Bulletin of the Karaganda University. Mathematics Series, No. 1(117), 2025, pp. 81–91

https://doi.org/10.31489/2025M1/81-91 *Research article*

Solution of nonlocal boundary value problems for the heat equation with discontinuous coefficients, in the case of two discontinuity points

U.K. Koilyshov[1,2,*], M.A. Sadybekov[1,2], K.A. Beisenbayeva[3]

[1] *Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan;*
[2] *Al-Farabi Kazakh National University, Almaty, Kazakhstan;*
[3] *Academy of Logistics and Transport, Almaty, Kazakhstan*
(E-mail: koylyshov@math.kz, sadybekov@math.kz, beisenbaeva@mail.ru)

In this paper, the solution of the initial-boundary value problem for the heat equation with a discontinuous coefficient under periodic or antiperiodic boundary conditions in the case of two discontinuity points is substantiated using the method of separation of variables. Using the replacement, the problem under consideration is reduced to a self-adjoint problem. By means of the Fourier method, this problem is reduced to the corresponding spectral problem. Then, the eigenvalues and eigenfunctions of this self-adjoint spectral problem are found. In conclusion, the main theorem on the existence and uniqueness of the classical solution to the problem under consideration is proved. The peculiarity of the problem under consideration is the non-local boundary conditions and the presence of two discontinuity points, which have not been considered before. The authors were able to find eigenvalues explicitly and construct eigenfunctions. This technique is also applicable in the case of more than two discontinuity points. The solution obtained in explicit form can be further used for numerical calculations.

Keywords: Heat equation with discontinuous coefficients, spectral problem, non-self-adjoint problem, Riesz basis, classical solution, Fourier method.

2020 Mathematics Subject Classification: 35K10, 35K20, 35P10.

Introduction. *Problem statement and research metods*

We consider an initial boundary value problem for the heat equation with a discontinuity constant coefficient

$$\frac{\partial u_j}{\partial t} = k_j^2 \frac{\partial^2 u_j}{\partial x^2} \tag{1}$$

in the domain $\Omega = \cup\Omega_j$, $\Omega_j = \{(x,t) : l_{j-1} < x < l_j,\ 0 < t < T\}$ $(j = 1, 2, 3)$, with the initial condition

$$u(x, 0) = \varphi(x), \quad l_0 \le x \le l_3, \tag{2}$$

boundary conditions of the form

$$\begin{cases} u_1(l_0,t) + e^{i\pi\theta} u_3(l_3,t) = 0, \\ k_1 \dfrac{\partial u_1(l_0,t)}{\partial x} + e^{i\pi\theta} k_3 \dfrac{\partial u_3(l_3,t)}{\partial x} = 0, \end{cases} \quad 0 \le t \le T \tag{3}$$

and with conjugation conditions

$$u_j(l_j - 0, t) = u_{j+1}(l_j + 0, t), \quad 0 \le t \le T, \quad j = 1, 2, \tag{4}$$

---

[*] Corresponding author. *E-mail: koylyshov@math.kz*

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19679487).

Received: 12 July 2024; *Accepted:* 26 November 2024.

$$k_j \frac{\partial u_1(l_j - 0, t)}{\partial x} = k_{j+1} \frac{\partial u_{j+1}(l_j + 0, t)}{\partial x}, \quad 0 \le t \le T, \quad j = 1, 2, \tag{5}$$

where $l_0 < l_1 < l_2 < l_3$, the coefficients $k_j > 0$, $\theta = 1, 2$.

Parabolic equations with discontinuous coefficients have been studied quite well [1–3]. In these works, the correctness of various initial-boundary value problems for a parabolic equation with discontinuous coefficients was proved using the Green function method and method of thermal potentials. In the absence of a discontinuity, the spectral theory of these problems has been constructed almost completely [4–6]. In [7], some properties of the eigenfunctions of the Sturm-Liouville operator with discontinuous coefficients were studied. In the case of a discontinuous coefficient, the spectral theory of such problems is considered in [8–12].

Works devoted to solving problems of multilayer diffusion should be especially noted. Mathematical models of diffusion in layered materials arise in many industrial, ecological, biological, medical applications and the theory of thermal conductivity of composite materials. Diffusion in several layers is used in a wide range of heat and mass transfer areas [13–21].

Let W be the linear variety of functions from the class $u(x,t) \in C(\overline{\Omega}) \cap C^{2,1}(\overline{\Omega_1}) \cap C^{2,1}(\overline{\Omega_2}) \cap C^{2,1}(\overline{\Omega_3})$ which satisfy all conditions (2)–(4). A function $u(x,t)$ from the class $u(x,t) \in W$ will be called a classical solution to problem (1)–(5) if: 1) it is continuous in the domain $\overline{\Omega}$; 2) it has continuous first-order derivatives with respect to t and continuous second-order derivatives with respect to x in the domain; 3) it satisfies equation (1) and all conditions (2)–(5) in the usual, continuous sense.

First let's consider the case $\theta = 1$. After the next replacement $u_j(x,t) = v_j(y,t)$, where

$$y = \begin{cases} \dfrac{x - l_0}{k_1}, & l_0 < x < l_1, \\ \dfrac{x - l_1}{k_2}, & l_1 < x < l_2, \\ \dfrac{x - l_2}{k_3}, & l_2 < x < l_3, \end{cases} \tag{6}$$

problem (1)–(5) take the following form:

$$\frac{\partial v_j}{\partial t} = \frac{\partial^2 v_j}{\partial y^2} \tag{7}$$

in the domain $D_j = \{(y,t) : 0 < y < h_j, 0 < t < T\}$ $(j = 1, 2, 3)$,

$$v_j(y, 0) = \psi_j(y), \quad 0 \le y \le h_j, \tag{8}$$

$$\begin{cases} v_1(0,t) - v_3(h_3,t) = 0, \\ \dfrac{\partial v_1(0,t)}{\partial y} - \dfrac{\partial v_3(h_3,t)}{\partial y} = 0, \end{cases} \quad 0 \le t \le T, \tag{9}$$

$$v_1(h_1,t) = v_2(0,t), \quad v_2(h_2,t) = v_3(0,t), \quad 0 \le t \le T, \tag{10}$$

$$\frac{\partial v_1(h_1,t)}{\partial y} = \frac{\partial v_2(0,t)}{\partial y}, \quad \frac{\partial v_2(h_2,t)}{\partial y} = \frac{\partial v_3(0,t)}{\partial y}, \quad 0 \le t \le T, \tag{11}$$

where

$$h_j = \frac{l_j - l_{j-1}}{k_j}, \quad \psi_j(y) = \varphi_j(k_j y + l_{j-1}), \quad j = 1, 2, 3. \tag{12}$$

To solve problem (7)–(11), we apply the Fourier method: $v_j(y,t) = Y_j(y) \cdot T(t) \neq 0$.

Substituting $v_j(y,t) = Y_j(y) \cdot T(t)$ into equation (7) and conditions (8)–(11), and separating the variables, we obtain the following spectral problem

$$LY(y)=\begin{Bmatrix}-Y''(x), & 0<y<h_1\\ -Y''(x), & 0<y<h_2\\ -Y''(x), & 0<y<h_3\end{Bmatrix}=\lambda Y(y), \tag{13}$$

$$\begin{cases}Y_1(0)-Y_3(h_3)=0,\\ Y_1'(0)-Y_3'(h_3)=0,\end{cases} \tag{14}$$

$$Y_1(h_1)=Y_2(0),\quad Y_2(h_2)=Y_3(0),\quad Y_1'(h_1)=Y_2'(0),\quad Y_2'(h_2)=Y_3'(0). \tag{15}$$

The function $T(t)$ is a solution to the equation

$$T'(t)+\lambda T(t)=0.$$

The following holds:

Lemma 1. Spectral problem (13)–(15) is self-adjoint.

The proof is carried out by direct calculation.

Now we will find the eigenvalues and construct the eigenfunctions of spectral problem (13)–(15). The general solution to equation (13) has the form:

$$\begin{cases}Y_1(y)=c_1 cos\sqrt{\lambda}y+c_2 sin\sqrt{\lambda}y, & 0<y<h_1,\\ Y_2(y)=c_3 cos\sqrt{\lambda}y+c_4 sin\sqrt{\lambda}y, & 0<y<h_2,\\ Y_3(y)=c_5 cos\sqrt{\lambda}(h_3-y)+c_6 sin\sqrt{\lambda}(h_3-y), & 0<y<h_3,\end{cases} \tag{16}$$

where c_j are arbitrary constants ($j=1,2,3,4,5,6$).

Substituting general solution (16) into boundary conditions (14) and conjugation conditions (15) we obtain the following system

$$\begin{cases}c_1=c_5,\\ c_2=-c_6,\\ c_1\cos\sqrt{\lambda}h_1+c_2\sin\sqrt{\lambda}h_1=c_3,\\ -c_1\sin\sqrt{\lambda}h_1+c_2\cos\sqrt{\lambda}h_1=c_4,\\ c_3\cos\sqrt{\lambda}h_2+c_4\sin\sqrt{\lambda}h_2=c_5\cos\sqrt{\lambda}h_3+c_6\sin\sqrt{\lambda}h_3,\\ -c_3\sin\sqrt{\lambda}h_2+c_4\cos\sqrt{\lambda}h_2=c_5\sin\sqrt{\lambda}h_3-c_6\cos\sqrt{\lambda}h_3.\end{cases}$$

The characteristic determinant of the system has the form:

$$\Delta(\lambda)=2-2\cos(s_3\sqrt{\lambda})=0,$$

where $s_3=\sum\limits_{j=1}^{3}h_j=\sum\limits_{j=1}^{3}\dfrac{l_j-l_{j-1}}{k_j}$. From the last equation we find the eigenvalues of problem (13)–(15):

$$\lambda_n=\left(\frac{2\pi n}{s_3}\right)^2,\quad n=0,1,2,... \tag{17}$$

Since these eigenvalues are twofold, the following eigenfunctions correspond to them:

$$Y_n(y)=C\begin{cases}cos\left(\dfrac{2\pi n}{s_3}y\right), & 0<y<h_1,\\ cos\left(\dfrac{2\pi n}{s_3}(h_2+h_3-y)\right), & 0<y<h_2,\\ cos\left(\dfrac{2\pi n}{s_3}(h_3-y)\right), & 0<y<h_3,\end{cases} \tag{18}$$

$$\widetilde{Y}_n(y) = C \begin{cases} sin\left(\frac{2\pi n}{s_3}y\right), & 0 < y < h_1, \\ sin\left(\frac{2\pi n}{s_3}(y - h_2 - h_3)\right), & 0 < y < h_2, \\ -sin\left(\frac{2\pi n}{s_3}(h_3 - y)\right), & 0 < x < h_3. \end{cases} \tag{19}$$

Lemma 2. The system of eigenfunctions (18)-(19) forms an orthonormal basis.

The proof follows from the general theory of self-adjoint problems. From the normalization condition it is not difficult to find $C = \sqrt{\frac{2}{s_3}}$.

From Lemma 2 it follows that the solution to problem (7)–(11) can be written in the following form:

$$v_j(y,t) = \sum_{n=0}^{\infty} \left(\varphi_n Y_n(y) + \widetilde{\varphi}_n \widetilde{Y}_n(y)\right) e^{-\lambda_n t},$$

where

$$\varphi_n = \int_0^{h_1} \psi_1(\eta) Y_n(\eta) d\eta + \int_0^{h_2} \psi_2(\eta) Y_n(\eta) d\eta + \int_0^{h_3} \psi_3(\eta) Y_n(\eta) d\eta, \tag{20}$$

$$\widetilde{\varphi}_n = \int_0^{h_1} \psi_1(\eta) \widetilde{Y}_n(\eta) d\eta + \int_0^{h_2} \psi_2(\eta) \widetilde{Y}_n(\eta) d\eta + \int_0^{h_3} \psi_3(\eta) \widetilde{Y}_n(\eta) d\eta. \tag{21}$$

Let us transform formula (20). In each integral we make the following replacements, respectively: $\eta = \frac{\xi - l_{j-1}}{k_j}$, $d\eta = \frac{d\xi}{k_j}$, $(j = 1,2,3)$. Taking into account formula (12), we obtain

$$\varphi_n = \frac{1}{k_1}\int_{l_0}^{l_1} \varphi_1(\xi) Y_n(\frac{\xi - l_0}{k_1}) d\xi + \frac{1}{k_2}\int_{l_1}^{l_2} \varphi_2(\xi) Y_n(\frac{\xi - l_1}{k_2}) d\xi + \frac{1}{k_3}\int_{l_2}^{l_3} \varphi_3(\xi) Y_n(\frac{\xi - l_2}{k_3}) d\xi. \tag{22}$$

Similarly, transforming formula (21), we have

$$\widetilde{\varphi}_n = \frac{1}{k_1}\int_{l_0}^{l_1} \varphi_1(\xi) \widetilde{Y}_n(\frac{\xi - l_0}{k_1}) d\xi + \frac{1}{k_2}\int_{l_1}^{l_2} \varphi_2(\xi) \widetilde{Y}_n(\frac{\xi - l_1}{k_2}) d\xi + \frac{1}{k_3}\int_{l_2}^{l_3} \varphi_3(\xi) \widetilde{Y}_n(\frac{\xi - l_2}{k_3}) d\xi. \tag{23}$$

If we move to the initial variable using formula (6), then formulas (18)–(19) take the form:

$$Y_n(y) = \sqrt{\frac{2}{s_3}} \begin{cases} Y_n(\frac{x - l_0}{k_1}), & l_0 < x < l_1, \\ Y_n(\frac{x - l_1}{k_2}), & l_1 < x < l_2, \\ Y_n(\frac{x - l_2}{k_3}), & l_2 < x < l_3, \end{cases} = \sqrt{\frac{2}{s_3}} \begin{cases} cos\left(\frac{2\pi n}{s_3}\left(\frac{x - l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ cos\left(\frac{2\pi n}{s_3}\left(\frac{l_2 - x}{k_2} + \frac{l_3 - l_2}{k_3}\right)\right), & l_1 < x < l_2, \\ cos\left(\frac{2\pi n}{s_3}\left(\frac{l_3 - x}{k_3}\right)\right), & l_2 < x < l_3, \end{cases}$$

$$\widetilde{Y}_n(y) = \sqrt{\frac{2}{s_3}} \begin{cases} \widetilde{Y}_n(\frac{x - l_0}{k_1}), & l_0 < x < l_1, \\ \widetilde{Y}_n(\frac{x - l_1}{k_2}), & l_1 < x < l_2, \\ \widetilde{Y}_n(\frac{x - l_2}{k_3}), & l_2 < x < l_3, \end{cases} = \sqrt{\frac{2}{s_3}} \begin{cases} sin\left(\frac{2\pi n}{s_3}\left(\frac{x - l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ sin\left(\frac{2\pi n}{s_3}\left(\frac{x - l_2}{k_2} + \frac{l_2 - l_3}{k_3}\right)\right), & l_1 < x < l_2, \\ sin\left(\frac{2\pi n}{s_3}\left(\frac{x - l_3}{k_3}\right)\right), & l_2 < x < l_3. \end{cases}$$

We redesignate the last formulas as follows: $Y_n(y) = X_n(x), \widetilde{Y}_n(y) = \widetilde{X}_n(x)$. Then

$$X_n(x) = \sqrt{\frac{2}{s_3}} \begin{cases} cos\left(\frac{2\pi n}{s_3}\left(\frac{x-l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ cos\left(\frac{2\pi n}{s_3}\left(\frac{l_2-x}{k_2} + \frac{l_3-l_2}{k_3}\right)\right), & l_1 < x < l_2, \\ cos\left(\frac{2\pi n}{s_3}\left(\frac{l_3-x}{k_3}\right)\right), & l_2 < x < l_3, \end{cases}$$

$$\widetilde{X}_n(x) = \sqrt{\frac{2}{s_3}} \begin{cases} sin\left(\frac{2\pi n}{s_3}\left(\frac{x-l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ sin\left(\frac{2\pi n}{s_3}\left(\frac{x-l_2}{k_2} + \frac{l_2-l_3}{k_3}\right)\right), & l_1 < x < l_2, \\ sin\left(\frac{2\pi n}{s_3}\left(\frac{x-l_3}{k_3}\right)\right), & l_2 < x < l_3. \end{cases}$$

Since the system of eigenfunctions $\{Y_n(y), \widetilde{Y}_n(y)\}$ forms a basis, the functions $\{X_n(x), \widetilde{X}_n(x)\}$ also form a basis. Formulas (22)-(23) have the form:

$$\varphi_n = \frac{1}{k_1}\int_{l_0}^{l_1} \varphi_1(\xi)X_n(\xi)d\xi + \frac{1}{k_2}\int_{l_1}^{l_2} \varphi_2(\xi)X_n(\xi)d\xi + \frac{1}{k_3}\int_{l_2}^{l_3} \varphi_3(\xi)X_n(\xi)d\xi, \tag{24}$$

$$\widetilde{\varphi}_n = \frac{1}{k_1}\int_{l_0}^{l_1} \varphi_1(\xi)\widetilde{X}_n(\xi)d\xi + \frac{1}{k_2}\int_{l_1}^{l_2} \varphi_2(\xi)\widetilde{X}_n(\xi)d\xi + \frac{1}{k_3}\int_{l_2}^{l_3} \varphi_3(\xi)\widetilde{X}_n(\xi)d\xi. \tag{25}$$

Now let's prove the main theorem.

Theorem. Let $\varphi(x)$ be a continuously differentiable function satisfying the conditions $\varphi(l_0) = \varphi(l_3)$, $k_1\varphi'(l_0) = k_3\varphi'(l_3)$, $\varphi(l_j - 0) = \varphi(l_j + 0)$, $k_j\varphi'(l_j - 0) = k_{j+1}\varphi'(l_j + 0)$ $(j = 1, 2)$.

Then the function

$$u(x,t) = \sum_{n=0}^{\infty}\left(\varphi_n X_n(x) + \widetilde{\varphi}_n\widetilde{X}_n(x)\right)e^{-\lambda_n t}, \tag{26}$$

where the coefficients are determined by formulas (24)-(25), is the only classical solution of (1)–(5).

Proof. First we prove the existence of solution (26). Since $\left\{X_n(x), \widetilde{X}_n(x)\right\}$ the eigenfunctions and λ_n eigenvalues of problem (13)–(15), then it is easy to verify that the function $u(x,t)$ determined by formula (26) satisfies the equation, initial condition, boundary conditions and pairing conditions of problem (1)–(5). Series (26) is the sum of functions

$$u_n(x,t) = \left(\varphi_n X_n(x) + \widetilde{\varphi}_n\widetilde{X}_n(x)\right)e^{-\lambda_n t}. \tag{27}$$

Let us show that when $t \geq \varepsilon > 0$ (ε is any positive number) the series $\sum\limits_{n=0}^{\infty} u_n(x,t)$, $\sum\limits_{n=0}^{\infty}\frac{\partial u_n}{\partial t}$, $\sum\limits_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2}$ converges uniformly. Obviously, $|\varphi| \leq M_1$ then from formula (27) it follows that $\left\{|\varphi_n|, |\widetilde{\varphi}_n|\right\} \leq M_2$. Then from equality (27) and from the following equalities

$$\frac{\partial u_n}{\partial t} = \left(-\lambda_n X_n(x)\varphi_n - \lambda_n\widetilde{X}_n(x)\widetilde{\varphi}_n\right)e^{-\lambda_n t}, \quad \frac{\partial^2 u_n}{\partial x^2} = \frac{\lambda_n}{k_j^2}\left(-X_n(x)\varphi_n - \widetilde{X}_n(x)\widetilde{\varphi}_n\right)e^{-\lambda_n t},$$

we get

$$|u_n(x,t)| \le M_3 e^{-\lambda_n \varepsilon}, \quad \left\{ \left|\frac{\partial u_n}{\partial t}\right|, \left|\frac{\partial^2 u_n}{\partial x^2}\right| \right\} \le M_4 \lambda_n e^{-\lambda_n \varepsilon},$$

where constants M_3, M_4 positive and does not depend on n. Taking into account formula (17), we have

$$\left\{ \sum_{n=1}^{\infty} \Big| u_n(x,t) \Big|, \sum_{n=1}^{\infty} \left|\frac{\partial u_n}{\partial t}\right|, \sum_{n=1}^{\infty} \left|\frac{\partial^2 u_n}{\partial x^2}\right| \right\} \le \sum_{n=1}^{\infty} M n^2 e^{-\left(\frac{2\pi n}{s_3}\right)^2 \varepsilon},$$

where constant $M > 0$, and does not depend on n. Since the series $\sum\limits_{n=1}^{\infty} M n^2 e^{-\left(\frac{2\pi n}{s_3}\right)^2 \varepsilon}$ an absolutely convergent series, hence, according to Weierstrass's test, the series $\left\{ \sum\limits_{n=0}^{\infty} \Big| u_n(x,t) \Big|, \sum\limits_{n=0}^{\infty} \left|\frac{\partial u_n}{\partial t}\right|, \sum\limits_{n=0}^{\infty} \left|\frac{\partial^2 u_n}{\partial x^2}\right| \right\}$ converge uniformly for $t \ge \varepsilon$ and are continuous for $t \ge \varepsilon$ the functions $u(x,t)$, $\frac{\partial u(x,t)}{\partial t}$, $\frac{\partial^2 u(x,t)}{\partial x^2}$.

Now we need to prove that series (26) converges uniformly everywhere in $\overline{\Omega}$. Note that the n-term of the series (26) is dominated by the sum $|\varphi_n| + |\widetilde{\varphi}_n|$. Integrating by parts the integral in formula (24), we obtain

$$|\varphi_n| \le \frac{C_1 s_3}{2\pi} \cdot \frac{|\alpha_n|}{n}, \quad |\widetilde{\varphi}_n| \le \frac{C_1 s_3}{2\pi} \cdot \frac{|\widetilde{\alpha}_n|}{n}, \quad C_1 = \max\left(\sqrt{k_1}, \sqrt{k_2}, \sqrt{k_3}\right),$$

where $\alpha_n = \frac{1}{\sqrt{k_1}} \int\limits_{l_0}^{l_3} \varphi'(\xi) X_n(\xi) d\xi$, $\widetilde{\alpha}_n = \frac{1}{\sqrt{k_2}} \int\limits_{l_0}^{l_3} \varphi'(\xi) \widetilde{X}_n(\xi) d\xi$ are Fourier coefficients of functions $\varphi'(x)$ on a segment $[l_0, l_3]$. Taking into account the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, we have

$$|\varphi_n| + |\widetilde{\varphi}_n| \le \frac{C_1 s_3}{4\pi} \cdot \left(\alpha_n^2 + \widetilde{\alpha}_n^2 + \frac{2}{n^2} \right).$$

Using the Bessel inequality

$$\sum_{n=0}^{\infty} \left(\alpha_n^2 + \widetilde{\alpha}_n^2 \right) \le \|\varphi'\|^2$$

and the well-known equality $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we get $\sum\limits_{n=0}^{\infty} \left(|\varphi_n| + |\widetilde{\varphi}_n| \right) \le C$.

Thus, the majorizing series is absolutely convergent, this means series (26) converges uniformly in $\overline{\Omega}$ and defines a continuous function $u(x,t)$ in $\overline{\Omega}$. Thus, we proved the existence of a solution. Now let's prove uniqueness. Let's assume there are two solutions $\widetilde{v}(x,t)$, $\widehat{v}(x,t)$. Then for the function $v(x,t) = \widetilde{v}(x,t) - \widehat{v}(x,t)$, we have the following *problem* C:

$$\frac{\partial v}{\partial t} = k_j^2 \frac{\partial^2 v}{\partial x^2},$$

$$v(x,0) = 0, \quad l_0 \le x \le l_3,$$

$$\begin{cases} v(l_0,t) - v(l_3,t) = 0, \\ k_1 \dfrac{\partial v(l_0,t)}{\partial x} - k_3 \dfrac{\partial v(l_3,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T,$$

$$\begin{cases} v(l_j - 0, t) = v(l_j + 0, t), \\ k_j \dfrac{\partial v(l_j - 0, t)}{\partial x} = k_{j+1} \dfrac{\partial v(l_j + 0, t)}{\partial x}, \end{cases} \qquad j = 1, 2.$$

The solution to this *problem* C can be represented in the form of an expansion in terms of the basis $\{X_n(x), \widetilde{X}_n(x)\}$ and it has the form:

$$v(x,t) = \sum_{n=0}^{\infty} \big(A_n(t) X_n(x) + \widetilde{A}_n(t) \widetilde{X}_n(x)\big). \tag{28}$$

The coefficients $A_n(t)$ and $\widetilde{A}_n(t)$ are determined by the formulas

$$A_n(t) = \frac{1}{k_1} \int_{l_0}^{l_1} v(\xi,t) X_n(\xi) d\xi + \frac{1}{k_2} \int_{l_1}^{l_2} v(\xi,t) X_n(\xi) d\xi + \frac{1}{k_3} \int_{l_2}^{l_3} v(\xi,t) X_n(\xi) d\xi, \tag{29}$$

$$\widetilde{A}_n = \frac{1}{k_1} \int_{l_0}^{l_1} v(\xi,t) \widetilde{X}_n(\xi) d\xi + \frac{1}{k_2} \int_{l_1}^{l_2} v(\xi,t) \widetilde{X}_n(\xi) d\xi + \frac{1}{k_3} \int_{l_2}^{l_3} v(\xi,t) \widetilde{X}_n(\xi) d\xi. \tag{30}$$

First, we transform formula (29). Differentiating with respect to the variable t, we obtain

$$A_n'(t) = \frac{1}{k_1} \int_{l_0}^{l_1} \frac{\partial v(\xi,t)}{\partial t} X_n(\xi) d\xi + \frac{1}{k_2} \int_{l_1}^{l_2} \frac{\partial v(\xi,t)}{\partial t} X_n(\xi) d\xi + \frac{1}{k_3} \int_{l_2}^{l_3} \frac{\partial v(\xi,t)}{\partial t} X_n(\xi) d\xi =$$

$$= k_1 \int_{l_0}^{l_1} \frac{\partial^2 v(\xi,t)}{\partial \xi^2} \cos\left(\frac{2\pi n}{s_3}\left(\frac{\xi - l_0}{k_1}\right)\right) d\xi + k_2 \int_{l_1}^{l_2} \frac{\partial^2 v(\xi,t)}{\partial \xi^2} \cos\left(\frac{2\pi n}{s_3}\left(\frac{l_2 - \xi}{k_2} + \frac{l_3 - l_2}{k_3}\right)\right) \xi +$$

$$+ k_3 \int_{l_2}^{l_3} \frac{\partial^2 v(\xi,t)}{\partial \xi^2} \cos\left(\frac{2\pi n}{s_3}\left(\frac{l_3 - \xi}{k_3}\right)\right) d\xi.$$

Integrating by parts twice and using the boundary conditions and conjugation conditions, we have

$$A_n'(t) = -\left(\frac{2\pi n}{s_3}\right)^2 \frac{1}{k_1} \int_{l_0}^{l_1} v(x,t) \cos\left(\frac{2\pi n}{s_3}\left(\frac{x - l_0}{k_1}\right)\right) dx -$$

$$-\left(\frac{2\pi n}{s_3}\right)^2 \frac{1}{k_2} \int_{l_1}^{l_2} v(x,t) \cos\left(\frac{2\pi n}{s_3}\left(\frac{l_2 - x}{k_2} + \frac{l_3 - l_2}{k_3}\right)\right) dx -$$

$$-\left(\frac{2\pi n}{s_3}\right)^2 \frac{1}{k_3} \int_{l_2}^{l_3} v(x,t) \cos\left(\frac{2\pi n}{s_3}\left(\frac{l_3 - x}{k_3}\right)\right) dx =$$

$$= -\lambda_n \int_{l_0}^{l_3} v(x,t) X_n(x) dx = -\lambda_n A_n(t).$$

Therefore $A_n(t) = c_n e^{-\lambda_n t}$. Transforming in a similar way, we obtain for the coefficient $\widetilde{A}_n(t)$.

$$\widetilde{A}'_n(t) = -\lambda_n \widetilde{A}_n(t) \quad \Rightarrow \quad \widetilde{A}_n(t) = \widetilde{c}_n e^{-\lambda_n t}.$$

Substituting the found $A_n(t)$ and $\widetilde{A}_n(t)$ into formula (29)-(30), we obtain

$$\frac{1}{k_1}\int\limits_{l_0}^{l_1} v(\xi,t)X_n(\xi)d\xi + \frac{1}{k_2}\int\limits_{l_1}^{l_2} v(\xi,t)X_n(\xi)d\xi + \frac{1}{k_3}\int\limits_{l_2}^{l_3} v(\xi,t)X_n(\xi)d\xi = c_n e^{-\lambda_n t}, \tag{31}$$

$$\frac{1}{k_1}\int\limits_{l_0}^{l_1} v(\xi,t)\widetilde{X}_n(\xi)d\xi + \frac{1}{k_2}\int\limits_{l_1}^{l_2} v(\xi,t)\widetilde{X}_n(\xi)d\xi + \frac{1}{k_3}\int\limits_{l_2}^{l_3} v(\xi,t)\widetilde{X}_n(\xi)d\xi = \widetilde{c}_n e^{-\lambda_n t}. \tag{32}$$

Passing to the limit $t \to 0$ in equality (31)-(32) what is possible due to continuity $v(x,t)$ in $\overline{\Omega}$, we have

$$0 = A_n(0) = c_n, \quad 0 = \widetilde{A}_n(0) = \widetilde{c}_n,$$

therefore $c_n = 0$, $\widetilde{c}_n = 0$.

Then from formula (28), we obtain $v(x,t) = 0$, it follows from this that $\widetilde{v}(x,t) = \widehat{v}(x,t)$. The theorem is proved.

Now consider the case $\theta = 2$.

Then, after applying the method of separation of variables, we obtain the following spectral problem

$$X''_j(x) + \frac{\lambda}{k_j^2} X_j(x) = 0, \quad l_{j-1} < x < l_j, \quad j = 1,2,3, \tag{33}$$

$$\begin{cases} X_1(l_0) + X_3(l_3) = 0, \\ k_1 X'_1(l_0) + k_3 X'_3(l_3) = 0, \end{cases} \tag{34}$$

$$X_j(l_j - 0) = X_{j+1}(l_j + 0), \quad k_j X'_j(l_j - 0) = k_{j+1} X'_{j+1}(l_j + 0), \quad j = 1,2. \tag{35}$$

The eigenvalues of problem (33)–(35) have the form: $\lambda_n = \left(\frac{(2n+1)\pi}{s_3}\right)^2$, $n = 0,1,2,...$

The following eigenfunctions correspond to these eigenvalues.

$$X_n(x) = \sqrt{\frac{2}{s_3}} \begin{cases} \cos\left(\frac{(2n+1)\pi}{s_3}\left(\frac{x-l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ -\cos\left(\frac{(2n+1)\pi}{s_3}\left(\frac{l_2-x}{k_2} + \frac{l_3-l_2}{k_3}\right)\right), & l_1 < x < l_2, \\ -\cos\left(\frac{(2n+1)\pi}{s_3}\left(\frac{l_3-x}{k_3}\right)\right), & l_2 < x < l_3, \end{cases}$$

$$\widetilde{X}_n(x) = \sqrt{\frac{2}{s_3}} \begin{cases} \sin\left(\frac{(2n+1)\pi}{s_3}\left(\frac{x-l_0}{k_1}\right)\right), & l_0 < x < l_1, \\ \sin\left(\frac{(2n+1)\pi}{s_3}\left(\frac{l_2-x}{k_2} + \frac{l_3-l_2}{k_3}\right)\right), & l_1 < x < l_2, \\ \sin\left(\frac{(2n+1)\pi}{s_3}\left(\frac{l_3-x}{k_3}\right)\right), & l_2 < x < l_3. \end{cases}$$

All other calculations, including the proof of the theorem, are carried out in a similar way.

Conclusion

The method proposed in this article can be used in the case of n break points, where $n \geq 3$, and for the more general case of the conjugation condition (in this work, the ideal contact condition is considered). The solution to the problem is found in explicit form, which allows it to be used for numerical solution.

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

Acknowledgments

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP19679487).

References

1 Samarskiy, A.A. (1958). Uravneniia parabolicheskogo tipa s razryvnymi koeffitsientami [Parabolic equations with discontinuous coefficients]. *Doklady AN SSSR — Reports of the USSR Academy of Sciences 2*(121), 225–228 [in Russian].

2 Kim, Ye.I., & Baymukhanov, B.B. (1961). O raspredelenii temperatury v kusochno-odnorodnoi polu-beskonechnoi plastinke [On the temperature distribution in a piecewise homogeneous semi-infinite plate]. *Doklady AN SSSR — Reports of the USSR Academy of Sciences, 2*(140), 333–336 [in Russian].

3 Kamynin, L.I. (1962). O metode potentsialov dlia parabolicheskogo uravneniia s razryvnymi koeffitsientami [On the potential method for a parabolic equation with discontinuous coefficients]. *Doklady AN SSSR — Reports of the USSR Academy of Sciences, 6*(145), 1213–1216 [in Russian].

4 Ionkin, N.I., & Moiseyev, Ye.I. (1979). O zadache dlia uravneniia teploprovodnosti s dvutochechnymi kraevymi usloviiami [On the problem for the heat equation with two-point boundary values conditions]. *Differentsialnye uravneniia — Differential equations, 7*(15), 1284–1295 [in Russian].

5 Orazov, I., & Sadybekov, M.A. (2012). Ob odnoi nelokalnoi zadache opredeleniia temperatury i plotnosti istochnikov tepla [About one nonlocal problem of determining the temperature and density of heat sources]. *Izvestiia vuzov. Matematika — News of universities. Mathematics, 2*, 70–75 [in Russian].

6 Sadybekov, M.A. (2017). Initial-Boundary Value Problem for a Heat Equation with not Strongly Regular Boundary Conditions. *Functional Analysis in Interdisciplinary Applications — Springer Proceedings in Mathematics Statistics, 216*, 330–348. https://doi.org/10.1007/978-3-319-67053-9_32

7 Il'in, V.A. (1977). O skhodimosti razlozhenii po sobstvennym funktsiiam v tochkakh razryva koeffitsientov differentsialnogo operatora [On the convergence of expansions in eigenfunctions at discontinuity points of the coefficients of a differential operator]. *Matematicheskie zametki — Math Notes, 5*(22), 679–698 [in Russian]. https://doi.org/10.1007/BF01098352

8 Sadybekov, M.A., & Koilyshov, U.K. (2022). Two-phase tasks thermal conductivity with boundary conditions of the Sturm type. *Sixth International Conference on Analysis and Applied Mathematics. Abstract book of the conference ICAAM*, Antalya, Turkey.

9 Sadybekov, M.A., & Koilyshov, U.K. (2021). Nachalno-kraevye zadachi dlia uravneniia teploprovodnosti s kusochno-postoiannym koeffitsientom [Initial boundary value problems for the heat equation with piecewise constant coefficient]. *Izvestiia Mezhdunarodnogo kazakhsko-turetskogo Universiteta im. Kh.A.Yasavi. Seriia matematika, fizika, informatika — News of the Akhmet Yassawi Kazakh-turkish international universitaty. Mathematics, physics, computer series, 16*(1), 7–16 [in Russian].

10 Koilyshov, U.K., & Sadybekov, M.A. (2023). Solution of initial-boundary value problems for the heat equation with discontinuous coefficients arising in financial mathematics. *International Mathematical Conference "Functional Analysis in Interdisciplinary Applications"*. Antalya, Turkiye.

11 Koilyshov, U.K., & Sadybekov, M.A. (2022). Solution of boundary value problems for the heat equation with a piecewise constant coefficient using the Fourier method. *Kazakh Mathematical Journal, 2*(22), 16–27. https://doi.org/10.70474/drms4076

12 Koilyshov, U.K., & Sadybekov, M.A. (2024). Two-phase heat conduction problems with Sturm-type boundary conditions. *Boundary Value Problems, 149*. https://doi.org/10.1186/s13661-024-01964-x

13 Carr, E.J., & March, N.G. (2018). Semi-analitical solution of multilayer diffusion problems with time-varying boundary conditions and general inter face conditions. *Applied Mathematics and Computation, 333*, 286–303. https://doi.org/10.1016/j.amc.2018.03.095

14 Rodrigo, M.R., & Worthy, A.L. (2016). Solution of multilayer diffusion problems via the Laplace transform. *Journal of Mathematical Analysis and Applications, 444*, 475–502. https://doi.org/10.1016/j.jmaa.2016.06.042

15 Carr, E.J., & Turner, I.W. (2016). A semi-analytical solution for multilayer diffusion in a composite medium consisting of a large number of layers. *Applied Mathematical Modelling, 15-16*(40), 7034–7050. https://doi.org/10.1016/j.apm.2016.02.041

16 Carr, E.J., Turner, I.W., & Perre, P. (2017). Macroscale modelling of multilayer diffusion: Using volume averaging to correct the boundary conditions. *Applied Mathematical Modelling, 47*, 600–618. https://doi.org/10.1016/j.apm.2017.03.044

17 Carr, E.J., Perre, P., & Turner, I.W. (2016). The extended distributed microstructure model for gradient-driven transport: a two-scale model for bypassing effective parameters. *J. Comput. Phys., 327*, 810–829. https://doi.org/10.1016/j.jcp.2016.10.004

18 Mantzavinos, D., Papadomanolaki, M.G., Saridakis,Y.G., & Sifalakis, A.G. (2016). Fokas transform method for a brain tumor invasion model with heterogeneous diffusion in 1 + 1 dimensions. *Applied Numerical Mathematics, 104*, 47–61. https://doi.org/10.1016/j.apnum.2014.09.006

19 Sheils, N.E., & Deconinck, B. (2016). Initial-to-Interface Maps for the Heat Equation on Composite Domains. *Studies in Applied Mathematics, 1*(137), 140–154. https://doi.org/10.1111/sapm.12138

20 Asvestas, M., Sifalakis, A.G., Papadopoulou, E.P., & Saridakis, Y.G. (2014). Fokas method for a multi-domain linear reaction-di usion equation with discontinuous di usivity. *Publishing Journal of Physics: Conference Series, 490*. https://doi.org/10.1088/1742-6596/490/1/012143

21 Itkin, A., Lipton, A., & Muravey, D. (2022). *Multilayer heat equations: Application to finance, 1*(1), 99–135. https://doi.org/10.3934/fmf.2021004

*Author Information**

Umbetkul Kurmankulovich Koilyshov (*corresponding author*) — Candidate of physical and mathematical sciences, Associate professor, Institute of Mathematics and Mathematical Modeling, 28 Shevshenko Avenue, Almaty, 050010, Kazakhstan; e-mail: *koylyshov@math.kz*; https://orcid.org/0000-0003-1752-7848

Makhmud Abdysametovich Sadybekov — Doctor of physical and mathematical sciences, professor, Institute of Mathematics and Mathematical Modeling, 28 Shevshenko Avenue, Almaty, 050010, Kazakhstan; e-mail: *sadybekov@math.kz*; https://orcid.org/0000-0001-8450-8191

Kulnyar Adambaevna Beisenbayeva — Candidate of physical and mathematical sciences, Associate professor, Academy of Logistics and Transport, 97 Shevshenko Avenue, Almaty, 050010, Kazakhstan; e-mail: *beisenbaeva@mail.ru*; https://orcid.org/0009-0004-3496-5289

*The author's name is presented in the order: First, Middle and Last Names.