

D.M. Zhangazinova*, A.S. Naurazbekova

*L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
(E-mail: dinara_pav@mail.ru, altynkul.82@mail.ru)*

Automorphisms of the universal enveloping algebra of a finite-dimensional Zinbiel algebra with zero multiplication

In recent years there has been a great interest in the study of Zinbiel (dual Leibniz) algebras. Let A be Zinbiel algebra over an arbitrary field K and let $e_1, e_2, \dots, e_m, \dots$ be a linear basis of A . In 2010 A. Naurazbekova, using the methods of Gröbner-Shirshov bases, constructed the basis of the universal (multiplicative) enveloping algebra $U(A)$ of A . Using this result, the automorphisms of the universal enveloping algebra of a finite-dimensional Zinbiel algebra with zero multiplication are described.

Keywords: Zinbiel (dual Leibniz) algebra, universal (multiplicative) enveloping algebra, basis, automorphism, affine automorphism.

Introduction

An algebra A over a field K is called (left) *dual Leibniz* or *Zinbiel* (Leibniz is written in reverse order) if it satisfies the identity

$$(xy)z = x(yz) + x(zx).$$

The Leibniz algebras form a Koszul operad in the sense of V. Ginzburg and M. Kapranov [1]. Under the Koszul duality, the operad of Lie algebras is dual to the operad of associative and commutative algebras. The notion of Zinbiel (dual Leibniz) algebra defined by J.-L. Loday [2] is precisely the dual operad of Leibniz algebras in their sense. Moreover, any dual Leibniz algebra A with respect to the symmetrization $a \circ b = ab + ba$ is an associative and commutative algebra [2].

Zinbiel algebras are also known as pre-commutative algebras [3] and chronological algebras [4]. A Zinbiel algebra is equivalent to the commutative dendriform algebra [5]. It plays an important role in the definition of pre-Gerstenhaber algebras [6]. The variety of Zinbiel algebras is a proper subvariety in the variety of right commutative algebras. Each Zinbiel algebra with the commutator multiplication gives a Tortkara algebra [7], which appeared in unexpected areas of mathematics [8, 9]. Recently, the notion of matching Zinbiel algebras was introduced in [10]. Zinbiel algebras also appeared in the study of rack cohomology [11], number theory [12] and in a construction of a Cartesian differential category [13]. In recent years there has been a great interest in the study of Zinbiel algebras.

J.-L. Loday (J.-L. Loday) [2] proved that the set of all non-associative words with right arranged parenthesis (right-normed words) form the basis of free Zinbiel algebra. It was shown that free Zinbiel algebras are precisely the shuffle product algebra [14]. A. Naurazbekova [15] proved that free Zinbiel algebras over a field of characteristic zero are the free associative-commutative algebras (without unity) with respect to the symmetrization multiplication and their free generators are found; also she constructed examples of subalgebras of the two-generated free Zinbiel algebra that are free Zinbiel algebras of countable rank. A. Dzhumadildaev and K. Tulenbaev [16] proved the analogue of Nagata-Higman's theorem [17] for the Zinbiel algebras (any Zinbiel nil-algebra is nilpotent). They also proved that every finite-dimensional Zinbiel algebra over an algebraically closed field is solvable and nilpotent

*Corresponding author.

E-mail: dinara_pav@mail.ru

over the complex number field. A. Naurazbekova and U. Umirbaev [18] proved that in characteristic 0 any proper subvariety of variety of Zinbiel algebras is nilpotent and, as a consequence, the variety of Zinbiel algebras is Spechtian and has base rank 1. D.A. Towers [19] showed that every finite-dimensional Zinbiel algebra over an arbitrary field is nilpotent, extending a previous result by other authors that they are solvable. Filiform Zinbiel algebras were described and classified in [20–22]. The classification of complex Zinbiel algebras up to dimension 4 was obtained in [16] and [23]. A partial classification of the 5-dimensional case was done in [24]. M.A. Alvarez, R.F. Junior, I. Kaygorodov [25] proved that the variety of complex 5-dimensional Zinbiel algebras has dimension 24, it is defined by 16 irreducible components and it has 11 rigid algebras.

This paper is devoted to the description of automorphisms of the universal (multiplicative) enveloping algebra of a finite-dimensional Zinbiel algebra with zero multiplication.

The paper is organized as follows. In section 1, for convenience, we rewrite A. Naurazbekova's result [26] on the basis of the universal enveloping algebra of a Zinbiel algebra in new notation. In section 2, we describe automorphisms of the universal enveloping algebra of a finite-dimensional Zinbiel algebra with zero multiplication.

1 The basis of the universal enveloping algebra

Let K be an arbitrary field. An algebra A over a field K is called *dual Leibniz* or *Zinbiel* if it satisfies the identity

$$(xy)z = x(yz) + x(zx).$$

In [2] J.-L. Loday proved that any Zinbiel algebra with respect to multiplication $x \circ y = xy + yx$ is an associative commutative algebra. A linear basis of free Zinbiel algebras is also given in [2].

Let A be an arbitrary Zinbiel algebra over K . Let $L_A = \{L_x | x \in A\}$ and $R_A = \{R_x | x \in A\}$ be two isomorphic copies of the vector space A with the fixed isomorphisms $A \rightarrow L_A (x \mapsto L_x)$ and $A \rightarrow R_A (x \mapsto R_x)$, respectively. *The universal (multiplicative) enveloping algebra* $U(A)$ [27] is an associative algebra with the identity 1 generated by the two vector spaces L_A and R_A satisfying the defining relations

$$\begin{aligned} R_x R_y &= R_{xy+yx}, \\ R_x L_y &= L_y R_x + L_y L_x, \\ L_{xy} &= L_x L_y + L_x R_y \end{aligned}$$

for all $x, y \in A$. Recall that every dual Leibniz A -bimodule M can be regarded as a left $U(A)$ -module with respect to the action

$$L_a m = am, R_a m = ma, a \in A, m \in M.$$

Conversely, every left $U(A)$ -module can be considered as a Zinbiel A -bimodule [27].

This definition of the universal enveloping algebra is suitable for algebras without identity element. If the identity element 1 is fixed in the signature, then we have to add the relations $L_1 = R_1 = \text{Id} = 1$ and consider only unital modules. It is easy to see that a Zinbiel algebra is an algebra without an identity element. Below we rewrite A. Naurazbekova's [26] result on the basis of $U(A)$ in new notation.

Theorem 1. Let A be a Zinbiel algebra over a field K and let

$$e_1, e_2, \dots, e_m, \dots$$

be a linear basis of A . Then the set of all associative words of the form

$$1, L_{e_i}, R_{e_j}, L_{e_i} R_{e_j}, \tag{1}$$

where $i, j \geq 1$, is a linear basis for $U(A)$.

Proof. We define a linear order \leq on the set of associative words in the alphabet $R_{e_i}, L_{e_i}, i \geq 1$. Set $R_{e_i} < R_{e_j}$ and $L_{e_i} < L_{e_j}$ if $i < j$ and $R_{e_i} < L_{e_j}$ for all $i, j \geq 1$. If u and v are two words in the variables R_{e_i}, L_{e_j} , then set $u < v$ if one of the following conditions hold:

(i) $\deg(u) < \deg(v)$, where \deg is the degree function with respect to the variables R_{e_i}, L_{e_j} ;

(ii) $\deg(u) = \deg(v)$, $\deg_L(u) < \deg_L(v)$, where \deg_L is the degree function with respect to the variables L_{e_i} ;

(iii) $\deg(u) = \deg(v)$, $\deg_L(u) = \deg_L(v)$, and u precedes v with respect to the lexicographical order.

The defining relations of the algebra $U(A)$ are

$$R_{e_i} R_{e_j} - R_{e_i o e_j} = 0, \quad (2)$$

$$R_{e_i} L_{e_j} - L_{e_j e_i} = 0, \quad (3)$$

$$L_{e_i} L_{e_j} + L_{e_i} R_{e_j} - L_{e_i e_j} = 0 \quad (4)$$

for all $i, j \geq 1$.

The leading terms of these relations are $R_{e_i} R_{e_j}$, $R_{e_i} L_{e_j}$ and $L_{e_i} L_{e_j}$ for all $i, j \geq 1$. Consequently, the relations (2), (3) and (4) form three types of compositions.

Case 1. Set $w = (R_{e_i} R_{e_j}) R_{e_k} = R_{e_i} (R_{e_j} R_{e_k})$. Then the relations (2) form a composition

$$f = (R_{e_i} R_{e_j} - R_{e_i o e_j}) R_{e_k} - R_{e_i} (R_{e_j} R_{e_k} - R_{e_j o e_k}) = -R_{e_i o e_j} R_{e_k} + R_{e_i} R_{e_j o e_k}$$

with base w . Denote by \equiv the comparison in the free associative algebra in the variables $R_{e_i}, L_{e_i}, i \geq 1$, modulo linear combinations of elements of the form ugv , where g is one of the left hand side of the relations (2), (3) and (4), u and v are associative words, and the leading monomial of ugv is less than w . We have

$$f \equiv -R_{e_i o e_j o e_k} + R_{e_i o e_j o e_k} = 0.$$

Case 2. Set $w = (R_{e_i} R_{e_j}) L_{e_k} = R_{e_i} (R_{e_j} L_{e_k})$. The relations (2) and (3) form a composition

$$g = (R_{e_i} R_{e_j} - R_{e_i o e_j}) L_{e_k} - R_{e_i} (R_{e_j} L_{e_k} - L_{e_k e_j}) = -R_{e_i o e_j} L_{e_k} + R_{e_i} L_{e_k e_j}$$

with base w . We have

$$g \equiv -L_{e_k(e_i o e_j)} + L_{(e_k e_j) e_i} = -L_{e_k(e_i o e_j)} + L_{e_k(e_j o e_i)} = 0.$$

Case 3. Set $w = (L_{e_i} L_{e_j}) L_{e_k} = L_{e_i} (L_{e_j} L_{e_k})$. The relations (4) form a composition

$$\begin{aligned} h &= (L_{e_i} L_{e_j} + L_{e_i} R_{e_j} - L_{e_i e_j}) L_{e_k} - L_{e_i} (L_{e_j} L_{e_k} + L_{e_j} R_{e_k} - L_{e_j e_k}) = \\ &= L_{e_i} R_{e_j} L_{e_k} - L_{e_i e_j} L_{e_k} - L_{e_i} L_{e_j} R_{e_k} + L_{e_i} L_{e_j e_k} \end{aligned}$$

with base w . We have

$$h \equiv L_{e_i} L_{e_k e_j} + L_{e_i e_j} R_{e_k} - L_{(e_i e_j) e_k} + L_{e_i} R_{e_j} R_{e_k} - L_{e_i e_j} R_{e_k} + L_{e_i} L_{e_j e_k}$$

$$\equiv -L_{e_i} R_{e_k e_j} + L_{e_i(e_k e_j)} - L_{e_i(e_j e_k)} - L_{e_i(e_k e_j)} + L_{e_i} R_{e_j e_k} + L_{e_i} R_{e_k e_j} - L_{e_i} R_{e_j e_k} + L_{e_i(e_j e_k)} = 0.$$

Consequently, the relations (2), (3) and (4) are closed with respect to composition [28, 29]. This implies [28, 29] that the set of all words that are not divisible by the leading terms is a linear basis of the algebra $U(A)$. Therefore, the set of words of the form (1) is a linear basis for $U(A)$. Theorem 1 is proved.

2 Automorphisms

Let A be a finite-dimensional Zinbiel algebra with zero multiplication over an arbitrary field K . Let e_1, e_2, \dots, e_n be a linear basis of A . Then the universal enveloping algebra $U(A)$ of A is generated by the operators $R_{e_1}, \dots, R_{e_n}, L_{e_1}, \dots, L_{e_n}$ and (2)–(4) imply the defining relations of $U(A)$

$$R_{e_i}R_{e_j} = R_{e_i}L_{e_j} = 0, \tag{5}$$

$$L_{e_i}L_{e_j} = -L_{e_i}R_{e_j} \tag{6}$$

for all i, j . By these relations and Theorem 1, the set of all associative words of the form

$$1, L_{e_i}, R_{e_j}, L_{e_i}R_{e_j},$$

where $i, j \in \{1, \dots, n\}$, is a linear basis of $U(A)$ and $U(A)$ is a nilpotent algebra over field K with nilpotency index 3.

Theorem 2. Let A be the finite-dimensional Zinbiel algebra with zero multiplication over an arbitrary field K and let e_1, \dots, e_n be a linear basis of A . Then the affine automorphism group of the universal enveloping algebra $U(A)$ of A consists of endomorphisms of the form

$$\begin{aligned} \varphi(L_{e_i}) &= \sum_{j=1}^n \alpha_{ij}L_{e_j} + \sum_{j=1}^n \beta_{ij}R_{e_j}, \\ \varphi(R_{e_i}) &= \sum_{j=1}^n (\alpha_{ij} - \beta_{ij})R_{e_j}, \end{aligned} \tag{7}$$

$1 \leq i \leq n$, $A = (\alpha_{ij}), D = (\delta_{ij})$, where $\delta_{ij} = \alpha_{ij} - \beta_{ij}$, are square matrices of order n over a field K , $\det A \neq 0$ and $\det D \neq 0$.

Proof. Let φ be an affine automorphism of the algebra $U(A)$ and let

$$\begin{aligned} \varphi(L_{e_i}) &= \sum_{j=1}^n \alpha_{ij}L_{e_j} + \sum_{j=1}^n \beta_{ij}R_{e_j} + \mu_i, \\ \varphi(R_{e_k}) &= \sum_{t=1}^n \gamma_{kt}L_{e_t} + \sum_{t=1}^n \delta_{kt}R_{e_t} + \nu_k, \end{aligned}$$

$i, k \in \{1, \dots, n\}$, $\alpha_{ij}, \beta_{ij}, \mu_i, \gamma_{kt}, \delta_{kt}, \nu_k \in K$ for all i, j, k, t . Since φ is an automorphism of $U(A)$, we have

$$\det \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} & \beta_{11} & \dots & \beta_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \dots & \alpha_{nn} & \beta_{n1} & \dots & \beta_{nn} \\ \gamma_{11} & \dots & \gamma_{1n} & \delta_{11} & \dots & \delta_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_{n1} & \dots & \gamma_{nn} & \delta_{n1} & \dots & \delta_{nn} \end{pmatrix} \neq 0 \tag{8}$$

and (5), (6) imply

$$\varphi(R_{e_i})\varphi(R_{e_k}) = \varphi(R_{e_i})\varphi(L_{e_k}) = 0, \tag{9}$$

$$\varphi(L_{e_i})\varphi(L_{e_k}) = -\varphi(L_{e_i})\varphi(R_{e_k}) \tag{10}$$

for all i, j .

It is easy to see that if $i = j$, (9) and (10) give

$$\nu_i = \mu_i = 0 \text{ for all } i.$$

Using (5) and (6), it follows from (9) and (10) that

$$\varphi(R_{e_i})\varphi(R_{e_k}) = \sum_{j=1}^n \sum_{t=1}^n \gamma_{ij} (-\gamma_{kt} + \delta_{kt}) L_{e_j} R_{e_t} = 0,$$

$$\varphi(R_{e_i})\varphi(L_{e_k}) = \sum_{j=1}^n \sum_{t=1}^n \gamma_{ij} (-\alpha_{kt} + \beta_{kt}) L_{e_j} R_{e_t} = 0,$$

$$\varphi(L_{e_i})\varphi(L_{e_k}) + \varphi(L_{e_i})\varphi(R_{e_k}) = \sum_{j=1}^n \sum_{t=1}^n \alpha_{ij} (-\alpha_{kt} + \beta_{kt} - \gamma_{kt} + \delta_{kt}) L_{e_j} R_{e_t} = 0$$

for all $i, k \in \{1, \dots, n\}$. Hence

$$\gamma_{ij} (-\gamma_{kt} + \delta_{kt}) = \gamma_{ij} (-\alpha_{kt} + \beta_{kt}) = \alpha_{ij} (-\alpha_{kt} + \beta_{kt} - \gamma_{kt} + \delta_{kt}) = 0 \tag{11}$$

for all $i, j, k, t \in \{1, \dots, n\}$.

Suppose that $\gamma_{ij} \neq 0$ for some i, j . It follows from (11) that

$$\gamma_{kt} = \delta_{kt}, \alpha_{kt} = \beta_{kt} \text{ for all } k, t.$$

This contradicts (8). Consequently, $\gamma_{ij} = 0$ for all i, j . Using this and (8), we obtain

$$\det A \neq 0, \det D \neq 0,$$

where $A = (\alpha_{ij}), D = (\delta_{ij})$ are square matrices of order n over a field K . It is clear that there exists i, j such that $\alpha_{ij} \neq 0$. It follows from (11) that

$$\delta_{kt} = \alpha_{kt} - \beta_{kt} \text{ for all } k, t.$$

Consequently, if ϕ is an affine automorphism of $U(A)$, then ϕ has the form (7).

It is obvious that any endomorphism of the form (7) is an automorphism of the algebra $U(A)$. Theorem 2 is proved.

Lemma 1. Let $A = (a_{ij})$ and $B = (b_{ks})$ be non-zero square matrices of orders n and m , respectively. Then

$$\det \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{pmatrix} = (\det A)^m \cdot (\det B)^n.$$

Proof. Prove the statement of the lemma by induction on $n + m$. Without loss of generality, assume $a_{11} \neq 0$. By the induction proposition, we get

$$\det \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{n1}B & a_{n2}B & \dots & a_{nn}B \end{pmatrix} = \det \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ 0 & \left(\frac{a_{21}a_{12}}{a_{11}} - a_{22}\right)B & \dots & \left(\frac{a_{21}a_{1n}}{a_{11}} - a_{2n}\right)B \\ \dots & \dots & \dots & \dots \\ 0 & \left(\frac{a_{n1}a_{12}}{a_{11}} - a_{n2}\right)B & \dots & \left(\frac{a_{n1}a_{1n}}{a_{11}} - a_{nn}\right)B \end{pmatrix} =$$

$$\begin{aligned}
 &= \det(a_{11}B) \cdot \det \left(\begin{pmatrix} \left(\frac{a_{21}a_{12}}{a_{11}} - a_{22}\right) B & \dots & \left(\frac{a_{21}a_{1n}}{a_{11}} - a_{2n}\right) B \\ \dots & \dots & \dots \\ \left(\frac{a_{n1}a_{12}}{a_{11}} - a_{n2}\right) B & \dots & \left(\frac{a_{n1}a_{1n}}{a_{11}} - a_{nn}\right) B \end{pmatrix} \right) = \\
 &= a_{11}^m \det B \cdot \left(\det \left(\begin{pmatrix} \left(\frac{a_{21}a_{12}}{a_{11}} - a_{22}\right) & \dots & \left(\frac{a_{21}a_{1n}}{a_{11}} - a_{2n}\right) \\ \dots & \dots & \dots \\ \left(\frac{a_{n1}a_{12}}{a_{11}} - a_{n2}\right) & \dots & \left(\frac{a_{n1}a_{1n}}{a_{11}} - a_{nn}\right) \end{pmatrix} \right) \right)^m (\det B)^{n-1} = \\
 &= a_{11}^m \det B \cdot \left(\frac{1}{a_{11}} \det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & \left(\frac{a_{21}a_{12}}{a_{11}} - a_{22}\right) & \dots & \left(\frac{a_{21}a_{1n}}{a_{11}} - a_{2n}\right) \\ \dots & \dots & \dots & \dots \\ 0 & \left(\frac{a_{n1}a_{12}}{a_{11}} - a_{n2}\right) & \dots & \left(\frac{a_{n1}a_{1n}}{a_{11}} - a_{nn}\right) \end{pmatrix} \right)^m (\det B)^{n-1} = \\
 &= a_{11}^m \det B \cdot \frac{1}{a_{11}^m} (\det A)^m (\det B)^{n-1} = (\det A)^m (\det B)^n.
 \end{aligned}$$

Lemma 1 is proved.

Theorem 3. Let A be the finite-dimensional Zinbiel algebra with zero multiplication over an arbitrary field K and let e_1, e_2, \dots, e_n be a linear basis of A . Then the automorphism group of the universal enveloping algebra $U(A)$ of A consists of endomorphism of the form

$$\begin{aligned}
 \varphi(L_{e_i}) &= f_i + \sum_{j=1}^n \alpha_{ij} L_{e_j} + \sum_{j=1}^n \beta_{ij} R_{e_j}, \\
 \varphi(R_{e_i}) &= g_i + \sum_{j=1}^n \tau_{ij} R_{e_j},
 \end{aligned} \tag{12}$$

$1 \leq i \leq n$, f_i, g_i are any homogeneous elements of degree 2 of $U(A)$, $A = (\alpha_{ij}), T = (\tau_{ij})$, where $\tau_{ij} = \alpha_{ij} - \beta_{ij}$, are square matrices of order n over a field K , $\det A \neq 0$ and $\det T \neq 0$.

Proof. Let φ be any automorphism of the algebra $U(A)$. By Theorem 2, the affine part of φ has the form (7). Since φ is an automorphism of $U(A)$, φ satisfies the equalities (9) and (10). Using (5) and (6), it is easy to see that φ has the form (12).

Let φ be an endomorphism of $U(A)$ of the form (12) and let

$$\begin{aligned}
 f_i &= \sum_{k=1}^n \sum_{t=1}^n \gamma_{kt}^{(i)} L_{e_k} R_{e_t}, \\
 g_i &= \sum_{k=1}^n \sum_{t=1}^n \delta_{kt}^{(i)} L_{e_k} R_{e_t},
 \end{aligned}$$

$1 \leq i \leq n$. Prove that φ is an automorphism of $U(A)$, i.e., prove that φ has an inverse endomorphism φ' . To find the endomorphism φ' in the following form

$$\begin{aligned}
 \varphi'(L_{e_i}) &= \sum_{k=1}^n \sum_{t=1}^n \gamma'_{kt}{}^{(i)} L_{e_k} R_{e_t} + \sum_{j=1}^n \alpha'_{ij} L_{e_j} + \sum_{j=1}^n \beta'_{ij} R_{e_j}, \\
 \varphi'(R_{e_i}) &= \sum_{k=1}^n \sum_{t=1}^n \delta'_{kt}{}^{(i)} L_{e_k} R_{e_t} + \sum_{j=1}^n \tau'_{ij} R_{e_j}.
 \end{aligned}$$

Since the affine part of φ' is inverse to the affine part of φ , it is easy to find all coefficients $\alpha'_{ij}, \beta'_{ij}, \tau'_{ij}$.

Let $h \in U(A)$. Denote by \bar{h} the homogeneous part of degree 2 of the element h .

Using (5) and (6), we get

$$\begin{aligned} 0 = \overline{\varphi' \circ \varphi(L_{e_i})} &= \sum_{k=1}^n \sum_{t=1}^n \gamma_{kt}^{(i)} \varphi'(L_{e_k}) \varphi'(R_{e_t}) + \sum_{j=1}^n \alpha_{ij} \varphi'(L_{e_j}) + \sum_{j=1}^n \beta_{ij} \varphi'(R_{e_j}) = \\ &= \sum_{k=1}^n \sum_{t=1}^n \gamma_{kt}^{(i)} \left(\sum_{p=1}^n \alpha'_{kp} L_{e_p} \right) \left(\sum_{s=1}^n \tau'_{ts} R_{e_s} \right) + \\ &+ \sum_{j=1}^n \alpha_{ij} \left(\sum_{a=1}^n \sum_{b=1}^n \gamma_{ab}^{(j)} L_{e_a} R_{e_b} \right) + \sum_{j=1}^n \beta_{ij} \left(\sum_{a=1}^n \sum_{b=1}^n \delta_{ab}^{(j)} L_{e_a} R_{e_b} \right) \end{aligned}$$

and

$$\begin{aligned} 0 = \overline{\varphi' \circ \varphi(R_{e_i})} &= \sum_{k=1}^n \sum_{t=1}^n \delta_{kt}^{(i)} \varphi'(L_{e_k}) \varphi'(R_{e_t}) + \sum_{j=1}^n \tau_{ij} \varphi'(R_{e_j}) = \\ &= \sum_{k=1}^n \sum_{t=1}^n \delta_{kt}^{(i)} \left(\sum_{p=1}^n \alpha'_{kp} L_{e_p} \right) \left(\sum_{s=1}^n \tau'_{ts} R_{e_s} \right) + \sum_{j=1}^n \tau_{ij} \left(\sum_{a=1}^n \sum_{b=1}^n \delta_{ab}^{(j)} L_{e_a} R_{e_b} \right). \end{aligned}$$

Since all the coefficients α'_{ij}, τ'_{ij} are known, it follows from these equalities

$$\begin{aligned} \sum_{j=1}^n \left(\alpha_{ij} \gamma_{ab}^{(j)} + \beta_{ij} \delta_{ab}^{(j)} \right) L_{e_a} R_{e_b} &= \mu_{ab}^{(i)} L_{e_a} R_{e_b}, \\ \sum_{j=1}^n \tau_{ij} \delta_{ab}^{(j)} L_{e_a} R_{e_b} &= \nu_{ab}^{(i)} L_{e_a} R_{e_b}, \end{aligned}$$

for all $i, a, b \in \{1, \dots, n\}$, where $\mu_{ab}^{(i)}, \nu_{ab}^{(i)}$ are some elements of K . For each a, b we obtain the following system of $2n$ linear equations with unknowns $\gamma_{ab}^{(1)}, \dots, \gamma_{ab}^{(n)}, \delta_{ab}^{(1)}, \dots, \delta_{ab}^{(n)}$:

$$\begin{cases} \sum_{j=1}^n \left(\alpha_{ij} \gamma_{ab}^{(j)} + \beta_{ij} \delta_{ab}^{(j)} \right) = \mu_{ab}^{(i)} \\ \sum_{j=1}^n \tau_{ij} \delta_{ab}^{(j)} = \nu_{ab}^{(i)}, \end{cases}$$

$1 \leq i \leq n$. Since $\det A \neq 0, \det T \neq 0$, this system has the solution for each a, b . Consequently, there exists a left inverse of φ .

Using (5) and (6), we also get

$$\begin{aligned} 0 = \overline{\varphi \circ \varphi'(L_{e_i})} &= \sum_{k=1}^n \sum_{t=1}^n \gamma_{kt}^{(i)} \varphi(L_{e_k}) \varphi(R_{e_t}) + \sum_{j=1}^n \alpha'_{ij} \varphi(L_{e_j}) + \sum_{j=1}^n \beta'_{ij} \varphi(R_{e_j}) = \\ &= \sum_{k=1}^n \sum_{t=1}^n \gamma_{kt}^{(i)} \left(\sum_{p=1}^n \alpha_{kp} L_{e_p} \right) \left(\sum_{s=1}^n \tau_{ts} R_{e_s} \right) + \\ &+ \sum_{j=1}^n \alpha'_{ij} \left(\sum_{a=1}^n \sum_{b=1}^n \gamma_{ab}^{(j)} L_{e_a} R_{e_b} \right) + \sum_{j=1}^n \beta'_{ij} \left(\sum_{a=1}^n \sum_{b=1}^n \delta_{ab}^{(j)} L_{e_a} R_{e_b} \right) \end{aligned}$$

and

$$0 = \overline{\varphi \circ \varphi'(R_{e_i})} = \overline{\sum_{k=1}^n \sum_{t=1}^n \delta_{kt}^{(i)} \varphi(L_{e_k}) \varphi(R_{e_t}) + \sum_{j=1}^n \tau_{ij}' \varphi(R_{e_j})} =$$

$$= \sum_{k=1}^n \sum_{t=1}^n \delta_{kt}^{(i)} \left(\sum_{p=1}^n \alpha_{kp} L_{e_p} \right) \left(\sum_{s=1}^n \tau_{ts} R_{e_s} \right) + \sum_{j=1}^n \tau_{ij}' \left(\sum_{a=1}^n \sum_{b=1}^n \delta_{ab}^{(j)} L_{e_a} R_{e_b} \right).$$

Since all the coefficients $\alpha'_{ij}, \beta'_{ij}, \tau'_{ij}$ are known, it follows from these equalities

$$\sum_{k=1}^n \sum_{t=1}^n \gamma_{kt}^{(i)} \alpha_{kp} \tau_{ts} L_{e_p} R_{e_s} = \lambda_{ps}^{(i)} L_{e_p} R_{e_s},$$

$$\sum_{k=1}^n \sum_{t=1}^n \delta_{kt}^{(i)} \alpha_{kp} \tau_{ts} L_{e_p} R_{e_s} = \sigma_{ps}^{(i)} L_{e_p} R_{e_s},$$

for all $i, p, s \in \{1, \dots, n\}$, $\lambda_{ps}^{(i)}, \sigma_{ps}^{(i)}$ are some elements of K . For each i we obtain the following two systems of n^2 linear equations with unknowns $\gamma_{11}^{(i)}, \dots, \gamma_{1n}^{(i)}, \dots, \gamma_{n1}^{(i)}, \dots, \gamma_{nn}^{(i)}$ and $\delta_{11}^{(i)}, \dots, \delta_{1n}^{(i)}, \dots, \delta_{n1}^{(i)}, \dots, \delta_{nn}^{(i)}$, respectively:

$$\left\{ \begin{aligned} \gamma_{11}^{(i)} \alpha_{1p} \tau_{1s} + \dots + \gamma_{1n}^{(i)} \alpha_{1p} \tau_{ns} + \dots + \gamma_{n1}^{(i)} \alpha_{np} \tau_{1s} + \dots + \gamma_{nn}^{(i)} \alpha_{np} \tau_{ns} = \lambda_{ps}^{(i)} \end{aligned} \right.$$

and

$$\left\{ \begin{aligned} \delta_{11}^{(i)} \alpha_{1p} \tau_{1s} + \dots + \delta_{1n}^{(i)} \alpha_{1p} \tau_{ns} + \dots + \delta_{n1}^{(i)} \alpha_{np} \tau_{1s} + \dots + \delta_{nn}^{(i)} \alpha_{np} \tau_{ns} = \sigma_{ps}^{(i)}, \end{aligned} \right.$$

$p, s \in \{1, \dots, n\}$. The coefficient matrices of these systems have the form

$$C = \begin{pmatrix} \alpha_{11}T & \alpha_{21}T & \dots & \alpha_{n1}T \\ \alpha_{12}T & \alpha_{22}T & \dots & \alpha_{n2}T \\ \dots & \dots & \dots & \dots \\ \alpha_{1n}T & \alpha_{2n}T & \dots & \alpha_{nn}T \end{pmatrix}.$$

By Lemma 1, $\det C = (\det A)^n (\det T)^n$. Since $\det A \neq 0, \det T \neq 0$, we have $\det C \neq 0$. It follows that this systems has the solutions for each $1 \leq i \leq n$. Consequently, there exists a right inverse of φ . Since in groups the left and right inverses coincide, there exists an inverse of φ . Hence φ is an automorphism of the algebra $U(A)$. Theorem 3 is proved.

Acknowledgments

This work was supported by the grant of the Ministry of Science and Higher Education of the Republic of Kazakhstan (project AP09261086).

References

- 1 Ginzburg V. Koszul duality for operads / V. Ginzburg, M. Kapranov // Duke Math Journal. — 1994. — 76. — No. 1. — P. 203–272. <https://doi.org/10.48550/arXiv.0709.1228>
- 2 Loday J.-L. Cup-product for Leibniz cohomology and dual-Leibniz algebras / J.-L. Loday // Mathematica Scandinavica. — 1995. — 77. — No. 2. — P. 189–196. <https://doi.org/10.7146/math.scand.a-12560>

- 3 Kolesnikov P.S. Commutator algebras of pre-commutative algebras / P.S. Kolesnikov // *Mathematical Journal*. — 2016. — 16. — No. 2. — P. 145–158.
- 4 Kawski M. Chronological algebras: Combinatorics and control / M. Kawski // *Journal of Mathematical Sciences*. — 2001. — 103. — No. 6. — P. 725–744. <https://doi.org/10.1023/A:1009502501461>
- 5 Aguiar M. Pre-Poisson algebras / M. Aguiar // *Letters in Mathematical Physics*. — 2000. — 54. — P. 263–277.
- 6 Aloulou W. Algèbre pré-Gerstenhaber à homotopie près / W. Aloulou, D. Arnal, R. Chatbouri // *Journal of Pure Applied Algebra*. — 2017. — 221. — No. 11. — P. 2666–2688. <https://doi.org/10.1016/j.jpaa.2017.01.005>
- 7 Dzhumadil'daev A.S. Zinbiel algebras under q -commutators / A.S. Dzhumadil'daev // *Journal of Mathematical Sciences*. — 2007. — 144. — No. 2. — P. 3909–3925.
- 8 Diehl J. Time-warping invariants of multidimensional time series / J. Diehl, K. Ebrahimi-Fard, N. Tapia // *Acta Applicandae Mathematicae*. — 2020. — 170. — No. 1. — P. 265–290. <https://doi.org/10.1007/s10440-020-00333-x>
- 9 Diehl J. Areas of areas generate the shuffle algebra / J. Diehl, T. Lyons, R. Preis, J. Reizenstein // arXiv preprint arXiv:2002.02338. — 2021. <https://doi.org/10.48550/arXiv.2002.02338>
- 10 Gao X. Commutative matching Rota-Baxter operators, shuffle products with decorations and matching Zinbiel algebras / X. Gao, L. Guo, Yi. Zhang // *Journal of Algebra*. — 2021. — 586. — P. 402–432. <https://doi.org/10.1016/j.jalgebra.2021.06.032>
- 11 Covez S. Bialgebraic approach to rack cohomology / S. Covez, M. Farinati, V. Lebed, D. Manchon // *Algebraic and Geometric Topology*. — 2023. — 23. — No. 4. — P. 1551–1582.
- 12 Chapoton F. Zinbiel algebras and multiple zeta values / F. Chapoton // arXiv preprint arXiv:2109.00241. — 2021.
- 13 Ikonikoff S. Cartesian Differential Comonads and New Models of Cartesian Differential Categories / S. Ikonikoff, J.-S. Pacaud Lemay. — 2023. <https://doi.org/10.48550/arXiv.2108.04304>
- 14 Loday J.-L. On the algebra of quasi-shuffles / J.-L. Loday // *Manuscripta mathematica*. — 2007. — 123. — No. 1. — P. 79–93. <https://doi.org/10.48550/arXiv.math/0506498>
- 15 Naurazbekova A. On the structure of free dual Leibniz algebras / A. Naurazbekova // *Eurasian Mathematical Journal*. — 2019. — 10. — No. 3. — P. 40–47. <https://doi.org/10.32523/2077-9879-2019-10-3-40-47>
- 16 Dzhumadil'daev A.S. Nilpotency of Zinbiel algebras / A.S. Dzhumadil'daev, K.M. Tulenbaev // *Journal of Dynamical and Control Systems*. — 2005. — 11. — No. 2. — P. 195–213. <https://doi.org/10.1007/s10883-005-4170-1>
- 17 Higman G. On a conjecture of Nagata / G. Higman // *Proc. Cambridge Philos. Soc.* — 1956. — 52. — P. 1–4.
- 18 Naurazbekova A.S. Identities of dual Leibniz algebras / A.S. Naurazbekova, U.U. Umirbaev // *TWMS Journal of Pure and Applied Mathematics*. — 2010. — 1. — No. 1. — P. 86–91.
- 19 Towers D.A. Zinbiel algebras are nilpotent / D.A. Towers // *Journal of Algebra and Its Applications*. — 2023. — 22. — No. 8. — 2350166. <https://doi.org/10.1142/S0219498823501669>
- 20 Adashev J.Q., Omirov B.A., Khudoyberdiyev A.Kh. On some nilpotent classes of Zinbiel algebras and their applications // *Third International Conference on Research and Education in Mathematic*. — Malaysia, 2007. — P. 45–47.
- 21 Camacho L.M. Central extensions of filiform Zinbiel algebras / L.M. Camacho, I. Karimjanov, I. Kaygorodov, A.Kh. Khudoyberdiyev // *Linear and Multilinear Algebra*. — 2020. — 70. — No. 4. — P. 1–17. <https://doi.org/10.1080/03081087.2020.1764903>

- 22 Camacho L.M. p -Filiform Zinbiel algebras / L.M. Camacho, E.M. Canete, S. Gomez, B.A. Omirov // Linear Algebra and its Applications. — 2013. — 438. — No. 7. — P. 2958–2972. <https://doi.org/10.1016/j.laa.2012.11.030>
- 23 Omirov B.A. Classification of two-dimensional complex Zinbiel algebras / B.A. Omirov // Uzbek. Mat. Zh. — 2002. — 2. — P. 55–59.
- 24 Kaygorodov I. Central extensions of 3-dimensional Zinbiel algebras / I. Kaygorodov, M.A. Alvarez, T.C.d. Mello // Ricerche di Matematica. — 2023. — 72. — P. 921–947. <https://doi.org/10.1007/s11587-021-00604-1>
- 25 Alvarez M.A. The algebraic and geometric classification of Zinbiel algebras / M.A. Alvarez, R.F. Junior, I. Kaygorodov // Journal of Pure and Applied Algebra. — 2022. — 226. — No. 11. — 107106. <https://doi.org/10.1016/j.jpaa.2022.107106>
- 26 Науразбекова А.С. Универсальные мультипликативные обертывающие алгебры дуальных алгебр Лейбница / А.С. Науразбекова // Вестн. Евраз. нац. ун-та им. Л.Н. Гумилева. — 2010. — 75. — № 2. — С. 307–316.
- 27 Jacobson N. Structure and Representations of Jordan Algebras / N. Jacobson // American Mathematical Society, Providence, 1968. — 453 p.
- 28 Бокуть Л.А. Вложения в простые ассоциативные алгебры / Л.А. Бокуть // Алгебра и логика. — 1976. — 15. — № 2. — С. 117–142.
- 29 Bergman G.M. The diamond lemma for ring theory / G.M. Bergman // Advances in Mathematics. — 1978. — 29. — No. 2. — P. 178–218. [https://doi.org/10.1016/0001-8708\(78\)90010-5](https://doi.org/10.1016/0001-8708(78)90010-5)

Д.М. Жангазинова, А.С. Науразбекова

Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Астана, Қазақстан

Нөлдік көбейтіндісі бар ақырлыөлшемді Зинбил алгебрасының универсалды ораушы алгебрасының автоморфизмдері

Соңғы жылдары Зинбил алгебраларын (дуалды Лейбниц алгебраларын) зерттеуге үлкен қызығушылық бар. Айталық A кез келген K өрісіндегі құрастырылған Зинбил алгебрасы және $e_1, e_2, \dots, e_m, \dots$ A алгебрасының сызықты базисі. 2010 жылы А. Науразбекова Грёбнер-Ширшов базистерінің әдістерін қолданып, A алгебрасының $U(A)$ универсалды (мультипликативті) ораушы алгебрасының базисін құрастырды. Осы нәтижені пайдаланып, нөлдік көбейтіндісі бар ақырлыөлшемді Зинбил алгебрасының универсалды ораушы алгебрасының автоморфизмдері сипатталған.

Кілт сөздер: Зинбил (дуалды Лейбниц) алгебрасы, универсалды (мультипликативті) ораушы алгебра, базис, автоморфизм, аффинді автоморфизм.

Д.М. Жангазинова, А.С. Науразбекова

Евразийский национальный университет имени Л.Н. Гумилева, Астана, Казахстан

Автоморфизмы универсальной обертывающей алгебры конечномерной алгебры Зинбиля с нулевым умножением

В последние годы наблюдается большой интерес к изучению алгебр Зинбиля (дуальных алгебр Лейбница). Пусть A алгебра Зинбиля над произвольным полем K и пусть $e_1, e_2, \dots, e_m, \dots$ линейный базис алгебры A . В 2010 году А. Науразбекова, применяя методы базисов Грёбнера–Ширшова, построила базис универсальной (мультипликативной) обертывающей алгебры $U(A)$ алгебры A . Используя данный результат, описаны автоморфизмы универсальной обертывающей алгебры конечномерной алгебры Зинбиля с нулевым умножением.

Ключевые слова: алгебра Зинбиля (дуальная алгебра Лейбница), универсальная (мультипликативная) обертывающая алгебра, базис, автоморфизм, аффинный автоморфизм.

References

- 1 Ginzburg, V., & Kapranov, M. (1994). Koszul duality for operads. *Duke Math. J.*, 76(1), 203–272. <https://doi.org/10.48550/arXiv.0709.1228>
- 2 Loday, J.-L. (1995). Cup-product for Leibniz cohomology and dual-Leibniz algebras. *Mathematica Scandinavica*, 77(2), 189–196. <https://doi.org/10.7146/math.scand.a-12560>
- 3 Kolesnikov, P.S. (2016). Commutator algebras of pre-commutative algebras. *Matematicheskii Zhurnal*, 16(2), 145–158.
- 4 Kawski, M. (2001). Chronological algebras: Combinatorics and control. *Journal of Mathematical Sciences*, 103(6), 725–744. <https://doi.org/10.1023/A:1009502501461>
- 5 Aguiar, M. (2000). Pre-Poisson algebras. *Letters in Mathematical Physics*, 54, 263–277.
- 6 Aloulou, W., Arnal, D., & Chatbouri, R. (2017). Algèbre pré-Gerstenhaber à homotopie près. *Journal of Pure Applied Algebra*, 221(11), 2666–2688. <https://doi.org/10.1016/j.jpaa.2017.01.005>
- 7 Dzhumadildaev, A. (2007). Zinbiel algebras under q -commutators. *Journal of Mathematical Sciences*, 144(2), 3909–3925.
- 8 Diehl, J., Ebrahimi-Fard, K., & Tapia, N. (2020). Time-warping invariants of multidimensional time series. *Acta Applicandae Mathematicae*, 170(1), 265–290. <https://doi.org/10.1007/s10440-020-00333-x>
- 9 Diehl, J., Lyons, T., Preis, R., & Reizenstein, J. (2021). Areas of areas generate the shuffle algebra. *arXiv preprint arXiv:2002.02338* <https://doi.org/10.48550/arXiv.2002.02338>
- 10 Gao, X., Guo, L., & Zhang, Yi. (2021). Commutative matching Rota-Baxter operators, shuffle products with decorations and matching Zinbiel algebras. *Journal of Algebra*, 586, 402–432. <https://doi.org/10.1016/j.jalgebra.2021.06.032>
- 11 Covez, S., Farinati, M., Lebed, V., & Manchon, D. (2023). Bialgebraic approach to rack cohomology. *Algebraic and Geometric Topology*, 23(4), 1551–1582.
- 12 Chapoton, F. (2021). Zinbiel algebras and multiple zeta values. *arXiv preprint arXiv:2109.00241*.
- 13 Ikonicoff, S., & Pacaud Lemay, J.-S. (2023). Cartesian Differential Comonads and New Models of Cartesian Differential Categories. <https://doi.org/10.48550/arXiv.2108.04304>
- 14 Loday, J.-L. (2007). On the algebra of quasi-shuffles. *Manuscripta mathematica*, 123(1), 79–93. <https://doi.org/10.48550/arXiv.math/0506498>

- 15 Naurazbekova, A. (2019). On the structure of free dual Leibniz algebras. *Eurasian Mathematical Journal*, 10(3), 40–47. <https://doi.org/10.32523/2077-9879-2019-10-3-40-47>
- 16 Dzhumadildaev, A., & Tulenbaev, K. (2005). Nilpotency of Zinbiel algebras. *Journal of Dynamical and Control Systems*, 11(2), 195–213. <https://doi.org/10.1007/s10883-005-4170-1>
- 17 Higman, G. (1956). On a conjecture of Nagata. *Proc. Cambridge Philos. Soc.* 52, 1–4.
- 18 Naurazbekova, A., & Umirbaev, U. (2010). Identities of dual Leibniz algebras. *TWMS Journal of Pure and Applied Mathematics*, 1(1), 86–91.
- 19 Towers, D.A. (2023). Zinbiel algebras are nilpotent. *Journal of Algebra and Its Applications*, 22(8), 2350166. <https://doi.org/10.1142/S0219498823501669>
- 20 Adashev, J.Q., Omirov, B.A., & Khudoyberdiyev, A.Kh. (2007). On some nilpotent classes of Zinbiel algebras and their applications. *Third International Conference on Research and Education in Mathematics*. Malaysia, 45–47.
- 21 Camacho, L.M., Karimjanov, I., Kaygorodov, I., & Khudoyberdiyev, A. (2020). Central extensions of filiform Zinbiel algebras. *Linear and Multilinear Algebra*, 70(4), 1–17.
- 22 Camacho, L.M., Canete, E.M., Gomez, S., & Omirov, B.A. (2013). p -Filiform Zinbiel algebras. *Linear Algebra and its Applications*, 438(7), 2958–2972. <https://doi.org/10.1016/j.laa.2012.11.030>
- 23 Omirov, B.A. (2002). Classification of two-dimensional complex Zinbiel algebras. *Uzbek. Mat. Zh.*, 2, 55–59.
- 24 Kaygorodov, I., Alvarez, M.A., & Mello, T.C.d. (2023). Central extensions of 3-dimensional Zinbiel algebras. *Ricerche di Matematica*, 72, 921–947. <https://doi.org/10.1007/s11587-021-00604-1>
- 25 Alvarez, M.A., Junior, R.F., & Kaygorodov, I. (2022). The algebraic and geometric classification of Zinbiel algebras. *Journal of Pure and Applied Algebra*, 226(11), 107106. <https://doi.org/10.1016/j.jpaa.2022.107106>
- 26 Naurazbekova, A.S. (2010). Universalnye multiplikativnye obertyvaiushchie algebrы dualnykh algebr Leibnitsa [Universal multiplicative enveloping algebras of dual Leibniz algebras]. *Vestnik Yevraziiskogo natsionalnogo universiteta imeni L.N. Gumilyova — Bulletin of L.N. Gumilyov Eurasian National University*, 75(2), 307–316 [in Russian].
- 27 Jacobson, N. (1968). Structure and Representations of Jordan Algebras. *American Mathematical Society, Providence, R.I.*
- 28 Bokut, L.A. (1976). Vlozheniia v prostye assotsiativnye algebrы [Imbeddings into simple associative algebras]. *Algebra i logika — Algebra and logic*, 15(2), 117–142 [in Russian].
- 29 Bergman, G.M. (1978). The diamond lemma for ring theory. *Advances in Mathematics*, 29(2), 178–218. [https://doi.org/10.1016/0001-8708\(78\)90010-5](https://doi.org/10.1016/0001-8708(78)90010-5)