.12)$$

We find roots of equation (2.2.11). If $|\lambda|\geq1$, we have $\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}=\lambda$ [85-89**?**]. Taking the logarithm, we get

$$\frac{2}{a}\sqrt{-p}=\ln|\lambda|+i(\arg\lambda+2k\pi);\ \ k=0,1,2,...$$

function $f_1(t)$, we get

$$\tilde{\varphi}(t)=\tilde{f}(t)+\lambda\int\limits_0^t\frac{1}{2a\sqrt{\pi(t-\tau)}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)\times$$

$$\times\tilde{\varphi}(\tau)\,d\tau+\lambda\int\limits_0^t r(t,\tau)\left(\tilde{f}(\tau)+\right.$$

$$\left.+\lambda\int\limits_0^\tau\frac{1}{2a\sqrt{\pi(\tau-\tau_1)}}\left(1-\exp\left\{-\frac{\tau\tau_1}{a^2(\tau-\tau_1)}\right\}\right)\tilde{\varphi}(\tau_1)\,d\tau_1\right)d\tau+$$

$$+\sum_{k=-N_1}^{N_2}C_k\cdot\frac{1}{\sqrt{t}}\cdot\exp\left\{\frac{p_k}{t}\right\}.$$

Changing the order of integration on the right side of this equation and changing roles τ and τ_1, we have

$$\tilde{\varphi}(t)=\lambda\int\limits_0^t\left\{\frac{1}{2a\sqrt{\pi(t-\tau)}}\left(1-\exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right)+\right.$$

$$+\lambda\int\limits_\tau^t r(t,\tau_1)\frac{1}{2a\sqrt{\pi(\tau_1-\tau)}}\times$$

$$\left.\times\left(1-\exp\left\{-\frac{\tau_1\tau}{a^2(\tau_1-\tau)}\right\}\right)d\tau_1\right\}\tilde{\varphi}(\tau)\,d\tau+\tilde{f}(t)+$$

$$+\lambda\int\limits_0^t r(t,\tau)\tilde{f}(\tau)\,d\tau+\sum_{k=-N_1}^{N_2}C_k\cdot\frac{1}{\sqrt{t}}\cdot\exp\left\{\frac{p_k}{t}\right\}. \qquad (2.3.1)$$

zeros of its denominator are numbers $p_k,\ k = 0, \pm1, \pm2, ...,$ that are bypassed twice in opposite direction. Therefore, according to [16] we have

$$r_{\lambda-}(y) = \sum_{n=0}^{\infty} \underset{p=p_n}{res}\ \overline{A}(p) = \frac{1}{a\sqrt{\pi}(-y)^{3/2}} \sum_{n=1}^{\infty} \frac{n}{\lambda^n} \exp\left\{-\frac{n^2}{a^2(-y)}\right\}.$$

So, the solution of nonhomogeneous equation (2.2.8) has the form:

$$\psi(y) = f_2(y) + \lambda \int_y^{\infty} r_{\lambda-}(y-x) f_2(x)\, dx + \sum_{k=-N_1}^{N_2} C_k \exp(p_k y), \tag{2.2.14}$$

where $C_k -$ const and the resolvent $r_{\lambda-}(y)$ is defined above.

By accomplishing reverse replacements (2.2.6) to (2.2.14), we obtain the solution of nonhomogeneous equation (2.2.4)

$$\tilde{\varphi}(t) = f_1(t) + \lambda \int_0^t r(t,\tau) f_1(\tau)\, d\tau + \sum_{k=-N_1}^{N_2} \frac{C_k}{\sqrt{t}} \exp\left(\frac{p_k}{t}\right) \tag{2.2.15}$$

where

$$r(t,\tau) = \frac{t}{a\sqrt{\pi}(t-\tau)^{3/2}} \sum_{n=1}^{\infty} \frac{n}{\lambda^n} \exp\left\{-n^2 \frac{t\tau}{a^2(t-\tau)}\right\}. \tag{2.2.16}$$

2.3 Reducing to Abel equation

Using the formula for the solution of the the characteristic equation (2.2.15), taking into account ratio (2.2.5) for the

$$-p_k = \frac{a^2}{4}\left(\ln^2|\lambda| - (\arg\lambda + 2k\pi)^2\right) + i\frac{a^2}{4}\ln|\lambda|^2(\arg\lambda + 2k\pi)\,; \tag{2.2.13}$$

$k = 0, 1, 2, ...$

For boundedness of function (2.2.12) at infinity it is necessary $\mathrm{Re}(-p_k) \geq 0$, i.e., $\ln^2|\lambda| \geq (\arg\lambda + 2k\pi)^2$ or

$$-\ln|\lambda| \leq \arg\lambda + 2k\pi \leq \ln|\lambda|.$$

Therefore, $-N_1 \leq k \leq N_2$, where

$$N_1 = \left[\frac{\ln|\lambda| + \arg\lambda}{2\pi}\right], \quad N_2 = \left[\frac{\ln|\lambda| - \arg\lambda}{2\pi}\right],$$

$N_1 + N_2 + 1$ is a number of eigenfunctions (2.2.12), and $[a]$ is the integral part of a number a. Obviously the more $|\lambda|$, the more eigenfunctions.

Thus, the following lemma holds.

Lemma 3.1

Eigenfunctions of an equation with a difference kernel

$$\psi(y) - \lambda \int_y^{\infty} \frac{1}{a\sqrt{\pi}\,(x-y)^{3/2}} \exp\left\{-\frac{1}{a^2(x-y)}\right\} \psi(x)\, dx = f_2(y),$$

$$y > 0,$$

have the form

$$\psi_k(y) = C_k \exp(p_k y), \quad C_k = \mathrm{const},$$

where p_k are roots of the equation

$$-p_k = \frac{a^2}{4}\left(\ln^2|\lambda| - (\arg\lambda + 2k\pi)^2\right) + i\frac{a^2}{4}\ln|\lambda|^2(\arg\lambda + 2k\pi)\,;$$

and $-N_1 \le k \le N_2$, *where*

$$N_1 = \left[\frac{\ln|\lambda| + \arg\lambda}{2\pi}\right], \quad N_2 = \left[\frac{\ln|\lambda| - \arg\lambda}{2\pi}\right],$$

$N_1 + N_2 + 1$ *is a number of is a number of eigenfunctions and* $[a]$ *is the integral part of a number a.*

In this way, $\forall\lambda,\ |\lambda| \ge 1$ we have

$$\psi_{\text{about}}(y) = \sum_{k=-N_1}^{N_2} C_k \exp\{p_k y\}.$$

Turning back to variables (2.2.6), we obtain the solution of homogeneous equation (2.2.4)

$$\tilde\varphi_{\text{об.одн.}}(t) = \sum_{k=-N_1}^{N_2} C_k \cdot \frac{1}{\sqrt{t}} \cdot \exp\left\{\frac{p_k}{t}\right\},$$

where $\operatorname{Re} p_k \le 0$ by virtue of (2.2.13).

Note that if $\lambda = 1$, then $p_0 = 0$. This case is detailed discussed in previous section.

So, the following result is obtained:

Lemma 3.2

The homogeneous integral equation

$$\tilde\varphi(t) - \lambda \int_0^t \frac{t}{a\sqrt{\pi}\,(t-\tau)^{3/2}} \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\} \tilde\varphi(\tau)\,d\tau = 0,$$

has a non-trivial solution

$$\tilde\varphi(t) = \sum_{k=-N_1}^{N_2} C_k \cdot \frac{1}{\sqrt{t}} \cdot \exp\left\{\frac{p_k}{t}\right\},$$

in a function class $\sqrt{t}\cdot\exp\{-t/(4a^2)\}\cdot\tilde\varphi(t) \in L_\infty(0,\infty)$, where

$$N_1 = \left[\frac{\ln|\lambda| + \arg\lambda}{2\pi}\right], \quad N_2 = \left[\frac{\ln|\lambda| - \arg\lambda}{2\pi}\right],$$

and p_k *are determined by the relation*

$$-p_k = \frac{a^2}{4}\left(\ln^2|\lambda| - (\arg\lambda + 2k\pi)^2\right) + i\frac{a^2}{4}\ln|\lambda|^2(\arg\lambda + 2k\pi).$$

We rewrite the nonhomogeneous operator equation in the form

$$\overline{\psi}(p) = \overline{f}_2(p) + \frac{\lambda\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}}{1 - \lambda\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}}, \quad \text{where } \operatorname{Re} p \le 0.$$

Introducing the notation

$$\overline{r}_{\lambda-}(p) = \frac{\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}}{1 - \lambda\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}},$$

we find the original of this image

$$r_{\lambda-}(y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}}{1 - \lambda\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}}\,dp,$$

where $r_{\lambda-}(y) \equiv 0$, if $y > 0$.

In the last integral, we integrate along the contour, bypassing on the left the points p_k, defined by formula (2.2.13). The integral is understood in sense of Cauchy principal value. Since $y \le 0$, then we close the contour on the right, cutting off the half-plane (the cut along the positive real semi-axis). For the function

$$\overline{A}(p) = \frac{\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}}{1 - \lambda\exp\left\{-\frac{2}{a}\sqrt{-p}\right\}}$$

$$+\frac{\lambda\sqrt{\pi}}{2a}\exp\left(\frac{\lambda^2 t}{4a^2}-\frac{\lambda\sqrt{-p_k}}{a}\right)\operatorname{erfc}\left(\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}\right),$$

where $\{\mathrm{IP}\} = \lim\limits_{t\to+0}\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}\in\mathbb{C}$ is an infinitely remote point, and the following formally designated expression

$$\operatorname{erfc}\left(\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}\right)\overset{\text{def}}{=}\frac{2}{\sqrt{\pi}}\int\limits_{\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}}^{\{\mathrm{IP}\}}\exp\{-\xi^2\}\,d\xi,$$

is an integral along an opened contour from the starting point $\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}$ to the infinitely remote pointt $\{\mathrm{IP}\}$.

So the function

$$\tilde{\varphi}_k(t)=\exp\left(\frac{p_k}{t}\right)\left\{\frac{1}{\sqrt{t}}+\frac{\lambda\sqrt{\pi}}{2a}\exp\left(\frac{\sqrt{-p_k}}{\sqrt{t}}-\frac{\lambda\sqrt{t}}{2a}\right)^2\times\right.$$

$$\left.\times\operatorname{erfc}\left(\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}\right)\right\} \tag{2.3.8}$$

is an eigenfunction of "simplified" equation (2.2.2) for each k; $-N_1\le k\le N_2$, where

$$N_1=\left[\frac{\ln|\lambda|+\arg\lambda}{2\pi}\right],\quad N_2=\left[\frac{\ln|\lambda|-\arg\lambda}{2\pi}\right],$$

and $[a]$ is an integral part of a number a.

Then the function

$$\tilde{\varphi}(t)=\sum_{k=-N_1}^{N_2}C_k\tilde{\varphi}_k(t) \tag{2.3.9}$$

is a solution of Abel equation (2.3.4) at $\tilde{f}_2(t)=0$, that is, the

$$=\frac{a\sqrt{\pi}}{nt\sqrt{t-\tau}}\exp\left\{-\frac{n^2t\tau}{a^2(t-\tau)}\right\},$$

$$I_n^{(2)}(t,\tau)=\frac{2}{t-\tau}\exp\left\{-\frac{(n^2+1)t\tau}{a^2(t-\tau)}\right\}\times$$

$$\times\int\limits_0^\infty\exp\left\{-\frac{n^2t^2z^2}{a^2(t-\tau)}-\frac{\tau^2}{a^2(t-\tau)z^2}\right\}dz=$$

$$=\frac{a\sqrt{\pi}}{nt\sqrt{t-\tau}}\exp\left\{-\frac{(n+1)^2t\tau}{a^2(t-\tau)}\right\}.$$

When calculating the integral $I_n^{(2)}(t,\tau)$ we have used formula (3.325) from [68, ?]:

$$\int\limits_0^\infty\exp\left\{-\mu x^2-\frac{\eta}{x^2}\right\}=\frac{\sqrt{\pi}}{2\sqrt{\mu}}\exp\left\{-2\sqrt{\mu\eta}\right\},\ \mu>0,\ \eta>0.$$

So for the difference $I_n^{(1)}(t,\tau)-I_n^{(2)}(t,\tau)$ we get

$$I_n^{(1)}(t,\tau)-I_n^{(2)}(t,\tau)=\frac{a\sqrt{\pi}}{nt\sqrt{t-\tau}}\times$$

$$\times\left(\exp\left\{-\frac{n^2t\tau}{a^2(t-\tau)}\right\}-\exp\left\{-\frac{(n+1)^2t\tau}{a^2(t-\tau)}\right\}\right).$$

Substituting into (2.3.2), we have

$$J(t,\tau;\lambda)=\frac{1}{2a\sqrt{\pi(t-\tau)}}\times$$

$$\times\sum_{n=1}^\infty\frac{1}{\lambda^n}\left(\exp\left\{-\frac{n^2t\tau}{a^2(t-\tau)}\right\}-\exp\left\{-\frac{(n+1)^2t\tau}{a^2(t-\tau)}\right\}\right)=$$

$$= \frac{1}{2a\lambda\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}.$$

Then equation (2.3.1) can be rewritten as

$$\tilde{\varphi}(t) = \lambda \int_0^t \left\{ \frac{1}{2a\sqrt{\pi(t-\tau)}} \left(1 - \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\}\right) + \right.$$

$$\left. + \frac{1}{2a\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{t\tau}{a^2(t-\tau)}\right\} \right\} \tilde{\varphi}(\tau)\, d\tau + \tilde{f}(t) +$$

$$+ \lambda \int_0^t r(t,\tau)\tilde{f}(\tau)\, d\tau + \sum_{k=-N_1}^{N_2} C_k \cdot \frac{1}{\sqrt{t}} \cdot \exp\left\{\frac{p_k}{t}\right\}.$$

Finally, after the introduction of the notation

$$\tilde{f}_2(t) = \tilde{f}(t) + \lambda \int_0^t r(t,\tau)\tilde{f}(\tau)\, d\tau, \qquad (2.3.3)$$

where $r(t,\tau)$ is defined by formula (2.2.16), we get

$$\tilde{\varphi}(t) - \frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{\tilde{\varphi}(\tau)}{\sqrt{t-\tau}}\, d\tau = \tilde{f}_2(t) + \sum_{k=-N_1}^{N_2} C_k \cdot \frac{1}{\sqrt{t}} \cdot \exp\left\{\frac{p_k}{t}\right\}, \qquad (2.3.4)$$

where a solution and a right side of equation (2.3.4) belong to classes (2.2.3).

Thus, initial "simplified" integral equation (2.2.2) is reduced to equation (2.3.4), i.e., to a nonhomogeneous Abel equation of the second kind.

According to [65, ?] the solution to an Abel equation of the

$$+ \frac{\lambda\sqrt{\pi}}{2a} \left[-\frac{4a^2}{\lambda^2} \exp\left(-\frac{\lambda^2}{4a^2}t\right) \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) + \right.$$

$$\left. \left. + \frac{8a^2\sqrt{-p_k}}{\lambda^2\sqrt{\pi}} \int_0^{\sqrt{t}} \exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right) \frac{1}{z^2}\, dz \right] \right\}.$$

After simple transformations we get

$$\tilde{\varphi}_k(t) = \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} +$$

$$+ \frac{\lambda^2}{2a^2} \exp\left(\frac{\lambda^2 t}{4a^2}\right) \int_0^{\sqrt{t}} \exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right) \left(1 + \frac{2a\sqrt{-p_k}}{\lambda z^2}\right) dz =$$

$$= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda}{a} \exp\left(\frac{\lambda^2 t}{4a^2}\right) \int_0^{\sqrt{t}} \exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right) \times$$

$$\times \left(\frac{\lambda}{2a} + \frac{\sqrt{-p_k}}{z^2}\right) dz = \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} -$$

$$- \frac{\lambda}{a} \exp\left(\frac{\lambda^2 t}{4a^2}\right) \int_0^{\sqrt{t}} \exp\left(\frac{p_k}{z^2} - \frac{\lambda^2}{4a^2}z^2\right) d\left(\frac{\sqrt{-p_k}}{z} - \frac{\lambda}{2a}z\right).$$

Introducing a replacement $\xi = \dfrac{\sqrt{-p_k}}{z} - \dfrac{\lambda}{2a}z$ we obtain

$$\tilde{\varphi}_k(t) = \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda}{a} \exp\left(\frac{\lambda^2 t}{4a^2} - \frac{\lambda\sqrt{-p_k}}{a}\right) \times$$

$$\times \int_{\frac{2a\sqrt{-p_k}-\lambda t}{2a\sqrt{t}}}^{\{\mathrm{IP}\}} \exp(-\xi^2)\, d\xi = \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} +$$

We calculate the integral $I_{2k}(t;\lambda)$ by parts:

$$\left\| \begin{array}{ll} u = \operatorname{erfc}\left(\dfrac{\sqrt{-p_k}}{\sqrt{\tau}}\right); & dv = \exp\left(-\dfrac{\lambda^2}{4a^2}\tau\right)\, d\tau; \\ du = \dfrac{\sqrt{-p_k}}{\sqrt{\pi}\tau^{3/2}} \exp\left\{\dfrac{p_k}{\tau}\right\}\, d\tau; & v = -\dfrac{4a^2}{\lambda^2}\exp\left(-\dfrac{\lambda^2}{4a^2}\tau\right). \end{array} \right\| \tag{2.3.7}$$

Then using (2.3.7), we have

$$I_{2k}(t;\lambda) = -\frac{4a^2}{\lambda^2}\exp\left(-\frac{\lambda^2}{4a^2}t\right)\cdot \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) +$$

$$+\frac{4a^2\sqrt{-p_k}}{\lambda^2\sqrt{\pi}}\int\limits_0^t \exp\left(-\frac{\lambda^2}{4a^2}\tau + \frac{p_k}{\tau}\right)\frac{1}{\tau^{3/2}}\, d\tau.$$

After a replacement $z = \sqrt{\tau}$ we get

$$I_{2k}(t;\lambda) = -\frac{4a^2}{\lambda^2}\exp\left(-\frac{\lambda^2}{4a^2}t\right)\cdot \operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) +$$

$$+\frac{8a^2\sqrt{-p_k}}{\lambda^2\sqrt{\pi}}\int\limits_0^{\sqrt{t}} \exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right)\frac{1}{z^2}\, dz.$$

Substituting expressions for $I_{1k}(t;\lambda)$ and $I_{2k}(t;\lambda)$ into (2.3.6) we have

$$\tilde{\varphi}_k(t) = \frac{1}{\sqrt{t}}\exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a}\operatorname{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) +$$

$$+\frac{\lambda^2}{4a^2}\exp\left(\frac{\lambda^2 t}{4a^2}\right)\left\{2\int\limits_0^{\sqrt{t}} \exp\left(-\frac{\lambda^2}{4a^2}z^2 + \frac{p_k}{z^2}\right)\, dz + \right.$$

second kind

$$y(x) + \mu\int\limits_a^x \frac{y(t)}{\sqrt{x-t}}\, dt = g(x)$$

has the form

$$y(x) = G(x) + \pi\mu^2 \int\limits_a^x \exp\left[\pi\mu^2(x-t)\right] G(t)\, dt, \tag{2.3.5}$$

where

$$G(x) = g(x) - \mu\int\limits_a^x \frac{g(t)}{\sqrt{x-t}}\, dt.$$

We find a solution to Abel equation (2.3.4) at $\tilde{f}_2(t) = 0$, that is, we find a solution to corresponding homogeneous equation (2.2.2) for each k; $-N_1 \le k \le N_2$ (eigenfunctions). Under this condition, for each k; $-N_1 \le k \le N_2$, equation (2.3.4) has the form

$$\tilde{\varphi}_k(t) - \frac{\lambda}{2a\sqrt{\pi}}\int\limits_0^t \frac{\tilde{\varphi}_k(\tau)}{\sqrt{t-\tau}}\, d\tau = \frac{1}{\sqrt{t}}\exp\left\{\frac{p_k}{t}\right\}.$$

We write a solution of the last equation in the form (see (2.3.5))

$$\tilde{\varphi}_k(t) = G_k(t) + \frac{\lambda^2}{4a^2}\int\limits_0^t \exp\left(\frac{\lambda^2(t-\tau)}{4a^2}\right) G_k(\tau)\, d\tau,$$

where

$$G_k(t) = \frac{1}{\sqrt{t}}\exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda}{2a\sqrt{\pi}}\int\limits_0^t \frac{\exp\left\{\frac{p_k}{t}\right\}}{\sqrt{\tau(t-\tau)}}\, d\tau =$$

$$= \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \text{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right).$$

Calculating the last integral, we have used formulas 3.471 (2), 9.224 from [68]. Indeed, according to the first of these equations, we obtain

$$\frac{\lambda}{2a\sqrt{\pi}} \int_0^t \frac{\exp\left\{\frac{p_k}{t}\right\}}{\sqrt{\tau(t-\tau)}} d\tau =$$

$$= \frac{\lambda}{2a} \left(\frac{-p_k}{t}\right)^{-1/4} \exp\left\{\frac{p_k}{2t}\right\} \cdot W_{-1/4,1/4}\left(\frac{-p_k}{t}\right),$$

and for the second formula we have:

$$W_{-1/4,1/4}\left(\frac{-p_k}{t}\right) = \sqrt{\pi}\left(\frac{-p_k}{t}\right)^{1/4} \exp\left\{\frac{-p_k}{2t}\right\} \cdot \text{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right),$$

where $W_{\alpha,\beta}(z)$ is the Whittaker function.

Function $G_k(t)$ is bounded $\forall t \in [0; +\infty)$ as $t \to +\infty$ and $G_k(0) = 0$.

So, the eigenfunctions of equation (2.2.2) have the form

$$\tilde{\varphi}_k(t) = \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \text{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) +$$

$$+ \frac{\lambda^2}{4a^2} \int_0^t \exp\left(\frac{\lambda^2(t-\tau)}{4a^2}\right) \times$$

$$\times \left\{\frac{1}{\sqrt{\tau}} \exp\left\{\frac{p_k}{\tau}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \text{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}}\right)\right\} d\tau.$$

We rewrite the last function in the form

$$\tilde{\varphi}_k(t) = \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \text{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) +$$

$$+ \frac{\lambda^2}{4a^2} \exp\left(\frac{\lambda^2 t}{4a^2}\right) \left\{ \int_0^t \exp\left(\frac{-\lambda^2}{4a^2}\tau + \frac{p_k}{\tau}\right) \frac{1}{\sqrt{\tau}} d\tau + \right.$$

$$\left. + \frac{\lambda\sqrt{\pi}}{2a} \int_0^t \text{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}}\right) \exp\left(-\frac{\lambda^2}{4a^2}\tau\right) d\tau \right\}$$

or

$$\tilde{\varphi}_k(t) = \frac{1}{\sqrt{t}} \exp\left\{\frac{p_k}{t}\right\} + \frac{\lambda\sqrt{\pi}}{2a} \text{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{t}}\right) + \\ + \frac{\lambda^2}{4a^2} \exp\left(\frac{\lambda^2 t}{4a^2}\right) \left\{ I_{1k}(t;\lambda) + \frac{\lambda\sqrt{\pi}}{2a} I_{2k}(t;\lambda) \right\}, \quad (2.3.6)$$

where

$$I_{1k}(t;\lambda) = \int_0^t \exp\left(-\frac{\lambda^2}{4a^2}\tau + \frac{p_k}{\tau}\right) \frac{1}{\sqrt{\tau}} d\tau,$$

$$I_{2k}(t;\lambda) = \int_0^t \text{erfc}\left(\frac{\sqrt{-p_k}}{\sqrt{\tau}}\right) \cdot \exp\left(-\frac{\lambda^2}{4a^2}\tau\right) d\tau.$$

After a replacement $z = \sqrt{\tau}$ the integral $I_{1k}(t;\lambda)$ can be rewritten as

$$I_{1k}(t;\lambda) = 2 \int_0^{\sqrt{t}} \exp\left(-\frac{\lambda^2}{4a^2} z^2 + \frac{p_k}{z^2}\right) dz.$$

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solution of the "simplified homogeneous equation (2.2.2), and functions $\tilde{\varphi}_k(t)$ and values p_k are defined by formulas (2.3.8) and (2.2.13) respectively.

Note that multiplying equality (2.3.9) by the $\exp\left(-t/(4a^2)\right)$, we obtain a solution of the homogeneous equation corresponding to initial equation (2.1.1):

$$\varphi(t) = \sum_{k=-N_1}^{N_2} C_k \left\{ \frac{1}{\sqrt{t}} \exp\left(\frac{p_k}{t} - \frac{t}{4a^2}\right) + \right.$$

$$\left. + \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\lambda^2 - 1}{4a^2} t - \frac{\lambda\sqrt{-p_k}}{a}\right) \cdot \operatorname{erfc}\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}\right) \right\}. \tag{2.3.10}$$

A function $\sqrt{t} \cdot \varphi(t)$ belongs to the class $L_\infty(0, \infty)$. Indeed, for the first summand in curly brackets of (2.3.10)

$$\exp\left(\frac{p_k}{t} - \frac{t}{4a^2}\right) \in L_\infty(0, \infty).$$

For the second summand in curly brackets of (2.3.10) the following inclusion is valid:

$$\sqrt{t} \cdot \frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\lambda^2 - 1}{4a^2} t - \frac{\lambda\sqrt{-p_k}}{a}\right) \cdot \operatorname{erfc}\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}\right) \in$$

$$\in L_\infty(0, \infty).$$

It's enough to take into account that the numbers p_k, $k \in [-N_1, N_2]$, are roots of equation (2.2.11) for each fixed complex spectral parameter $\lambda \in \mathcal{C}$, and to use the asymptotics of the function $\operatorname{erfc}(z)$ for large values z (see 8.254^8 in [68, p.890] or [64, p.708]). Obviously, there is a limit relation

$$z = \frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}} \to \{\text{IP}\} \quad \text{as} \quad t \to \infty \text{ and for each } \quad |\lambda| > 1.$$

2.4 Main result

Thus, the following theorem holds.

Теорема 2.1

Nonhomogeneous integral equation (2.1.1) *is solvable in the class*

$$\sqrt{t}\,\varphi(t) \in L_\infty(0,\infty)$$

for each right side $\sqrt{t}f(t) \in L_\infty(0;\infty)$ *and for each*

$$|\lambda| > \exp(|\arg\lambda|), \quad \arg\lambda \in [-\pi;\,\pi]$$

The corresponding homogeneous equation has $(N_1 + N_2 + 1)$ *eigenfunctions of the form*

$$\varphi_k(t) = \frac{1}{\sqrt{t}} \exp\left(\frac{p_k}{t} - \frac{t}{4a^2}\right) +$$

$$+\frac{\lambda\sqrt{\pi}}{2a} \exp\left(\frac{\lambda^2-1}{4a^2}t - \frac{\lambda\sqrt{-p_k}}{a}\right) \cdot \mathrm{erfc}\left(\frac{2a\sqrt{-p_k} - \lambda t}{2a\sqrt{t}}\right),$$

and the general solution of integral equation (2.1.1) can be written as

$$\varphi(t) = F(t) + \frac{\lambda^2}{4a^2} \int\limits_0^t \exp\left(\frac{\lambda^2(t-\tau)}{4a^2}\right) F(t)\, d\tau + \sum_{k=-N_1}^{N_2} C_k \varphi_k(t),$$

where

$$F(t) = \tilde{f}_2(t) - \frac{\lambda}{2a\sqrt{\pi}} \int\limits_0^t \frac{\tilde{f}_2(\tau)}{\sqrt{t-\tau}}\, d\tau,$$

and function $\sqrt{t} \cdot \exp\{-t/(4a^2)\} \cdot \tilde{f}_2(t) \in L_\infty(0,\infty)$ *is defined by formula* (2.3.3).

The results presented in this section are published in [27], [28].

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We studied the solvability of a singular Volterra integral equation of the second kind in the weight space of essentially bounded functions. It is proved that at

$$|\lambda| > \exp(|\arg\lambda|),\ \arg\lambda \in [-\pi;\ \pi],$$

the homogeneous equation corresponding (2.1.1) has a continuous spectrum, and the multiplicity of characteristic numbers increases with increasing modulus of the spectral parameter $|\lambda|$. Initial equation (2.1.1) is reduced to Abel equation (2.3.4) by Carleman-Vekua regularization method. This method was developed to solve singular integral equations. Eigenfunctions of equation (2.1.1) are found explicitly and their multiplicity is calculated, depending on the absolute value of the characteristic number λ.

CONCLUSION

In degenerating domains we considered the existence and uniqueness issues of a solution to the boundary value problem for the heat equation in the weight class of essentially bounded functions. These problems were reduced to singular Volterra integral equations of the second kind; for solving integral equations the Carleman-Vekua regularization method was used.

The main results are concerned the solvability of boundary value problems for the heat equation in domains degenerating to a point at the initial time. These results were reduced to the following:

1^0 Formulations of direct and conjugate boundary value problems in weight functional classes were given, solving the problems was reduced to the study and solving a singular Volterra integral equation;

2^0 The characteristic equation was introduced and solved according to the Carleman-Vekua regularization method;

3^0 The singular Volterra integral equation of the second kind was reduced to the Abel equation of the second kind;

4^0 Using the integral representation of the solution to the posed boundary value problem, the Noetherian property of this problem in the defined functional spaces is established;

5^0 The multiplicity of eigenvalues and eigenfunctions for the Volterra integral operator is found depending on the value of the spectral parameter;

6^0 Weight solution uniqueness classes for the studied boundary value problems are found.

The results obtained in the monograph have further continuation and development in Refs [73]–[86]. There are attempts to develop the obtained results for boundary value problems in the two-dimensional domain in [82], [86]. Study of singular Volterra integral equations of the second kind with other kernels is carried out in [83]–[85]. Some results of the

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