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On boundary value problems for essentially loaded parabolic equations in bounded domains

In the paper we study issues of a strong solution for "essentially" loaded differential equations of the parabolic type in bounded domains. Features of the problems under consideration: for example, in the $L_2(Q)$ space the corresponding differential operators are not closure operators, since firstly, the load does not obey the corresponding differential part of the considered operator, that is, for its differential part the load is not a weak perturbation. Secondly, it is obvious that load operators in the spaces $L_2(0,1)$ and $L_2(Q)$ are not closure operators. This indicates that it is impossible to directly investigate the issues of the strong solution to boundary value problems for non-closed loaded differential equations. However, the study of equations [1-4] give theoretical character, but also a clear applied [5-7] character.

Keywords: "essentially" loaded parabolic equations, Volterra integral equation, boundary value problem, strong solution, load operator.

1 Statement of boundary value problems

Statement of the first boundary value problem. Consider the following boundary value problem in the domain $Q = \{x,t|0 < x < 1, 0 < t < 2\pi\}$

$$L_1 u \equiv \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \alpha \cdot x \cdot \frac{\partial^2 u(x,t)}{\partial x^2}|_{x\equiv\overline{x}} = f(x,t), \{x,t\} \in Q; \tag{1}$$

$$u(0,t) = u(1,t) = 0, u(x,0) = u(x,2\pi), \tag{2}$$

where $\overline{x} \in (0,1)$ is a given point; $\alpha \in C$ is a given number;

$$f \in L_2\left(0, 2\pi; \dot{W}_2^1(0,1)\right) \tag{3}$$

is a given function.

Statement of the second boundary value problem. Consider the following boundary value problem in the domain $Q = \{x,t|0 < x < 1, 0 < t < 2\pi\}$

$$L_2 u \equiv \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \alpha(x) \cdot \frac{\partial^k u(x,t)}{\partial x^k}|_{x\equiv\overline{x}} = f(x,t), \{x,t\} \in Q; \tag{4}$$

$$u(0,t) = u(1,t) = 0, u(x,0) = u(x,2\pi), \tag{5}$$

where

$$\begin{cases} \overline{x} \in (0,1) \text{ is a fixed point;} \alpha \in W_2^{2m}(0,1), \\ f \in L_2\left(0, 2\pi; W_2^{2m}(0,1) \cap \dot{W}_2^m(0,1)\right) \text{ are the given functions,} \\ k \geq 2, m = \begin{cases} \frac{k}{2}, \text{ if k is an even number,} \\ \frac{k-1}{2} \text{ if k is an odd number.} \end{cases} \end{cases} \quad (6)$$

Remark 1. The loaded differential operator L_1 defined by problem (1) - (3) is not closed in the $L_2(Q)$ space, so for considering problem (1)-(3) we introduce the following an auxiliary problem:

$$L_3 u \equiv \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial^2 f}{\partial x^2}, \{x,t\} \in Q; \quad (7)$$

$$u(0,t) = u(1,t) = 0, u(x,0) = u(x,2\pi); \quad (8)$$

$$\frac{\partial^2 u(0,1)}{\partial x^2} = 0, \frac{\partial^2 u(1,t)}{\partial x^2} - \alpha \frac{\partial^2 u(\overline{x},t)}{\partial x^2} = 0. \quad (9)$$

Note that in the operator L_3 boundary value problem (7)-(9) (except the L_1 operator) is closed in the $L_2(Q)$ space. It is also obvious that boundary value problems (1)-(3) and (7)-(9) are connected. In fact, a regular solution to problem (7)-(9) is also a solution to problem (1)-(2). And visa versa, if the regular solution to problem (1)-(2) contains a derivative of the required order, then it is a regular solution to problem (7)-(9) [8].

Remark 2. For considering problem (4)-(6), in the domain Q we introduce a non-contiguous auxiliary problem

$$L_4 u \equiv \frac{\partial^{2m}}{\partial x^{2m}}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) + \alpha^{(2m)}(x)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = \frac{\partial^{2m} f}{\partial x^{2m}}; \quad (10)$$

$$u(0,t) = u(1,t) = 0, u(x,0) = u(x,2\pi); \quad (11)$$

$$\frac{\partial^2 u(0,t)}{\partial x^2} - \alpha(0)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = 0; \frac{\partial^2 u(1,t)}{\partial x^2} - \alpha(1)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = 0; \quad (12)$$

$$\frac{\partial^{j+1} u(0,t)}{\partial x^j \partial t} - \frac{\partial^{j+2} u(0,t)}{\partial x^{j+2}} + \alpha^{(j)}(0)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = 0; \quad (13)$$

$$\frac{\partial^{j+1} u(1,t)}{\partial x^j \partial t} - \frac{\partial^{j+2} u(1,t)}{\partial x^{j+2}} + \alpha^j(1)\frac{\partial^k u(\overline{x},t)}{\partial x^k} = 0 \quad (14)$$

$$j = 1, \ldots, m-1.$$

Note that boundary problem (4)-(5) and (10)-(14) are connected. In fact, a regular solution of problem (10)-(14) is a solution to problem (4)-(5). And visa versa if a regular solution to problem (4)-(5) contains a derivative of the required order, then it is a regular solution to problem (10)-(14).

There are some necessary definitions [9].

Supposing that $\tilde{C} = \{u \mid u \in C_{X,t}^{2,1}(\overline{Q}), u_t, u_{xx} \in C_{x,t}^{2,0}(Q),\}$ and conditions (8)-(9) are implemented [9].

Definition 1. If there exists a sequence of functions $\{u_n(x,t)\}_{(n=3)}^{\infty} \subset \tilde{C}$ such that the following conditions 1^0 and 2^0 are implemented:

1^0. In $L_2(Q) \lim_{n\to\infty} u_n(x,t) = u(x,t)$;

2^0. In $L_2(Q) \lim_{n\to\infty} L_3 u_n(x,t) = \frac{\partial^2 f}{\partial x^2}$

then the function $u(x,t)$ is called a strong solution to boundary value problem (7)-(9).

Definition 2. A strong solution to boundary value problem (7)-(9) are called a strong solution to boundary value problem (1)-(2).

2 Theorems on uniqueness and existence of a strong solution

First we consider the first boundary value problem, and show that the following statements are valid.

Theorem 1. Let

$$\delta_s = 1 - \frac{\alpha \cdot sh\{\lambda\overline{x}\}}{sh\{\lambda\}} \neq 0, \forall_s \in v, \tag{15}$$

in the case, $v = \{s|s = 0; \pm1; \pm2; \dots\}, \lambda^2 = is$, $i = \sqrt{(-1)}$. Than for any function

$$f \in L_2 \left(0, 2\pi; W_2^2(0,1) \cap \dot{W}_2^1(0,1)\right)$$

boundary value problem (1) - (2) has a strong solution $u(x,t)$.

Corollary 1. Let $\alpha \in R^1$. The statement in this case is true for Theorem 1, iff the following condition is valid

$$1 - \alpha\overline{x} \neq 0 \tag{16}$$

The statement is a simple consequence of following fact: in (15) the imaginary part of the expression $\frac{sh\{\lambda\overline{x}\}}{sh\{\lambda\}}$ is not equal to zero at any point $s \in v\backslash\{0\}$, since the real and imaginary parts of this expression have a value that is always different from zero.

Corollary 2. Let (16) be not satisfied, i.e. $1 - \alpha\overline{x} = 0$. Then the operator of boundary value problem (1)-(2) is equal to zero, and according to this, the function is equal to

$$w_0(x) = x(1 - x^2) \tag{17}$$

Proof of the first theorem. In proving this theorem, we refer to the proving by A.A. Desin [8]. We are looking for a solution to problem (7) - (9) based on the following series:

$$u(x,t) = \sum_{s\in v} u_s(x)e^{is\cdot t}, f(x,t) = \sum_{s\in v} f_s(x)e^{is\cdot t} \tag{18}$$

Then from boundary value problems (1) - (2) taking into account the Fourier coefficients defined from (18), we obtain boundary problems for an ordinary differential equation

$$\begin{cases} (isu_s(x) - u_s''(x) + \alpha x u_s''(\overline{x}) = f_s(x), x \in (0,1), \\ u_s(0) = u_s(1) = 0, \end{cases} \quad \forall_s \in v. \tag{19}$$

A unique solution to (19) can be represented as follows:

$$\begin{cases} u_s(x) = \alpha\delta_s^{-1}\left[\int_0^1 G_s(\overline{x},\xi)f_s(\xi)d\xi - \frac{1}{\lambda^2}f_s(\overline{x})\right] \times \\ \times\left[\frac{sh(\lambda x)}{sh(\lambda)} - x\right] + \int_0^1 G_s(x,\xi)f_s(\xi)d\xi, \forall_s \in v\backslash\{0\}; \\ u_0(x) = 6^{-1}\delta_0^{-1}\alpha x(x^2-1)f_0(\overline{x}) + \int_0^1 G_0(x,\xi)f_0\{\xi\}d\xi \end{cases} \tag{20}$$

where

$$G_s(x,\xi) = \begin{cases} \frac{sh(\lambda\xi)sh\{\lambda(1-x)\}}{\lambda sh(\lambda)}, 0 \le \xi \le x \le 1, \\ \frac{sh(\lambda\xi)sh\{\lambda(1-\xi)\}}{\lambda sh(\lambda)}, 0 \le x \le \xi \le 1, \end{cases} \quad \forall_s \in v \tag{21}$$

and

$$\delta_s = 1 - \frac{\alpha sh\{\lambda\overline{x}}{sh\{\lambda\}} \neq 0, \ \forall_s \in v \tag{22}$$

Expressions for $G_0(x,\xi)$ and δ_0 can be obtained directly at $s = 0$ or passing to the limit from formulas (21) and (22) as $\lambda \to 0 (s \to 0)$

$$G_0(x,\xi) = \begin{cases} \xi(1-x), 0 \le \xi \le x \le 1, \\ x(1-\xi), 0 \le x \le \xi \le 1, \end{cases}$$

$$\delta_0 = 1 - \alpha\overline{x}.$$

Formula (20) can define a regular solution to boundary value problem (7)-(9) for the Fourier coefficients with sufficient smoothness of the function $f_s(x)$. Therefore, for the correctness of the function $f_s(x)$ according to the functions $u_s(x)$ found on the basis of the formula (20), any combinations in the form

$$u^N(x,t) = \sum_{s=-N}^{s=N} u_s(x)e^{is\cdot x}$$

defines a regular solution to boundary value problem (7)-(9).

Based on formula (20) we obtain the following a'priori estimates

$$\|u_s(x)\|_{L_2(0,1)} \leq K \cdot \|f_s''(x)\|_{L_2(0,1)}, s \in v, \tag{23}$$

where k is a constant that independent of s, so estimates (23) are constant relative to the s [10-12].

Furthermore, proving estimate (23),we establish that the following estimate for the Green function $G(x,\xi)$ is fair

$$\int_0^1 \int_0^1 |G_s(x,\xi)|^2 dx d\xi \leq \frac{C}{|\lambda|^3} \leq K = const, \forall_s \in v\backslash\{0\}, (\lambda^2 = is).$$

Really, taking into account that $\lambda = \lambda_1 + i\lambda_1$, we get

$$\int_0^1 |G_s(x,\xi)|^2 d\xi \leq \frac{1}{|\lambda|^2|sh\lambda|^2}\left[|sh\lambda(1-x)|^2 \int_0^x |sh\lambda\xi|^2 d\xi + |sh\lambda x|^2 \int_x^1 |sh\lambda(1-\xi)|^2 d\xi\right] =$$

$$= \frac{1}{2|\lambda|^2|sh\lambda|^2}\left[|sh\lambda(1-x)|^2 \int_0^x |ch2\lambda_1\xi - cos2\lambda_1\xi)d\xi + |sh\lambda x|^2 \int_x^1 |ch2\lambda_1(1-\xi) - cos2\lambda_1(1-\xi))d\xi\right] =$$

$$= \frac{1}{8\lambda_1|\lambda|^2|sh\lambda|^2} \cdot \{[ch2\lambda_1(1-x) - cos2\lambda_1(1-x)](sh2\lambda_1 x - sin2\lambda_1 x) +$$

$$(ch2\lambda_1 x - cos2\lambda_1 x) \times [sh2\lambda_1(1-x) - sin2\lambda_1(1-x)]\} =$$

$$= \frac{1}{8\lambda_1|\lambda|^2|sh\lambda|^2} \cdot [sh2\lambda_1 x + sin2\lambda_1 x - ch2\lambda_1(1-x)sin2\lambda_1 x -$$

$$-sh2\lambda_1 x cos2\lambda_1(1-x) - ch2\lambda_1 x sin2\lambda_1(1-x) - sh2\lambda_1(1-x)cos2\lambda_1 x].$$

As a result, we get the following estimate

$$\int_0^1 \int_0^1 |G_s(x,\xi)|^2 dx d\xi = \frac{1}{8\lambda_1|\lambda|^2|sh\lambda|^2}\left(sh2\lambda_1 x + sin2\lambda_1 x - \frac{ch2\lambda_1}{\lambda_1} + \frac{cos2\lambda_1}{\lambda_1}\right) \leq \frac{C}{|\lambda|^3},$$

$$\lambda_1 = Re\lambda = Im\lambda, 2|\lambda_1|^2 = |\lambda|^2.$$

To receive estimate (23) for $s \neq 0$ (20) we obtain the following equalities

$$\lambda^2 \cdot \frac{d^2u_s}{dx^2} = -\lambda^2 f_s(x) + \lambda^4 \int_0^1 G_s(x,\xi)f_s(\xi)d\xi + \alpha \cdot \delta_s^{-1}\lambda^4 \frac{sh\lambda x}{sh\lambda} \cdot \left[\int_0^1 G_s(\overline{x},\xi)f_s(\xi)d\xi - \frac{f_s(\overline{x})}{\lambda^2}\right]; \tag{24}$$

$$\frac{d^4u_s(x)}{dx^4} = -\frac{d^2f_s(x)}{dx^2} + \lambda^4 \int_0^1 G_s(x,\xi)f_s(\xi)d\xi - \lambda^2 f_s(x) +$$

$$+\alpha \cdot \delta_s^{-1}\left[\int_0^1 G_s(\overline{x},\xi)f_s(\xi)d\xi - \frac{f_s(\overline{x})}{\lambda^2}\right] \cdot \lambda^4 \frac{sh\lambda x}{sh\lambda}. \tag{25}$$

For some terms of solution (20) and their derivatives (24)–(25) we get the following inequality

$$\left\|\lambda^2 f_s(x)\right\|^2_{L_2(0,1)} = \left\||\lambda|^2 f_s(x)\right\|^2_{L_2(0,1)} \leq \left\||\lambda|^{\frac{5}{2}} f_s(x)\right\|^2_{L_2(0,1)}.$$

We take into account $\lambda^2 = is, s = \pm1, \pm2, \ldots$

$$\int_0^1 \left|\lambda^4 \int_0^1 G_s(x,\xi)f_s(\xi)d\xi\right|^2 dx \leq \left\||\lambda|^{\frac{5}{2}} f_s\right\|^2_{L_2(0,1)} \times \int_0^1 \left\||\lambda|^{\frac{3}{2}} G_s\right\|^2_{L_2(0,1)} dx \leq \| |\lambda|^{\frac{5}{2}} f_s\|^2_{L_2(0,1)} \cdot |\lambda|^3 \cdot \frac{C^2}{|\lambda|^3} =$$

$$= C^2 \left\| |\lambda|^{\frac{5}{2}} f_s \right\|^2_{L_2(0,1)}$$

$$\int_0^1 \left| \lambda^4 \frac{sh\lambda x}{sh\lambda} \int_0^1 G_s(\overline{x}, \xi) f_s(\xi) d\xi \right|^2 dx = \frac{|\lambda|^8}{|sh\lambda|^2} \int_0^1 |sh\lambda x|^2 dx \left(\int_0^1 |G_s(\overline{x}, \xi) f_s(\xi)| d\xi \right)^2 \leq$$

$$\leq K_1 |\lambda|^4 \|f_s(x)\|^2_{L_2(0,1)} = K_1 \left\| |\lambda|^2 f_s(x) \right\|^2_{L_2(0,1)},$$

there, we are used the following

$$\int_0^1 |sh\lambda|^2 dx = \frac{1}{2|sh\lambda^2|} \int_0^1 (ch2\lambda_1 x - cos2\lambda_1 x) dx = \frac{sh2\lambda_1 - sin2\lambda_1}{4\lambda_1 |sh\lambda|^2} \leq \frac{C}{|\lambda|}.$$

We get estimate

$$\left\| \lambda^4 \frac{sh\lambda x}{sh\lambda} \cdot \frac{f_s(\overline{x})}{\lambda^2} \right\|^2_{L_{2(0,1)}} = \frac{|\lambda|^4}{|sh\lambda|^2} \int_0^1 |sh\lambda x|^2 dx \cdot \left(\int_0^{\overline{x}} |f_s'(\xi)| d\xi \right)$$

$$\leq C \cdot \left\| |\lambda|^{\frac{3}{2}} f_s'(x) \right\|^2_{L_2(0,1)} \leq C \cdot \left\| |\lambda|^{\frac{3}{2}} f_s(x) \right\|^2_{W_2^2(0,1)}$$

or

$$\left\| |\lambda|^2 \cdot \frac{d^2 u_s}{dx^2} \cdot \frac{f_s(\overline{x})}{\lambda^2} \right\|^2_{L_{2(0,1)}} \leq K_1 \left[|| |\lambda|^{\frac{5}{2}} f_s(x) \|^2_{L_2(0,1)} + \| |\lambda|^{\frac{3}{2}} f_s(x) \|^2_{W_2^2(0,1)} \right].$$

Taking into account (24), and terms on the right side of equality (25) are covered by the right side of the equality, the next assessment will not be difficult to obtain

$$\left\| \frac{d^4 u_s(x)}{dx^4} \cdot \frac{f_s(\overline{x})}{\lambda^2} \right\|^2_{L_{2(0,1)}} \leq K_2 \left[|| |\lambda|^{\frac{5}{2}} f_s(x) \|^2_{L_2(0,1)} + \| |\lambda|^{\frac{3}{2}} f_s(x) \|^2_{W_2^2(0,1)} + \|f_s(x)\|^2_{W_2^2(0,1)} \right].$$

Now, we can determine that the estimate is valid. Really,

$$||u_s(x)\|^2_{L_2(0,1)} \leq K \left[\| \frac{f_s(x)}{|\lambda|} \|^2_{L_{2(0,1)}} + \| \frac{f_s(x)}{|\lambda|^{\frac{3}{2}}} \|^2_{L_{2(0,1)}} + \| \frac{f_s(x)}{|\lambda|^2} \|^2_{L_{2(0,1)}} \right] \leq$$

$$K_1 \left\| \frac{f_s(x)}{|\lambda|} \right\|^2_{L_{2(0,1)}} \leq ||f_s(x)\|^2_{L_2(0,1)} \leq K_3 ||f_s''(x)\|^2_{L_2(0,1)},$$

for $s = 0$. Based on formula (20), this calculation is taken in a simple form.

Taking into account estimate (23), on the basis of the results [8] (p. 118-119) we proved uniqueness of strong solution to boundary value problem (7)-(9). The theorem is proved.

In addition, from the above estimates, the following uniform estimate for $s \in v$ is derived by the formula

$$\left\| |\lambda|^2 \cdot \frac{d^2 u_s(x)}{dx^2} \right\|^2_{L_{2(0,1)}} + \left\| \frac{d^4 u_s(x)}{dx^4} \right\|^2_{L_{2(0,1)}} \leq$$

$$\leq K \left[\left\| |\lambda|^{\frac{5}{2}} f_s(x) \right\|^2_{L_{2(0,1)}} + \left\| |\lambda|^{\frac{3}{2}} f_s(x) \right\|^2_{W^2_{2(0,1)}} + \|f_s(x)\|^2_{W^2_{2(0,1)}} \right], \qquad (26)$$

in addition, For derivative the next estimate is valid

$$\left\| \frac{\partial^3 u}{\partial x^2 \partial e} \right\|_{L_2(Q)} \leq K \left[\|f(x,t)\|_{W^{\frac{5}{2}}_{2(0.2\pi; L_2(0,1))}} + \| |f_x(x,t)\|^2_{W^{\frac{3}{4}}_{2(0.2\pi; L_2(0,1))}} + \|f(x,t)\|_{L_{2(0.2\pi; W_2^2(0,1) \cap W_2^1(0,1))}} \right]$$

Estimate (26) presents that the strong solution to the boundary value problem has the differential property that is given by estimate (23).

Thus, in condition (3), the requirement for the function $f(x,t)$ can be replaced by the following: $-f \in L_2(0,2\pi; W_2^2(0,1))$. In this case (23), the estimate has the following form

$$\|u_s(x)\|_{L_2(0,1)} \leq K \cdot \|f_s(x)\|_{W^2_{2(0,1)}}, s \in v.$$

Definition 3. Let $u_n(x,t)_{n=1}^{\infty} \subset \widetilde{C}$ be a sequence of functions and

$$1^0.L_2(Q)in \lim_{n\to\infty} u_n(x,t) = u(x,t);$$

$$2^0.L_2(Q)in \lim_{n\to\infty} L_4 u_n(x,t) = \frac{\partial^{2m} f}{\partial x^{2m}}.$$

Then the function $u(x,t)$ is-(14) are called a strong solution to boundary value problem (10)-(14).

If

$$u \in C(\widetilde{Q}), u \in C(0,2\pi; C^{2m+2}(0,1) \cap C^{m+1}[0,1]), \frac{\partial u(x,t)}{\partial t} \in C(0,2\pi; C^{2m}(0,1) \cap C^m[0,1]),$$

we assume that the conditions $u \in \widetilde{C}$ and (11)-(14) are satisfied.

Definition 4. The strong solution to boundary value problem (10)-(14) is called a strong solution to boundary value problem (4)-(5).

From these definitions it follows that the domains of the closed operators L_2 and L_4 are equal.

We get the following

$$D(L_2) \equiv D(L_4) \equiv$$

$$\left\{u | u \in L_2\left(0,2\pi; W_2^{2m+2}(0,1)\right), \frac{\partial u}{\partial t} \in L_2\left(0,2\pi; W_2^{2m}(0,1)\right),\right\} \times \textit{ boundary conditions}(11)-(14) \quad (27)$$

For the second boundary value problem, the following statement is valid [13-15].

Theorem 2. Let

$$\delta_s \equiv 1 + \frac{\partial^k}{\partial x^k} \int_0^1 G_s(\overline{x},\xi)\alpha(\xi)d\xi \neq 0, \forall_s \in v, \quad (28)$$

where $v = \{s | s = 0; \pm 1; \pm 2; \ldots\}, \lambda^2 = is, i = \sqrt{-1}, G_s(\overline{x},\xi)$ is the function defined by the formula from conditions (21), then for any $f \in L_2\left(0,2\pi; W_2^{2m}(0,1) \cap \dot{W}_2^m(0,1)\right), \alpha \in W_2^{2m}(0,1)$ the function $u(x,t)$ is a strong solution to boundary value problem (4)-(5). The proof of the second theorem is similar to the proof of the first theorem.

The validity of the statements follows from (27) and (28).

3 Conjugated problem

Consider conjugate problem (1)-(2),

$$L_1^* \psi \equiv -\frac{\partial \psi}{\partial t} - \frac{\partial^2 \psi}{\partial t^2} + \delta''(x-\overline{x}) \bigotimes \int_0^1 \alpha \cdot \xi \cdot \psi(\xi,t)d\xi = g(x,t), \{x,t\} \in Q, \quad (29)$$

$$\psi(0,t) = \psi(1,t) = 0; \psi(x,0) = \psi(x,2\pi), \overline{x} \in (0,1), \quad (30)$$

here it is taken into account that the value $supp\{\psi(x,1)\} \subseteq \overline{Q}$. A weak solution $\psi \in L_2(Q)$ of this problem we define the following integral equality: for any $\omega \in \widetilde{C}$ (from the first definition) $(\omega, L_1^*\psi) = (L_1, \omega, \psi) = (\omega, g)$.

First, we show that the operator L_1^* is conjugate with the operator L_1. To do this, it is enough to make sure that the following relation is valid

$$\int_0^{2\pi}\int_0^1 x \frac{\partial^2 u(\overline{x},t)}{(\partial x^2)} \psi(x,t)dxdt = \int_0^{2\pi}\int_0^1 \delta''(x-\overline{x}) \left(\int_0^1 \xi\psi(\xi,t)d\xi\right) u(x,t)dxdt.$$

Indeed,

$$\int_0^{2\pi}\int_0^1 x \frac{\partial^2 u(\overline{x},t)}{(\partial x^2)} \psi(x,t)dxdt = \int_0^{2\pi}\int_0^1 x\psi(x,t) \left[\int_0^1 \delta(\xi-\overline{x}) \frac{\partial^2 u(\xi,t)}{\partial \xi^2} d\xi\right] dxdt =$$

$$= \int_0^{2\pi}\int_0^1 \delta(\xi-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)\frac{\partial^2 u(x,t)}{(\partial x^2)}dxdt = \int_0^{2\pi}\delta(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)\frac{\partial u(x,t)}{(\partial x)}|_0^1 dt -$$

$$-\int_0^{2\pi}\int_0^1 \delta'(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)\frac{\partial u(x,t)}{\partial x}dxdt = -\int_0^1 \delta(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)u(x,t)|_0^1 dt +$$

$$+\int_0^{2\pi}\int_0^1 \delta'(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)u(x,t)dxdt = \int_0^{2\pi}\int_0^1 \delta''(x-\overline{x})\left(\int_0^1 \xi\psi(\xi,t)d\xi\right)u(x,t)dxdt.$$

Using the method of separation of variables from (29) - (30), we get the corresponding system of problems for the Fourier coefficient $\psi_s(x), s \in v\backslash\{0\}$

$$\begin{cases} -is\psi_s(x) - \psi_s''(x) + \delta(x-\overline{x})\int_0^1 \alpha\cdot\xi\cdot\psi_s(x)d\xi = g_s(x), x \in (0,1), \\ \psi_s(0) = \psi_s(1) = 0, \forall s \in \backslash\{0\}. \end{cases}$$

The solutions to these problems have the form: {the Fourier coefficient of the function $g(x,t)$ according to $g_s(x)$}

$$\psi_s(x) = \int_0^1 \widetilde{G}_S(x,\xi)g_s(\xi)d\xi + \int_0^1 \xi\psi_s(\xi)d\xi\cdot[\lambda^2\cdot\widetilde{G}_s(x,\overline{x})]$$

where

$$\widetilde{G}_S(x,\xi) = \begin{cases} \frac{sin\{\lambda\xi\}sin\{\lambda(1-x)\}}{\lambda\sin(\lambda)}, 0 \le \xi \le x \le 1, \\ \frac{sin\{\lambda\xi\}sin\{\lambda(1-\xi)\}}{\lambda sin(\lambda)}, 0 \le x \le \xi \le 1, \end{cases} \quad \forall s \in v \tag{31}$$

iff

$$\widetilde{\delta}_s = 1 + \alpha\overline{x} - \alpha\frac{sin(\lambda\overline{x})}{sin\lambda} \neq 0, \forall s \in v(\lambda^2 = is). \tag{32}$$

The expressions for $G_0(x,\xi)$ and δ_0 can be obtained directly or passing to the limit as $s \to 0(\lambda \to 0)$ in formulas (31) and (32):

$$\widetilde{G}_0(x,\xi) = \begin{cases} \xi(1-x), 0 \le \xi \le x \le 1; \\ x(1-\xi), 0 \le x \le \xi \le 1; \end{cases}$$

$$\widetilde{\delta}_0 = 1.$$

Remark 3. Let $\alpha \in R^1$. If the function $\omega_0(x)$ (17) given by (1)-(2) is orthogonal to all functions $g(x,t)$ from conjugate problem (29)-(30) (by Corollary 2) then $\omega_0(x)$ (17) is a univocal weak solution. In this case, condition (32) is valid for all $s \in v$.

Remark 4. If $g(x,t) \equiv 0$ then (29)-(30) has a unique solution $\psi(x,t) = \delta(x-\overline{x})$.

Note that to study the integral equation to which the problem for a parabolic equation has been reduced, we can use the Laplace transform by applying the model solution method [16].

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Шектелген аймақтардағы елеулі жүктелген параболалық теңдеулерге арналған шекаралық есептер туралы

Мақалада шектелген аймақтағы елеулі жүктелген параболалық дифференциалдық теңдеулерге арналған әлді шешім сұрақтары зерттелген. Қарастырылған есептердің ерекшеліктері: мысалы, $L_2(Q)$ кеңістігіне сәйкес дифференциалдық операторлар тұйықтаушы болмайды, себебі, біріншіден, жүктеме қарастырылып отырған оператордың сәйкес дифференциалдық бөлігіне бағынбайды, яғни оның дифференциалдық бөлігі үшін әлсіз ауытқу болып табылмайды. Екіншіден $L_2(0,1)$ және $L_2(Q)$ кеңістіктерінде жүктеме операторларының өздері тұйықтаушы операторлар болып табылмайтыны белгілі. Осының барлығы тұйықталмайтын жүктелген дифференциалдық теңдеулерге арналған әлді шешімді шекаралық есептер сұрақтарын тікелей зерттеу мүмкін емес екенін көрсетеді. Алайда [1-4] теңдеулерін зерттеу теориялық қана емес, анық қолданбалы [5-7] сипат береді.

Кілт сөздер: елеулі жүктелген дифференциалдық теңдеулер, Вольтерр интегралдық теңдеуі, шекаралық есеп, әлді шешім, оператор.

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О граничных задачах для существенно нагруженных параболических уравнений в ограниченных областях

В статье изучены вопросы сильного решения для существенно нагруженных дифференциальных уравнений параболического типа в ограниченных областях. Особенности рассматриваемых задач: например, в $L_2(Q)$ пространстве соответствующие дифференциальные операторы не являются операторами замыкания, поскольку, во-первых, нагрузка не подчиняется соответствующей дифференциальной части рассматриваемого оператора, то есть для его дифференциальной части не является слабым возмущением. Во-вторых, очевидно, что в пространствах $L_2(0,1)$ и $L_2(Q)$ операторы нагрузки сами не являются операторами замыкания. Все это указывает на то, что невозможно непосредственно исследовать вопросы сильного решения граничных задач для незамкнутых нагруженных дифференциальных уравнений. Однако исследование уравнений [1–4] дает не только теоретический, но и выраженный прикладной [5–7] характер.

Ключевые слова: существенно нагруженные дифференциальные уравнения, интегральное уравнение Вольтерра, граничная задача, сильное решение, оператор нагрузки.