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A.E. Mirzakulova[1], N. Atakhan[2], N. Asset[3], A. Rysbek[1]

[1] *Abay Kazakh National Pedagogical University, Almaty, Kazakhstan;*
[2] *Kazakh State Women's Teacher Training University, Almaty, Kazakhstan;*
[3] *Nazarbayev Intellectual Schools, Almaty, Kazakhstan*
(E-mail: mirzakulovaaziza@gmail.com)

Asymptotic convergence of the solution for singularly perturbed boundary value problem with boundary jumps

The article is devoted to study of boundary value problem with boundary jumps for third order linear integro-differential equation with a small parameter at the highest derivatives, provided that additional characteristic equation's roots have opposite signs. The modified unperturbed boundary value problem is constructed. The solution of modified unperturbed problem is obtained. Initial jumps' values of the integral term and solution are defined. An estimate difference of solution for singularly perturbed and modified unperturbed boundary value problems is obtained. The convergence of solution for singularly perturbed boundary value problem to solution of modified unperturbed boundary value problem is proved.

Keywords: singular perturbation, small parameter, the boundary jump, the initial jump, boundary functions, asymptotic.

Introduction

The theory of singular perturbations has been with us, in one form or another, for a little over a century (although the term 'singular perturbation' dates from the 1940s). The subject, and the techniques associated with it, have evolved over this period as a response to the need to find approximate solutions (in an analytical form) to complex problems. Typically, such problems are expressed in terms of differential equations which contain at least one small parameter, and they can arise in many fields: fluid mechanics, particle physics and combustion processes, to name but three. The essential hallmark of a singular perturbation problem is that a simple and straightforward approximation (based on the smallness of the parameter) does not give an accurate solution throughout the domain of that solution.

Mathematical problems that make extensive use of a small parameter were probably first described by J.H. Poincare (1854–1912) as part of his investigations in celestial mechanics. (The small parameter, in this context, is usually the ratio of two masses.) Although the majority of these problems were not obviously 'singular'—and Poincare did not dwell upon this—some are; for example, one is the earth-moon-spaceship problem mentioned. Nevertheless, Poincare did lay the foundations for the technique that underpins our approach: the use of asymptotic expansions. The notion of a singular perturbation problem was first evident in the seminal work of L. Prandtl (1874–1953) on the viscous boundary layer (1904). Here, the small parameter is the inverse Reynolds number and the equations are based on the classical Navier-Stokes equation of fluid mechanics. This analysis, coupled with small-Reynolds-number approximations that were developed at about the same time (1910), prepared the ground for a century of singular perturbation work in fluid mechanics. But other fields over the century also made important contributions, for example: integration of differential equations, particularly in the context of quantum mechanics; the theory of nonlinear oscillations; control theory; the theory of semiconductors.

Theory of asymptotic integration of singularly perturbed equations has become purposefully developed starting with the works of L. Schlesinger, G.D. Birkhoff, P. Noaillon. In a further development of the main trends of the theory W. Wasow, A.H. Nayfeh, M. Nagumo, A.N. Tikhonov, M.I. Vishik, L.A. Lusternik, N.N. Bogolyubov, U.A Mitropolsky, A.B. Vasilieva and V.F. Butuzov, R.E. O'Malley, D.R. Smith, W. Eckhaus, K. W. Chang and F. A. Howes, J. Kevorkian and J.D. Cole, Sanders and F. Verhulst, E.F. Mischenko and N.X. Rozov, S.A. Lomov, K.A. Kassymov and others have made a significant contribution. For a broad class of singularly perturbed problems effective asymptotic methods to build a uniform approximation with any degree of accuracy in the small parameter were developed.

For the first time, boundary value problems with initial jumps for singularly perturbed linear ordinary differential and integro-differential equations of the second order was studied by K.A. Kassymov [1, 2]. A systematic study of boundary value problems with initial jumps Kassymov and his students began in the nineties of the last century. He developed methods for qualitative research and the construction of an asymptotic expansion of solutions of boundary value problems with initial jumps for singularly perturbed ordinary differential equations [3, 4]. General boundary-value problems for singularly perturbed ordinary differential equations of higher orders are investigated by D.N. Nurgabyl. He singled out a class of singularly perturbed boundary value problems with an initial jump and developed an algorithm for constructing and investigating the asymptotic behavior of solutions of general boundary value problems [5, 6]. K.A. Kassymov and M.K. Dauylbaev for singularly perturbed higher-order integro-differential equations studied problems of a special type, when the presence of integral terms leads to a qualitative change in the behavior of the solution [7–9].

M. K. Dauylbaev [10–12] studied boundary value problems with two boundary layers possessing the phenomena of initial jumps. The novelty of these studies is that when the small parameter tends to zero, the fast solution variable grows unlimitedly, not only at one, the so-called initial point, but also at the other end of the considered segment. Thus, a class of singularly perturbed integro-differential equations with initial jump phenomena at both ends of the given segment is singled out. He also developed a method for studying and constructing the asymptotic of the solution of the Cauchy problem with initial jump for singularly perturbed linear differential equations with impulse action [13].

Consider the singularly perturbed integro-differential equation

$$L_\varepsilon y \equiv \varepsilon^2 y''' + \varepsilon A_0(t)y'' + A_1(t)y' + A_2(t)y = F(t) + \int_0^1 \sum_{i=0}^{1} H_i(t,x)y^{(i)}(x,\varepsilon)dx, \tag{1}$$

with integral boundary conditions

$$y(0,\varepsilon)=\alpha, \quad y'(0,\varepsilon)=\beta, \quad y(1,\varepsilon)=\gamma+\int_0^1\sum_{i=0}^{1} a_i(x)y^{(i)}(x,\varepsilon)dx, \tag{2}$$

where $\varepsilon > 0$ is a small parameter, α, β, γ are known constants independent of ε.

We will need the following assumptions:

C1) $A_i(t), i=\overline{0,2},\ F(t),\ a_j(x), j=0,1$ are sufficiently smooth functions defined on the interval $[0,1]$, $H_0(t,x), H_1(t,x)$ are sufficiently smooth functions defined in the domain $D=\{0\le t\le 1, 0\le x\le 1\}$.

C2) The roots $\mu_i(t),\ i=1,2$ of «additional characteristic equation» $\mu^2 + A_0(t)\mu + A_1(t) = 0$ satisfy the following inequalities $\mu_1(t) < -\gamma_1 < 0,\ \ \mu_2(t) > \gamma_2 > 0$.

C3) 1 is not an eigenvalue of the kernel

$$H(t,s) = \frac{H_1(t,s)}{A_1(s)} + \int_s^1 \frac{1}{A_1(s)}\left(H_0(t,x) - H_1(t,x)\frac{A_2(x)}{A_1(x)}\right)\exp\left(-\int_s^x \frac{A_2(p)}{A_1(p)}dp\right)dx.$$

C4) $a_1(1) \ne 1$.

C5)$\bar\delta \ne 0$, where

$$\bar\delta = \int_0^1 \frac{H_1(s,1)}{(1-a_1(1))A_1(s)y_{30}(s)}\left(y_{30}(1) - a_1(s)y_{30}(s) - \int_0^1\sum_{i=0}^{1} a_i(x)y_{30}^{(i)}(x)dx\right)ds.$$

For the solution of the boundary value problem (1),(2) are valid the following asymptotic estimations [10] as $\varepsilon \to 0$:

$$\left|y^{(q)}(t,\varepsilon)\right| \le C\left(|\alpha| + \varepsilon|\beta| + \max_{0\le t\le 1}\left|F(t) + \frac{\gamma}{1-a_1(1)}H_1(t,1)\right|\right) +$$
$$+C\varepsilon^{1-q}e^{-\gamma_1\frac{t}{\varepsilon}}\left(|\alpha| + |\beta| + \max_{0\le t\le 1}\left|F(t) + \frac{\gamma}{1-a_1(1)}H_1(t,1)\right|\right) +$$
$$+\frac{C}{\varepsilon^i}e^{-\gamma_2\frac{1-t}{\varepsilon}}\left(\left|\frac{\alpha}{1-a_1(1)}\right| + \varepsilon\left|\frac{\beta}{1-a_1(1)}\right| + \max_{0\le t\le 1}\left|F(t) + \frac{\gamma}{1-a_1(1)}H_1(t,1)\right|\right), \quad q=0,1,2. \tag{3}$$

Consider the following modified unperturbed problem as $\varepsilon = 0$:

$$L_0\overline{y} \equiv A_1(t)\overline{y}'(t) + A_2(t)\overline{y}(t) = F(t) + \int_0^1\sum_{i=0}^{1} H_i(t,x)\overline{y}^{(i)}(x)dx + \Delta(t), \tag{4}$$

$$\overline{y}(0) = \alpha, \quad \overline{y}(1) = \gamma + \int\limits_0^1 \sum_{i=0}^{1} a_i(x)\overline{y}^{(i)}(x)dx + \Delta_1, \tag{5}$$

where $\Delta(t)$ and Δ_1 are respectively unknown initial jumps of the integral term and the solution.

Let us denote by

$$u(t,\varepsilon) = y(t,\varepsilon) - \overline{y}(t), \Rightarrow y(t,\varepsilon) = u(t,\varepsilon) + \overline{y}(t), \tag{6}$$

where $y(t,\varepsilon)$ is a solution of singularly perturbed problem (1), (2) and $\overline{y}(t)$ is a solution of the unperturbed problem (4), (5).

Substituting (6) into (1), (2), we obtain the problem for $u(t,\varepsilon)$:

$$L_\varepsilon u \equiv \varepsilon^2 u''' + \varepsilon A_0(t)u'' + A_1(t)u' + A_2(t)u = -\Delta(t) + \varepsilon^2\overline{y}'''-$$
$$-\varepsilon A_0(t)\overline{y}'' + \int\limits_0^1 \sum_{i=0}^{1} H_i(t,x)u^{(i)}(x,\varepsilon)dx, \tag{7}$$

with boundary conditions

$$u(0,\varepsilon) = 0, \quad u'(0,\varepsilon) = \beta - \overline{y}'(0), \quad u(1,\varepsilon) = -\Delta_1 + \int\limits_0^1 \sum_{i=0}^{1} a_i(x)u^{(i)}(x,\varepsilon)dx, \tag{8}$$

here $\Delta_1 = (1 - a_1(1))\Delta_0$.

The problem (7), (8) is of the same type as the problem (1), (2), applying the asymptotic estimations (3) for $u(t,\varepsilon)$, we get

$$\left|u^{(q)}(t,\varepsilon)\right| \le C\left(\varepsilon + \varepsilon\left|\beta - \overline{y}'(0)\right| + \left|\Delta(t) + H_1(t,1)\Delta_0\right|\right) +$$
$$+C\varepsilon^{1-q}e^{-\gamma_1\frac{t}{\varepsilon}}\left(\varepsilon + \left|\beta - \overline{y}'(0)\right| + \left|\Delta(t) + H_1(t,1)\Delta_0\right|\right) +$$
$$+\frac{C}{\varepsilon^i}e^{-\gamma_2\frac{1-t}{\varepsilon}}\left(\varepsilon + \varepsilon\left|\frac{\beta - \overline{y}'(0)}{1 - a_1(1)}\right| + \left|\Delta(t) + H_1(t,1)\Delta_0\right|\right), \quad q = 0,1,2.$$

We choose the unknown function $\Delta(t)$ that the solution of the problem (7), (8) approach zero as $\varepsilon \to 0$, i.e. if the equality

$$\Delta(t) = -H_1(t,1)\Delta_0 \tag{9}$$

is valid, then the solution of the problem (1), (2) approaches to the modified unperturbed problem (4), (5) as $\varepsilon \to 0$. Thus, if the initial jump of the integral term $\Delta(t)$ is defined by the formula (9), then the solution of the problem (1), (2) approaches to the solution of the following modified unperturbed problem:

$$L_0\overline{y} \equiv A_1(t)\overline{y}'(t) + A_2(t)\overline{y}(t) = F(t) + \int\limits_0^1 \sum_{i=0}^{1} H_i(t,x)\overline{y}^{(i)}(x)dx - H_1(t,1)\Delta_0; \tag{10}$$

$$\overline{y}(0) = \alpha, \quad \overline{y}(1) = \gamma + \int\limits_0^1 \sum_{i=0}^{1} a_i(x)\overline{y}^{(i)}(x)dx + (1 - a_1(1))\Delta_0. \tag{11}$$

At first, we consider the equation (10) with condition

$$\overline{y}(0) = \alpha.$$

We seek the solution of the problem (10), (11):

$$\overline{y}(t) = \alpha\exp\left(-\int\limits_0^t \frac{A_2(x)}{A_1(x)}dx\right) + \int\limits_0^t \frac{\overline{z}(s)}{A_1(s)}\exp\left(-\int\limits_s^t \frac{A_2(x)}{A_1(x)}dx\right)ds, \tag{12}$$

where

$$\overline{z}(t) = F(t) + \int\limits_0^1 \sum_{i=0}^1 H_i(t,x)\overline{y}^{(i)}(x)dx - H_1(t,1)\Delta_0. \tag{13}$$

Substituting (12) into the equation (13), we obtain that $\overline{z}(t)$ satisfies the following Fredholm integral equation of the second kind:

$$\overline{z}(t) = \varphi(t) + \int\limits_0^1 H(t,s)\overline{z}(s)ds, \tag{14}$$

where

$$\varphi(t) = F(t) - H_1(t,1)\Delta_0 + \alpha \int\limits_0^1 \left(H_0(t,x) - H_1(t,x)\frac{A_2(x)}{A_1(x)} \right) \exp\left(-\int\limits_0^x \frac{A_2(p)}{A_1(p)}dp \right) dx; \tag{15}$$

$$H(t,s) = \frac{H_1(t,s)}{A_1(s)} + \int\limits_s^1 \frac{1}{A_1(s)} \left(H_0(t,x) - H_1(t,x)\frac{A_2(x)}{A_1(x)} \right) \exp\left(-\int\limits_s^x \frac{A_2(p)}{A_1(p)}dp \right) dx.$$

In view of the condition (C3), integral equation (14) has an unique solution, that can be represented in the form:

$$\overline{z}(t) = \varphi(t) + \int\limits_0^1 R(t,s)\varphi(s)ds, \tag{16}$$

here $R(t,s)$ is a resolvent of the kernel $H(t,s)$, the function $\varphi(t)$ is defined by the formula (15). Substituting (16) into the function (12), by virtue of (15), we get the solution of the problem (10), (11):

$$\overline{y}(t) = \alpha \exp\left(-\int\limits_0^t \frac{A_2(x)}{A_1(x)}dx \right) + \int\limits_0^t \frac{1}{A_1(s)} \exp\left(-\int\limits_s^t \frac{A_2(x)}{A_1(x)}dx \right) \left[\overline{F}(s) - \overline{H}_1(s,1)\Delta_0 + \right.$$

$$\left. + \alpha \int\limits_0^1 \left(\overline{H}_0(s,x) - \overline{H}_1(s,x)\frac{A_2(x)}{A_1(x)} \right) \exp\left(-\int\limits_0^x \frac{A_2(p)}{A_1(p)}dp \right) dx \right] ds, \tag{17}$$

where

$$\overline{F}(s) = F(s) + \int\limits_0^1 R(s,p)F(p)dp;$$

$$\overline{H}_i(s,x) = H_i(s,x) + \int\limits_0^1 R(s,p)H_i(p,x)dp, \quad i = 0,1.$$

To determine the initial jump Δ_0 of solution, we substitute (17) into the second condition of (11). As a result, we can find the initial jump Δ_0 of solution:

$$\Delta_0 = \frac{\overline{y}(1) - \int\limits_0^1 \sum\limits_{i=0}^1 a_i(x)\overline{y}^{(i)}(x)dx - \gamma}{1 - a_1(1)}. \tag{18}$$

Theorem 1. Under the above assumptions (C1)–(C5), for the solution $y(t,\varepsilon)$ of the boundary value problem (1), (2) hold the following limiting equalities:

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = \overline{y}(t), \quad 0 \le t < 1;$$

$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = \overline{y}'(t), \quad 0 < t < 1;$$

$$\lim_{\varepsilon\to 0} y''(t,\varepsilon) = \overline{y}''(t), \quad 0 < t < 1,$$

where $\overline{y}(t)$ is a solution of the problem (10), (11) and expressed by (17), the initial jump Δ_0 is defined by the formula (18).

Example. Consider the following singularly perturbed boundary value problem with boundary jumps:

$$\varepsilon^2 y''' - \varepsilon y'' - 2y' = \delta \int_0^1 y'(x,\varepsilon)dx; \tag{19}$$

$$y(0,\varepsilon) = 1, \ y'(0,\varepsilon) = 0, \ \ y(1,\varepsilon) = \int_0^1 ay'(x,\varepsilon)dx, \ \ a \neq 1. \tag{20}$$

The fundamental system of solutions of the equation $\varepsilon^2 y''' - \varepsilon y'' - 2y' = 0$ have the form:

$$y_1(t,\varepsilon) = 1, \ \ y_2(t,\varepsilon) = e^{-\frac{t}{\varepsilon}}, \ \ y_3(t,\varepsilon) = e^{-\frac{2}{\varepsilon}(1-t)}.$$

Let us denote by the right-hand side of the equation (19)

$$z(\varepsilon) = \delta \int_0^1 y'(x,\varepsilon)dx. \tag{21}$$

Then general solution of the equation (19) has the form:

$$y(t,\varepsilon) = C_1 + C_2 e^{-\frac{t}{\varepsilon}} + C_3 e^{-\frac{2}{\varepsilon}(1-t)} - 0,5z(\varepsilon)t, \tag{22}$$

here $C_i, i = \overline{1,3}$ are unknown constants, $z(\varepsilon)$ is an unknown function.

Substituting the function (22) into the (21), we obtain the equality to define the function $z(\varepsilon)$

$$z(\varepsilon) = \frac{C_2\delta(e^{-\frac{1}{\varepsilon}} - 1) + C_3\delta(1 - e^{-\frac{2}{\varepsilon}})}{1 + 0,5\delta}.$$

Now, we determine the unknown constants $C_i, i = \overline{1,3}$ in (22), which satisfy the boundary conditions (20). Thus, we need to solve the system of algebraic equations:

$$\begin{cases} C_1 + C_2 + e^{-\frac{2}{\varepsilon}} C_3 = 1; \\ \frac{\varepsilon\delta\left(1 - e^{-\frac{1}{\varepsilon}}\right) - 2 - \delta}{\varepsilon(2+\delta)} C_2 + \frac{2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta\left(1 - e^{-\frac{2}{\varepsilon}}\right)}{\varepsilon(2+\delta)} C_3 = 0; \\ C_1 + \frac{2(1-a)e^{-\frac{1}{\varepsilon}} + \delta + 2a}{2+\delta} C_2 + \frac{2(1-a) + (\delta+2a)e^{-\frac{2}{\varepsilon}}}{2+\delta} C_3 = 0. \end{cases}$$

The main determinant of the linear algebraic system has the form:

$$\Delta(\varepsilon) = \frac{2(1-a)}{\varepsilon(2+\delta)^2}\left[(\varepsilon\delta(1 - e^{-\frac{1}{\varepsilon}}) - 2 - \delta)(1 - e^{-\frac{2}{\varepsilon}}) + (2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1 - e^{-\frac{2}{\varepsilon}}))(1 - e^{-\frac{1}{\varepsilon}}\right].$$

As a result, the solutions of the system are defined by the formula:

$$C_1(\varepsilon) = \frac{(\varepsilon\delta(1 - e^{-\frac{1}{\varepsilon}}) - 2 - \delta)(2(1-a) + (\delta + 2a)e^{-\frac{2}{\varepsilon}})}{2(1-a)\left[(\varepsilon\delta(1 - e^{-\frac{1}{\varepsilon}}) - 2 - \delta)(1 - e^{-\frac{2}{\varepsilon}}) + (2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1 - e^{-\frac{2}{\varepsilon}}))(1 - e^{-\frac{1}{\varepsilon}})\right]} -$$

$$\frac{-(2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1 - e^{-\frac{2}{\varepsilon}})(2(1-a)e^{-\frac{1}{\varepsilon}} + \delta + 2a))}{};$$

$$C_2(\varepsilon) = \frac{(2+\delta)(2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1 - e^{-\frac{2}{\varepsilon}}))}{2(1-a)\left[(\varepsilon\delta(1 - e^{-\frac{1}{\varepsilon}}) - 2 - \delta)(1 - e^{-\frac{2}{\varepsilon}}) + (2e^{-\frac{2}{\varepsilon}}(2+\delta) - \varepsilon\delta(1 - e^{-\frac{2}{\varepsilon}}))(1 - e^{-\frac{1}{\varepsilon}})\right]};$$

$$C_3(\varepsilon) = \frac{(2+\delta)(2+\delta-\varepsilon\delta(1-e^{-\frac{1}{\varepsilon}}))}{2(1-a)\left[(\varepsilon\delta(1-e^{-\frac{1}{\varepsilon}})-2-\delta)(1-e^{-\frac{2}{\varepsilon}})+(2e^{-\frac{2}{\varepsilon}}(2+\delta)-\varepsilon\delta(1-e^{-\frac{2}{\varepsilon}}))(1-e^{-\frac{1}{\varepsilon}}\right]}.$$

If as $\varepsilon \to 0$, then

$$C_1(\varepsilon) \to 1, \;\; C_2(\varepsilon) \to 0, \;\; C_3(\varepsilon) \to \frac{2+\delta}{2(a-1)}.$$

Consider the following modified unperturbed problem:

$$-2\overline{y}'(t) = \int_0^1 \delta\overline{y}(x)dx - \delta\Delta_0, \tag{23}$$

$$\overline{y}(0) = 1, \;\; \overline{y}(1) = \int_0^1 a\overline{y}(x)dx + (1-a)\Delta_0,$$

here Δ_0 is called the initial jump of the solution.

Let us denote by

$$\overline{z} = \int_0^1 \delta\overline{y}(x)dx - \delta\Delta_0. \tag{24}$$

The general solution of the equation (23) is as follows

$$\overline{y}(t) = \frac{\delta\Delta_0}{2+\delta}t + C. \tag{25}$$

Substituting (25) in (24), we define the unknown function $\overline{z}$:

$$\overline{z} = -\frac{\delta\Delta_0}{1+0,5\delta}.$$

As a result, the solution of the modified unperturbed problem (23) has the form:

$$\overline{y}(t) = \frac{\delta}{2(1-a)}t + 1.$$

The initial jump Δ_0 of solution is defined by the following formula

$$\Delta_0 = \frac{2+\delta}{2(1-a)}.$$

The results can be seen to perform the following limiting equalities:

$$\lim_{\varepsilon\to 0} y(t,\varepsilon) = 1 + \frac{\delta}{2(1-a)}t \equiv \overline{y}(t), \quad 0 \le t < 1;$$

$$\lim_{\varepsilon\to 0} y'(t,\varepsilon) = \frac{\delta}{2(1-a)} \equiv \overline{y}'(t), \quad 0 < t < 1;$$

$$\lim_{\varepsilon\to 0} y''(t,\varepsilon) = 0 \equiv \overline{y}''(t), \quad 0 < t < 1.$$

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А.Е. Мирзакулова, Н. Атахан, Н. Асет, А. Рысбек

Шекаралы секірісті сингулярлы ауытқыған шеттік есебі шешімінің асимптотикалық жинақтылығы

Мақала екі үлкен туындысының алдында кіші параметрі бар үшінші ретті сингулярлы ауытқыған сызықты интегралды-дифференциалдық теңдеу үшін қосымша сипаттауыш теңдеудің түбірлерінің таңбасы қарама-қарсы болған жағдайдағы шекаралы секірісті шеттік есебін зерттеуге арналған. Өзгертілген ауытқымаған есеп құрылды, оның шешімі алынды. Сингулярлы ауытқыған шеттік есеп шешімі мен өзгертілген ауытқымаған есеп шешімінің арасындағы айырым бағаланды. Интегралдық мүшенің және шешімнің бастапқы секірістерінің шамалары анықталды. Берілген сингулярлы ауытқыған шеттік есеп шешімінің өзгертілген ауытқымаған шеттік есеп шешіміне ұмтылатыны дәлелденді.

Кілт сөздер: сингулярлы ауытқу, кіші параметр, шекаралық секіріс, бастапқы секіріс, шекаралық функциялар.

А.Е. Мирзакулова, Н. Атахан, Н. Асет, А. Рысбек

Асимптотическая сходимость решения сингулярно возмущенной краевой задачи с граничными скачками

Статья посвящена изучению краевой задачи с граничными скачками для линейного интегродифференциального уравнения третьего порядка с малым параметром при старших производных при условии, что корни дополнительного характеристического уравнения имеют противоположные знаки. Построена модифицированная невозмущенная краевая задача. Получено решение модифицированной невозмущенной задачи. Определены значения начальных скачков интегрального члена и решения. Получена оценка разности решений сингулярно возмущенных и модифицированных невозмущенных краевых задач. Доказана сходимость решений сингулярно возмущенной краевой задачи к решению модифицированной невозмущенной краевой задачи.

Ключевые слова: сингулярное возмущение, малый параметр, граничный скачок, начальный скачок, граничные функции, асимптотика.

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