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WEIGHTED INEQUALITIES FOR SUPERPOSITION OF OPERATORS

Abek A.N.[1], **Gogatishvili A.**[2], **Bokayev N.A**[3], **Unver T.**[4]

[1]L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
[1]E-mail: azhar.abekova@gmail.com
[2]Institute of Mathematics of the Czech Academy of Sciences, Praha, Czech Republic
[2]E-mail: gogatish@math.cas.cz
[3]L.N. Gumilyov Eurasian National University, Astana, Kazakhstan
[3]E-mail: bokayev2011@yandex.kz
[4]Kirikkale University, Kirikkale, Turkey
[4]E-mail: tugceunver@kku.edu.tr

In this paper we consider the superposition of three operators: Copson, Hardy and Tandori. Denote by $\mathfrak{M}^+(0,\infty)$ the set of all non-negative measurable functions on $(0,\infty)$.

Let $1 \le p < \infty$, $0 < q < \infty$, u, v and w are weights, (i.e. locally integrable non-negative functions on $(0,\infty)$), φ is strictly increasing function $(0,\infty)$, and $\frac{\varphi}{U}$ is decreasing on $(0,\infty)$, where

$$U(s) = \int_0^s u(t)dt$$

Our goal in this paper is to characterize the following inequality

$$\left(\int_0^\infty \left(\sup_{t<s<\infty} \frac{1}{\varphi(s)} \int_0^s \left(\int_\tau^\infty h(y)dy\right) u(\tau)d\tau\right)^q w(t)dt\right)^{\frac{1}{q}} \le C\Big(\int_0^\infty h^p(s)v(s)ds\Big)^{\frac{1}{p}} \qquad (1)$$

for all $h \in \mathfrak{M}^+(0,\infty)$.

Using the Fubini theorem for non-negative functions, we have

$$\int_0^s u(\tau)\int_\tau^\infty h(t)dtdy \;\;=\;\; \int_0^s U(\tau)h(\tau) + U(s)\int_s^\infty h(\tau)d\tau.$$

Therefore the inequality (1) is equivalent with following inequality

$$\left(\int_0^\infty \left(\sup_{t<s<\infty} \frac{1}{\varphi(s)} \left(\int_0^s U(\tau)h(\tau)d\tau + U(s)\int_s^\infty h(\tau)d\tau\right)\right)^q w(t)dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty h^p(\tau)v(\tau)d\tau\right)^{\frac{1}{p}}. \qquad (2)$$

Throughout the paper, we always denote by c or C a positive constant which is independent of the main parameters, but it may vary from line to line. However a constant with subscript such as $_1$ does not change in different occurrences.

Let $q \in (0,\infty)$, $p \in (1,\infty)$, and let u, v, w be weights on $(0,\infty)$. φ is U -quasiconcave function on $(0,\infty)$, Then there exists a constant $C > 0$ such that the inequality (3) holds for all $f \in \mathfrak{M}^+(0,\infty)$ if and only if one of the following conditions is satisfied:

(i) $1 < p \le q$,

$$C_1 := \sup_{t\in(0,\infty)} \left(\int_0^\infty \frac{w(s)ds}{(\varphi(s)+\varphi(t))^q}\right)^{\frac{1}{q}} \left(\int_0^\infty \frac{U^{p'}(t)U^{p'}(s)}{U^{p'}(s)+U^{p'}(t)} v^{1-p'}(s)ds\right)^{\frac{1}{p'}} < \infty,$$

(ii) $1 < p, q < p$,

$$C_2 := \left(\int_0^\infty \left(\int_0^\infty \frac{w(s)ds}{(\varphi(s)+\varphi(t))^q}\right)^{\frac{r}{q}} d\nu_p(t)\right)^{\frac{1}{r}} < \infty,$$

where ν_p is the representation measure of

$$\varphi^r(t) \sup_{s\in(t,\infty)} \frac{1}{\varphi^r(s)} \left(\int_0^\infty \frac{U^{p'}(\tau)U^{p'}(s)}{U^{p'}(s)+U^{p'}(\tau)} v^{1-p'}(\tau)d\tau\right)^{\frac{r}{p'}},$$

i.e.

$$\varphi^r(t) \sup_{s\in(t,\infty)} \frac{1}{\varphi^r(s)} \left(\int_0^\infty \frac{U^{p'}(\tau)U^{p'}(s)}{U^{p'}(s)+U^{p'}(\tau)} v^{1-p'}(\tau)d\tau\right)^{\frac{r}{p'}} = \int_0^\infty \frac{\varphi^r(t)}{\varphi^r(s)+\varphi^r(t)} \nu_p(s)ds.$$

Moreover, the best constant in the inequality (3) satisfies

$$C \approx \begin{cases} C_1 & \text{in case (i),} \\ C_2 & \text{in case (ii).} \end{cases}$$

The proof of Theorem () is essentially based on the following theorem.

Let $q \in (0,\infty)$, $p \in (1,\infty)$, and let u, v, w be weights on $(0,\infty)$. φ is U -quasiconcave function on $(0,\infty)$, Then there exists a constant $C > 0$ such that the inequality (3) holds for all $f \in \mathfrak{M}^+(0,\infty)$ if and only if one of the following conditions is satisfied:

(i) $1 < p \le q$,

$$A_1 := \sup_{k\in Z} \sup_{x_k\le t\le x_{k+1}} \frac{G(t)}{\varphi(t)} \left(\int_{x_k}^t U^{p'}(s)v^{1-p'}(s)ds + U^{p'}(t)\int_t^{x_{k+1}} v^{1-p'}(s)ds,\right)^{\frac{1}{p'}} < \infty,$$

(ii) $1 < p, q < p$,

$$A_2 := \left(\sum_{k\in Z} \left(\sup_{x_k\le t\le x_{k+1}} \frac{G(t)}{\varphi(t)} \left(\int_{x_k}^t U^{p'}(s)v^{1-p'}(s)ds + U^{p'}(t)\int_t^{x_{k+1}} v^{1-p'}(s)ds\right)^{\frac{1}{p'}}\right)^r\right)^{\frac{1}{r}} < \infty,$$

where $\{x_k\}_{k=N}^{M+1}$ is discretization sequence for G and

$$G(t) = \int_0^t w(s)ds + \varphi(t)\int_t^\infty \varphi^{-1}(s)w(s)ds.$$

Moreover, the best constant in the inequality (3) satisfies

$$C \approx \begin{cases} A_1 & \text{in case (i),} \\ A_2 & \text{in case (ii).} \end{cases}$$

We present the following definitions and auxiliary statements from (1; 2; 3; 4; 5) that are used to prove the indicated theorems.

[2] Let φ be a continuous strictly increasing function on $[0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. Then we say that φ is admissible.

Let φ be an admissible function. We say that a function h is φ-quasiconcave if h is equivalent to an increasing function on $[0, \infty)$ and $\frac{h}{\varphi}$ is equivalent to a decreasing function on $(0, \infty)$. We say that a φ-quasiconcave function h is non-degenerate if

$$\lim_{t\to 0+} h(t) = \lim_{t\to\infty} \frac{1}{h(t)} = \lim_{t\to\infty} \frac{h(t)}{\varphi(t)} = \lim_{t\to 0+} \frac{\varphi(t)}{h(t)} = 0. \tag{3}$$

The family of non-degenerate φ -quasiconcave functions will be denoted by Ω_φ.

[3] Assume that φ is admissible and $h \in \Omega_\varphi$. We say that $\{\mu_k\}_{k\in\mathbb{Z}}$ is a discretizing sequence for h with respect to φ if

(i) $\mu_0 = 1$ and $\varphi(\mu_k) \upuparrows$;

(ii) $h(\mu_k) \upuparrows$ and $\frac{h(\mu_k)}{\varphi(\mu_k)} \downdownarrows$;

(iii) there is a decomposition $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$ such that $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$ an for every $t \in [\mu_k, \mu_{k+1}]$,

$$h(\mu_k) \approx h(t) \quad \text{if} \quad k \in \mathbb{Z}_1,$$
$$\frac{h(\mu_k)}{\varphi(\mu_k)} \approx \frac{h(t)}{\varphi(t)} \quad \text{if} \quad k \in \mathbb{Z}_2.$$

Let φ be an admissible function and let ν be a non-negative Borel measure on $[0, \infty)$. We say that the function h defined by

$$h(t) = \varphi(t) \int_{[0,\infty)} \frac{d\nu(s)}{\varphi(s) + \varphi(t)}, \ \ t \in (0, \infty),$$

is the fundamental function of the measure ν with respect to φ. We will also say that ν is a representation measure of h with respect to φ.

We say that ν is non-degenerate if the following conditions are satisfied for every $t \in (0, \infty)$:

$$\int_{[0,\infty)} \frac{d\nu(s)}{\varphi(s) + \varphi(t)} < \infty, \ t \in (0, \infty) \quad \text{and} \quad \int_{[0,1]} \frac{d\nu(s)}{\varphi(s)} = \int_{[1,\infty)} d\nu(s) = \infty.$$

Assume that φ is an admissible function, $f \in \Omega_\varphi$, ν is a non-negative non-degenerate Borel measure on $[0, \infty)$ and h is the fundamental function of ν with respect to φ. If $\{x_k\}$ is a discretizing sequence for h with respect to φ, then

$$\int_{[0,\infty)} \frac{f(t)}{\varphi(t)} d\nu(t) \approx \sum_{k\in Z} \left(\frac{f(x_k)}{\varphi(x_k)} \right) \varphi(x_k).$$

Assume that φ is an admissible function, ν is a nondegenerate nonnegative Borel measure on $[0,\infty)$, h is the fundamental function of ν and $f \in \mathfrak{M}^+(0,\infty)$. If $\{x_k\}$ is a discretizing sequence for h with respect to φ, then

$$\begin{aligned}
\int\limits_{[0,\infty)} \sup_{y\in(0,\infty)} \frac{|f(y)|}{\varphi(x)+\varphi(y)} d\nu(x) &\approx \sum_{k\in\mathbb{Z}} \left(\sup_{y\in(0,\infty)} \frac{|f(y)|}{\varphi(x_k)+\varphi(y)} \right) \varphi(x_k) \\
&\approx \sum_{k\in\mathbb{Z}} \left(\varphi^{-1}(x_k) \sup_{x_{k-1}\le y<x_k} |f(y)| + \sup_{x_k\le y<x_{k+1}} |f(y)|\varphi^{-1}(y) \right) h(x_k) \\
&\approx \sum_{k\in\mathbb{Z}} \sup_{x_k\le y<x_{k+1}} |f(y)|\varphi^{-1}(y)h(y).
\end{aligned}$$

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ANALYSIS OF LIMITING VALUES OF THE FRACTIONAL DERIVATE'S ORDER FOR A CAUCHY-TYPE PROBLEM.

Andryuchshenko Yuliya Alexandrovna

[1]Karaganda Buketov University, Karaganda, Kazakhstan
[1]E-mail: katy.02.02@mail.ru

In this paper we consider a Cauchy type problem for a differential equation with fractional Riemann-Liouville derivatives of different orders. Special attention is paid to the study of limiting cases of changing the order of fractional derivatives. The intervals of values for which the existence and uniqueness theorem holds are determined. The results show that when the order of the fractional derivative tends to an integer value, the solution continuously transitions to the classical case. This allows us to establish a connection between fractional and integer order equations, which confirms the correctness of the chosen approach. The methods of integral transformations